A Family of Finite De Morgan Algebras

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Type-2 Fuzzy Sets

- In 1975, Zadeh proposed a setting generalizing that of both type-1 and interval-valued fuzzy sets. The truth value algebra for this new fuzzy set theory has been studied extensively. Its definition follows.

Definition On $[0,1]^{[0,1]}$, let

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \wedge g(z))$$

$$(f \sqcap g)(x) = \bigvee_{y \wedge z = x} (f(y) \wedge g(z))$$

$$f^*(x) = \bigvee_{1-y=x} f(y) = f(1-x)$$

$$\bar{1}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases} \quad \bar{0}(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

Definition The **algebra of truth values for type-2 fuzzy sets** is

$$\mathbf{M} = ([0,1]^{[0,1]}, \sqcup, \sqcap, *, \bar{0}, \bar{1})$$

Definition For $f \in \mathbf{M}$, let f^L and f^R be the elements of $\mathbf{M}$ defined by

$$f^L(x) = \bigvee_{y \leq x} f(y)$$

$$f^R(x) = \bigvee_{y \geq x} f(y)$$

Theorem The following hold for all $f, g \in \mathbf{M}$.

$$
\begin{aligned}
f \sqcup g &= (f \wedge g^L) \vee (f^L \wedge g) \\
&= (f \vee g) \wedge (f^L \wedge g^L)
\end{aligned}
$$

$$
\begin{aligned}
f \sqcap g &= (f \wedge g^R) \vee (f^R \wedge g) \\
&= (f \vee g) \wedge (f^R \wedge g^R)
\end{aligned}
$$

Corollary Let $f, g, h \in \mathbf{M}$. The basic properties of $\mathbf{M}$ follow.

1. $f \sqcup f = f$; $f \sqcap f = f$

2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$

3. $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$

4. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$

5. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$

6. $f^{**} = f$; $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$

Problem Does $\mathbf{M}$ satisfy any equation not a consequence of these equations? That is, are these equations an equational base for the variety generated by $\mathbf{M}$?

Problem Is the variety generated by $\mathbf{M}$ generated by a finite algebra?

Definition An element f of $\mathbf{M}$ is **normal** if

$\sup\{f(x) : x \in [0,1]\} = 1$.

Proposition The normal functions form a subalgebra $\mathbf{N}$.

Definition An element f of $\mathbf{M}$ is **convex** if for $x \leq y \leq z$,

$f(y) \geq f(x) \wedge f(z)$. Equivalently, f is convex if $f = f^L \wedge f^R$.

Proposition The convex functions form a subalgebra $\mathbf{C}$.

Theorem The subalgebra $\mathbf{D} = \mathbf{C} \cap \mathbf{N}$ is a De Morgan algebra, and is a maximal lattice in $\mathbf{M}$.

- The basic theory goes through when $[0, 1]$ is replaced by any two finite chains.
- In that case, $\mathbf{D}$ is a finite De Morgan algebras.
- So any two finite chains give rise to a finite De Morgan algebra.
- This family of finite De Morgan algebras is the subject of this paper.

Notation and Terminology

- For a positive integer k, let $\mathbf{k}$ be the linearly ordered set with k elements.

- $[0,1]^{[0,1]}$ is replaced by $\mathbf{m^n}$, and the convex normal functions form a De Morgan algebra denoted $\mathbf{D(m^n)}$.

- The elements of elements $\mathbf{D(m^n)}$ are denoted by n-tuples from $\{1,2,\ldots,m\}$.

- To be normal requires that each n-tuple $(a_1,a_2,\ldots,a_n)$ contains m as an entry.

- To be convex requires that each n-tuple $(a_1, a_2, \ldots, a_n)$ be increasing until the first entry that is m, and be decreasing after that.

- The negation on n-tuples comes from the negation $\mathbf{n} \to \mathbf{n}$ given by $i^* = n - i + 1$. Thus $(a_1, a_2, \ldots, a_n)^* = (a_n, a_{n-1}, \ldots, a_1)$.

- The lattice operations $\sqcup$ and $\sqcap$ are as defined earlier.

D(2^2)

1	2
2	2
2	1

D(3^2)

1	3
2	3
3	3
3	2
3	1

D(4^2)

1	4
2	4
3	4
4	4
4	3
4	2
4	1

D(2^3)

1	1	2
1	2	2
2	2	2
2	2	1
1	2	1
2	1	1

D(3^3)

1	1	3
1	2	3
2	2	3
1	3	3
2	3	3
1	3	1
1	3	2
2	3	1
2	3	2
3	3	1
3	3	2
3	3	3
3	2	2
3	2	1
3	1	1

Other Representations of $\mathbf{D(m^n)}$

- The partial order on $\mathbf{D(m^n)}$ given by the lattice operations $\sqcup$ and $\sqcap$ is not the coordinate-wise partial order on the n-tuples.

- We give another representation of the bounded lattice $\mathbf{D(m^n)}$ as n-tuples in which the partial order is coordinate-wise.

Definition $\mathbf{D}_1(\mathbf{m^n})$ is the algebra whose elements are decreasing n-tuples of elements from $\{1, 2, \ldots, 2m-1\}$ which include m, and whose operations are given by pointwise $\max$ and $\min$ on these n-tuples.

$\mathbf{D}_1(\mathbf{m^n})$ is clearly a bounded lattice.

Theorem For $a = (a_1, a_2, \ldots, a_n) \in \mathbf{D}(\mathbf{m^n})$, let i be the smallest index i for which $a_i = m$. The mapping

$$a \to (2m - a_1, 2m - a_2, \ldots, 2m - a_{i-1}, a_i, a_{i+1}, \ldots, a_n)$$

is an isomorphism from $\mathbf{D}(\mathbf{m^n})$ to $\mathbf{D}_1(\mathbf{m^n})$.

- Endow $\mathbf{D}_1(\mathbf{m^n})$ with the negation given by this lattice isomorphism: $(\varphi(a))^*$ to be $\varphi(a^*)$.
- In $\mathbf{D}_1(\mathbf{m^n})$ the negation of $(b_1, b_2, \ldots, b_n)$, is $(2m - b_n, 2m - b_{n-1}, \ldots, 2m - b_2, 2m - b_1)$.
- $\mathbf{D}(\mathbf{m^n})$ and $\mathbf{D}_1(\mathbf{m^n})$ isomorphic as De Morgan algebras.

- For each n-tuple in $\mathbf{D}_1(\mathbf{m^n})$, remove the entry with the smallest index that is equal to m.

- This yields all decreasing n-1-tuples from $\{1, 2, \ldots, 2m-1\}$.

- With pointwise operations of $\max$ and $\min$ and negation $(b_1, b_2, \ldots, b_{n-1})^* = (2m - b_{n-1}, 2m - b_{n-2}, \ldots, 2m - b_2, 2m - b_1)$ this clearly yields a De Morgan algebra $\mathbf{D}_2(\mathbf{m^n})$ isomorphic to $\mathbf{D}_1(\mathbf{m^n})$.

- Of course, the elements of this algebra is the set of all decreasing maps from $\mathbf{n-1}$ into $\mathbf{2m-1}$.

- So $\mathbf{D}_2(\mathbf{m}^\mathbf{n})$ is the set of all anti-homomorphisms from the ordered set $\mathbf{n-1}$ into the ordered set $\mathbf{2m-1}$.

- In any case, as De Morgan algebras we have

$$\mathbf{D}(\mathbf{m}^\mathbf{n}) \approx \mathbf{D}_1(\mathbf{m}^\mathbf{n}) \approx \mathbf{D}_2(\mathbf{m}^\mathbf{n})$$

The Cardinality of $\mathbf{D(m^n)}$

Proposition The number of decreasing a-tuples from $\{1, 2, \ldots, i\}$ is $\frac{((i-1)+a)!}{(i-1)!a!}$.

Theorem $|\mathbf{D(m^n)}| = \dfrac{(2m-2+n-1)!}{(2m-2)!(n-1)!}$.

- $\dfrac{(2m-2+n-1)!}{(2m-2)!(n-1)!}$ is the number of subsets of $\{1, 2, \ldots, 2m-2+n-1\}$ of size $n-1$.

- This is the same as the number of strictly decreasing $n-1$ tuples from $\{1, 2, \ldots, 2m-2+n-1\}$. This is yet another representation of the *elements* of $\mathbf{D}(\mathbf{m}^n)$, but do the lattice operations correspond to pointwise $\max$ and $\min$?

Definition $\mathbf{D}_3(\mathbf{m}^\mathbf{n})$ is the algebra whose elements are the $n-1$ tuples of strictly decreasing sequences from $\{1, 2, \ldots, 2m-2+n-1\}$ with operations pointwise $\max$ and $\min$ the obvious constants, and $(a_1, a_2, \ldots, a_{n-1})^* = (2m-2+n-a_{n-1}, \ldots, 2m-2+n-a_1)$.

Theorem $\mathbf{D}_2(\mathbf{m^n}) \approx \mathbf{D}_3(\mathbf{m^n})$.

Proof The mapping $\mathbf{D}_2(\mathbf{m^n}) \rightarrow \mathbf{D}_3(\mathbf{m^n})$ given by

$$(a_1, a_2, \ldots, a_{n-1}) \rightarrow (a_1 + (n-2), a_2 + (n-3), \ldots, a_{n-2} + 1, a_n$$

is the desired isomorphism.

We now have four representations:

1. $\mathbf{D(m^n)}$, the normal convex functions of n-tuples from $\{1, 2, \ldots, m\}$,

2. $\mathbf{D}_1\mathbf{(m^n)}$, the decreasing n-tuples of elements from $\{1, 2, \ldots, 2m-1\}$, that have m as an entry,

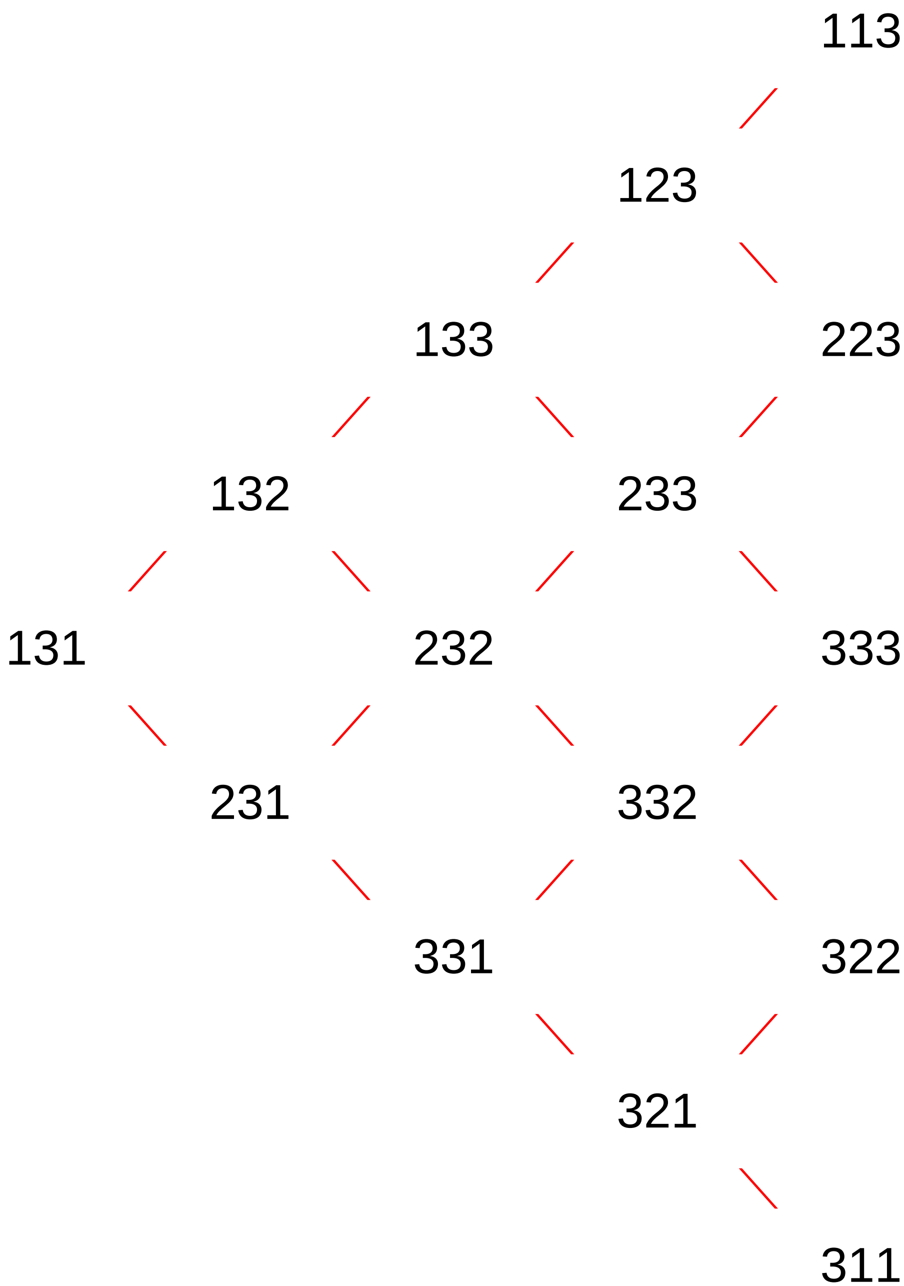
3. $\mathbf{D}_2\mathbf{(m^n)}$, the decreasing $(n-1)$-tuples of elements from $\{1, 2, \ldots, 2m-1\}$,

4. $\mathbf{D}_3\mathbf{(m^n)}$, the strictly decreasing $(n-1)$-tuples of elements from $\{1, 2, \ldots, (2m-2)+(n-1)\}$.

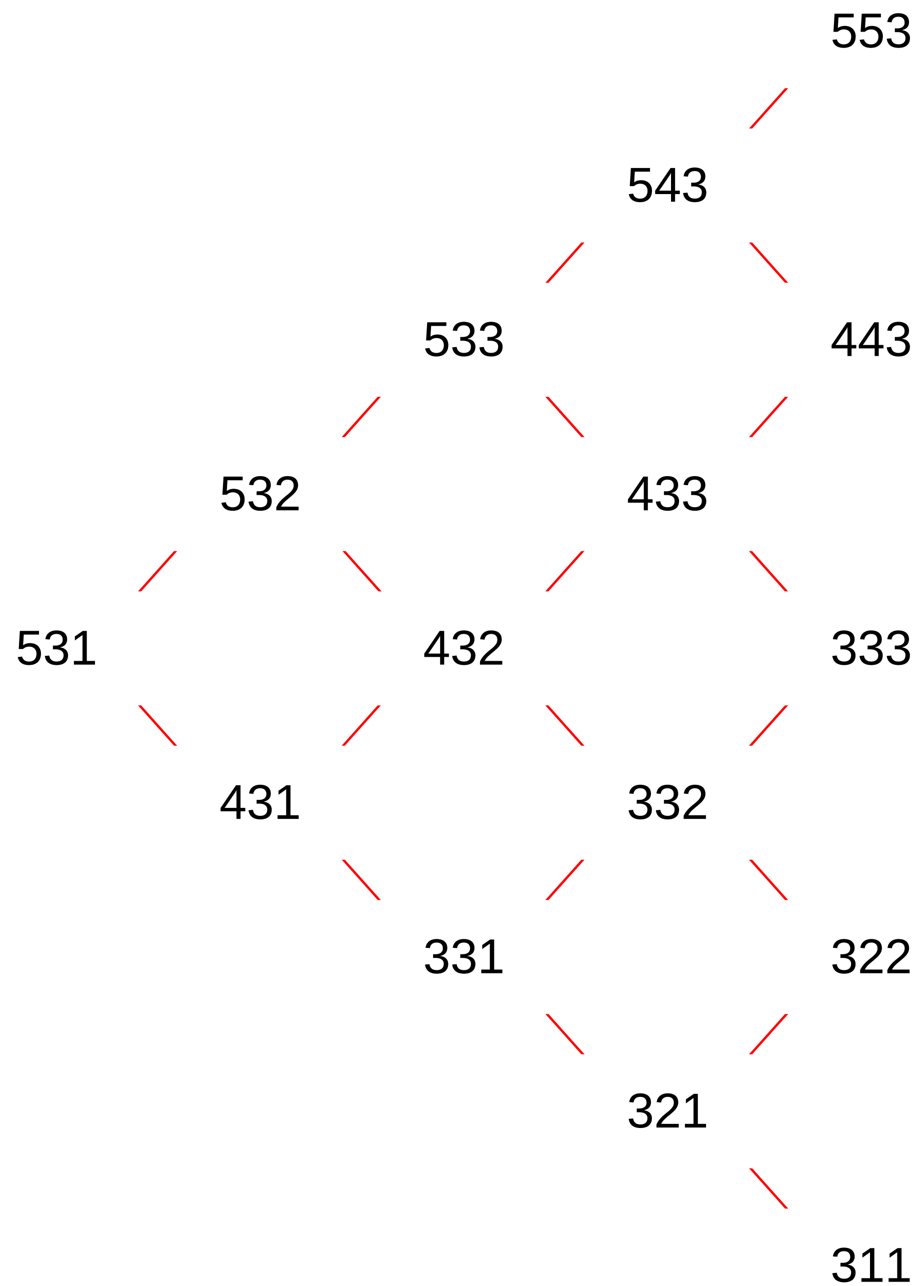
In the last three representations, the lattice operations are pointwise, and the negations are as indicated earlier. We will not use representation 4, but just note that it came about from the combinatorial result of the cardinality of $\mathbf{D}_2(\mathbf{m}^{\mathbf{n}})$.

To illustrate, below we show each representation for $m = n = 3$.

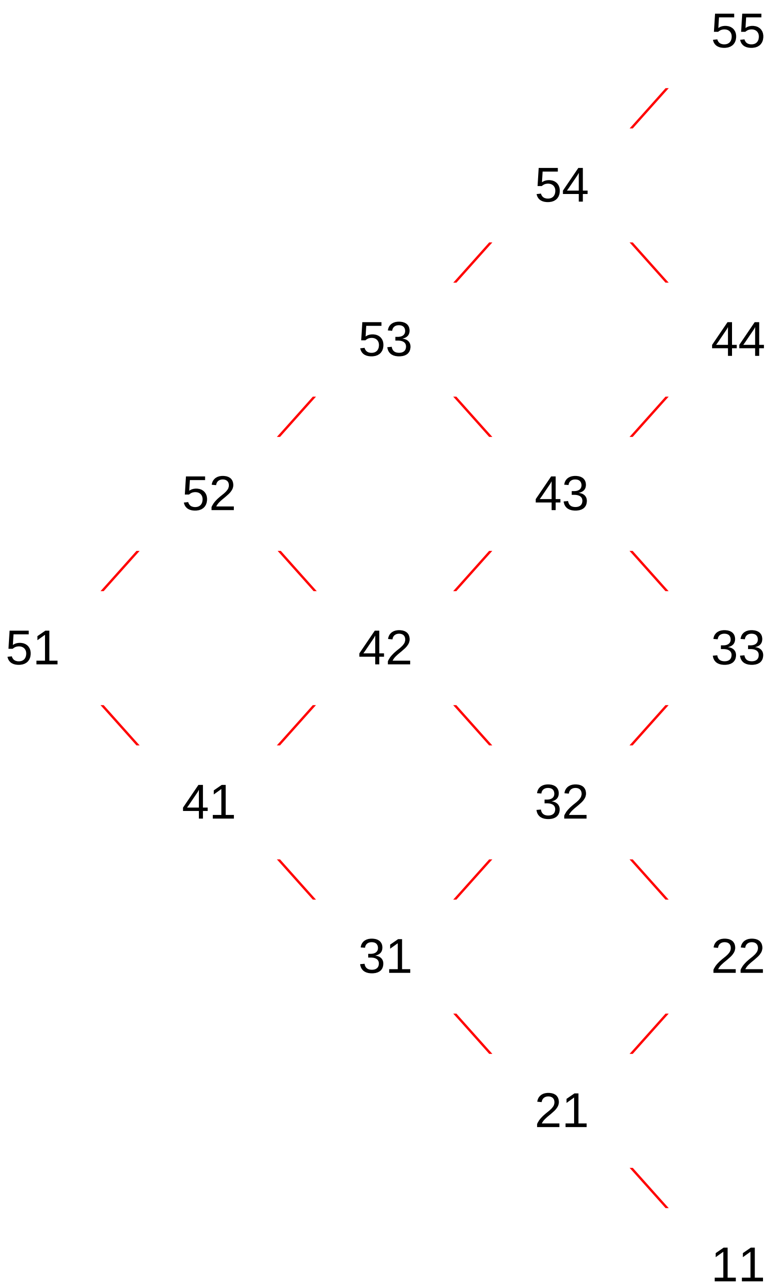
$\mathbf{D}(\mathbf{3}^3)$



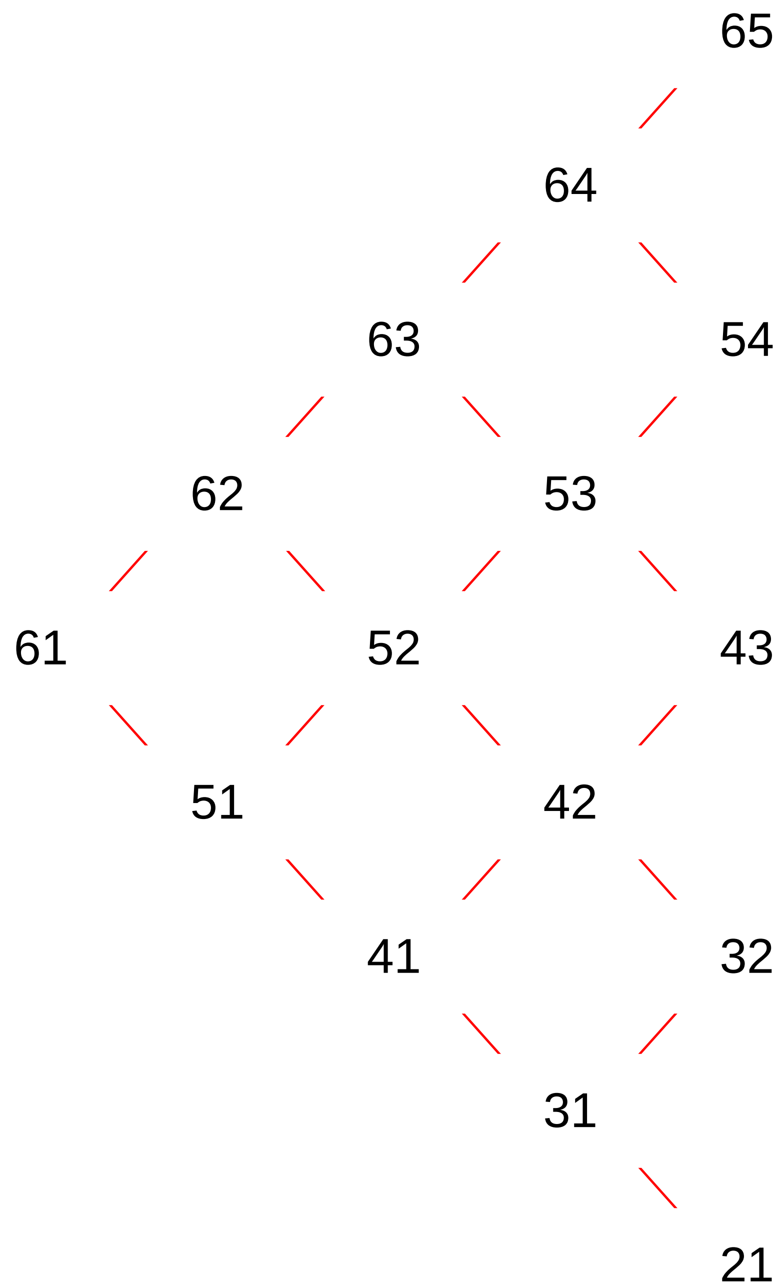
$\mathbf{D}_1(\mathbf{3}^3)$



$\mathbf{D}_2(\mathbf{3}^3)$



$\mathbf{D}_3(\mathbf{3}^3)$



There is difficulty in depicting such lattices as those above for larger m and n because of the following.

Proposition The De Morgan algebras $\mathbf{H(m^n)}$ are not planar if $m \geq 4$ and $n \geq 3$.

Proof A finite distributive is planar if and only if no element has 3 covers (Gratzer, page. 90, problem 45). For example, the 3 tuple $(3, 2, 1)$ has covers $(4, 2, 1)$,$(3, 3, 1)$, and $(3, 2, 2)$.

The De Morgan Algebras $\mathbf{H(m^n)}$

In the algebra $\mathbf{D}_2(\mathbf{m^n})$, the tuples can be of any positive integer length, but entries must come from a set with an odd number of elements, namely $\{1, 2, \ldots, 2m - 1\}$.

Definition For positive integers m and n, let $\mathbf{H(m^n)}$ be the algebra of all decreasing n-tuples from $\{1, 2, \ldots, m\}$, with pointwise operations $\vee$ and $\wedge$ of $\max$ and $\min$, with negation

$$(a_1, a_2, \ldots, a_n)^* =$$
$$(m + 1 - a_n, m + 1 - a_{n-1}, \ldots, m + 1 - a_2, m + 1 - a_1)$$

Theorem $|\mathbf{H}(\mathbf{m}^{\mathbf{n}})| = \frac{((m-1)+n)!}{(m-1)!n!}$.

- $|\mathbf{H}(\mathbf{m}^{\mathbf{n}})| = \left|\mathbf{H}((\mathbf{n}+\mathbf{1})^{\mathbf{m}-\mathbf{1}})\right|$.
- $\mathbf{H}(\mathbf{m}^{\mathbf{n}})$ is a De Morgan algebra.
- $\mathbf{H}(\mathbf{m}^{\mathbf{n}})$ is the set of all anti-homomorphisms from the poset $\mathbf{n}$ to the poset $\mathbf{m}$.
- $\mathbf{D}_2(\mathbf{m}^{\mathbf{n}}) = \mathbf{H}((\mathbf{2m}-\mathbf{1})^{\mathbf{n}-\mathbf{1}})$
- So, we now investigate the larger family $\mathbf{H}(\mathbf{m}^{\mathbf{n}})$.

The Join Irreducibles of $\mathbf{H(m^n)}$

Definition For $1 \le i \le n-1$, an n-tuple in $\mathbf{H(m^n)}$ has a **jump** at i if its $i+1$ entry is strictly less than its $i-th$ entry. It has a **jump** at n if the $n-th$ entry is at least 2.

- $(5,5,5,3,1)$ has jumps at 3 and 4
- $(8,7,2,2,2,2)$ has jumps at 1 and 2 and 6.
- $(5,5,5,5,1,1)$ has a jump at 4.
- The only n-tuple with no jumps is $(1,1,\ldots,1)$.

Theorem The join irreducibles of $\mathbf{H(m^n)}$ are those n tuples with exactly one jump.

Corollary The join irreducible elements of $\mathbf{H(m^n)}$ are of the form $(a, a, \ldots, a, 1, 1, \ldots, 1)$, with $a > 1$ and at least one a in the tuple.

Thus with each join irreducible, there is associated a pair of integers, the integer a and the index of the last a.

For example, we have the following associations.

$$(5, 5, 1, 1, 1) \rightarrow (5, 2)$$

$$(5, 5, 5, 5, 5) \rightarrow (5, 5)$$

This association gets a map from the join irreducibles of $\mathbf{H(m^n)}$ to the poset $(\mathbf{m} - \mathbf{1}) \times \mathbf{n}$. (Here, we are associating the poset $\{2, 3, \ldots, m\}$ with the poset $\mathbf{m} - \mathbf{1}$.) This is a one-to-one mapping of the join irreducibles of $\mathbf{H(m^n)}$ onto the poset $(\mathbf{m} - \mathbf{1}) \times \mathbf{n}$, and preserves component-wise order.

Theorem The poset of join irreducibles of $\mathbf{H(m^n)}$ is isomorphic to the poset $\mathbf{(m-1) \times n}$, and hence is a bounded distributive lattice.

Because of the categorical equivalence of finite distributive lattice and finite posets, with a finite distributive lattice corresponding to its poset of join irreducible elements, we get the following corollaries.

Corollary $\mathbf{H(m^n)} \approx \mathbf{H(p^q)}$ if and only if $m = p$ and $n = q$ or $m = q + 1$ and $n = p - 1$.

Corollary The automorphism group $Aut(\mathbf{H(m^n)})$ of the lattice $\mathbf{H(m^n)}$ has only one element unless $m - 1 = n$, in which case it has exactly two elements.

Corollary $Aut(\mathbf{D}(\mathbf{m}^n))$ has only one element unless $2m - 1 = n - 1$, in which case it has exactly two elements.

Since the poset of join irreducibles of $\mathbf{H(m^n)}$ is the lattice $(\mathbf{m} - \mathbf{1}) \times \mathbf{n}$, this lattice is in turn determined by its poset of join irreducibles. That poset is simply the disjoint chains $\mathbf{m} - \mathbf{2}$ and $\mathbf{n} - \mathbf{1}$, again not allowing the 0 element to be join irreducible. This again shows, for example that the automorphism group of $\mathbf{H(m^n)}$ has exactly one element unless $m - 2 = n - 1$, or equivalently unless $m - 1 = n$, in which case it has exactly two automorphisms.

- It is not true that $|\mathbf{H(m^n)}|$ determines $\mathbf{H(m^n)}$.
- $|\mathbf{H(8}^3\mathbf{)}| = |\mathbf{H(15^2)}| = 120$, yet the criteria for $\mathbf{H(m^n)} \approx \mathbf{H(p^q)}$ are not met.

A Kleene Subalgebra of $\mathbf{H(m^n)}$

Definition Let $\underline{n} = [\frac{n}{2}]$. Let $\mathbf{KH(m^n)}$ be those n-tuples of $\mathbf{H(m^n)}$ whose first $\underline{n}$ entries are m or whose last $\underline{n}$ entries are 1.

- Notice that this depends on the representation of $\mathbf{H(m^n)}$

Theorem $\mathbf{KH(m^n)}$ is a subalgebra of the De Morgan algebra $\mathbf{H(m^n)}$ and is Kleene. That is, for $a,b \in \mathbf{KH(m^n)}$, $a \vee a^* \geq b \wedge b^*$.

The proof of this theorem is entirely straightforward.

Theorem

$$|\mathbf{KH}(\mathbf{m}^{\mathbf{n}})| = 2|\mathbf{H}(\mathbf{m}^{\underline{\mathbf{n}}})| - 1 \text{ if } n \text{ is even}$$

$$|\mathbf{KH}(\mathbf{m}^{\mathbf{n}})| = 2|\mathbf{H}(\mathbf{m}^{\underline{\mathbf{n}}+1})| - m \text{ if } n \text{ is odd}$$

Any De Morgan algebra has subalgebras that are Kleene, for example the two constants is one such, and more generally, any chain that is closed under the negation operator. The Kleene algebra $\mathbf{KH(m^n)}$ is not necessarily a chain, and in fact is a very special subalgebra of $\mathbf{H(m}^{n}\mathbf{)}$ that is Kleene. Let $\mathbf{SH(m^n)}$ be the $n-tuples$ of $\mathbf{H(m}^{n}\mathbf{)}$ of the form $(m, m, \ldots, m, 1, 1 \ldots, 1)$, that is, those elements whose only entries are m and 1. This is a subalgebra of $\mathbf{H(m^n)}$, and, in fact, is Kleene.

Theorem Any subalgebra A of $\mathbf{H(m}^n\mathbf{)}$ which is Kleene and which contains $\mathbf{SH(m}^n\mathbf{)}$ is contained in $\mathbf{KH(m^n)}$.

Below are some tables of sizes of various of these algebras.

m	$\lvert\mathbf{H}(\mathbf{m}^2)\rvert$	$\lvert\mathbf{KH}(\mathbf{m}^2)\rvert$
2	3	3
3	6	5
4	10	7
5	15	9
6	21	11
7	28	13
8	36	15
9	45	17

m	$\lvert\mathbf{H}(\mathbf{m}^3)\rvert$	$\lvert\mathbf{KH}(\mathbf{m}^3)\rvert$
2	4	4
3	10	9
4	20	16
5	35	25
6	56	36
7	84	49
8	120	64
9	165	81

m	$\lvert\mathbf{H(m^4)}\rvert$	$\lvert\mathbf{KH(m^4)}\rvert$
2	5	5
3	15	11
4	35	19
5	70	29
6	126	41
7	210	55
8	330	71
9	495	89

m	$\lvert\mathbf{H(m^5)}\rvert$	$\lvert\mathbf{KH(m^5)}\rvert$
2	6	6
3	21	17
4	56	36
5	126	65
6	252	106
7	462	161
8	792	232
9	1287	321

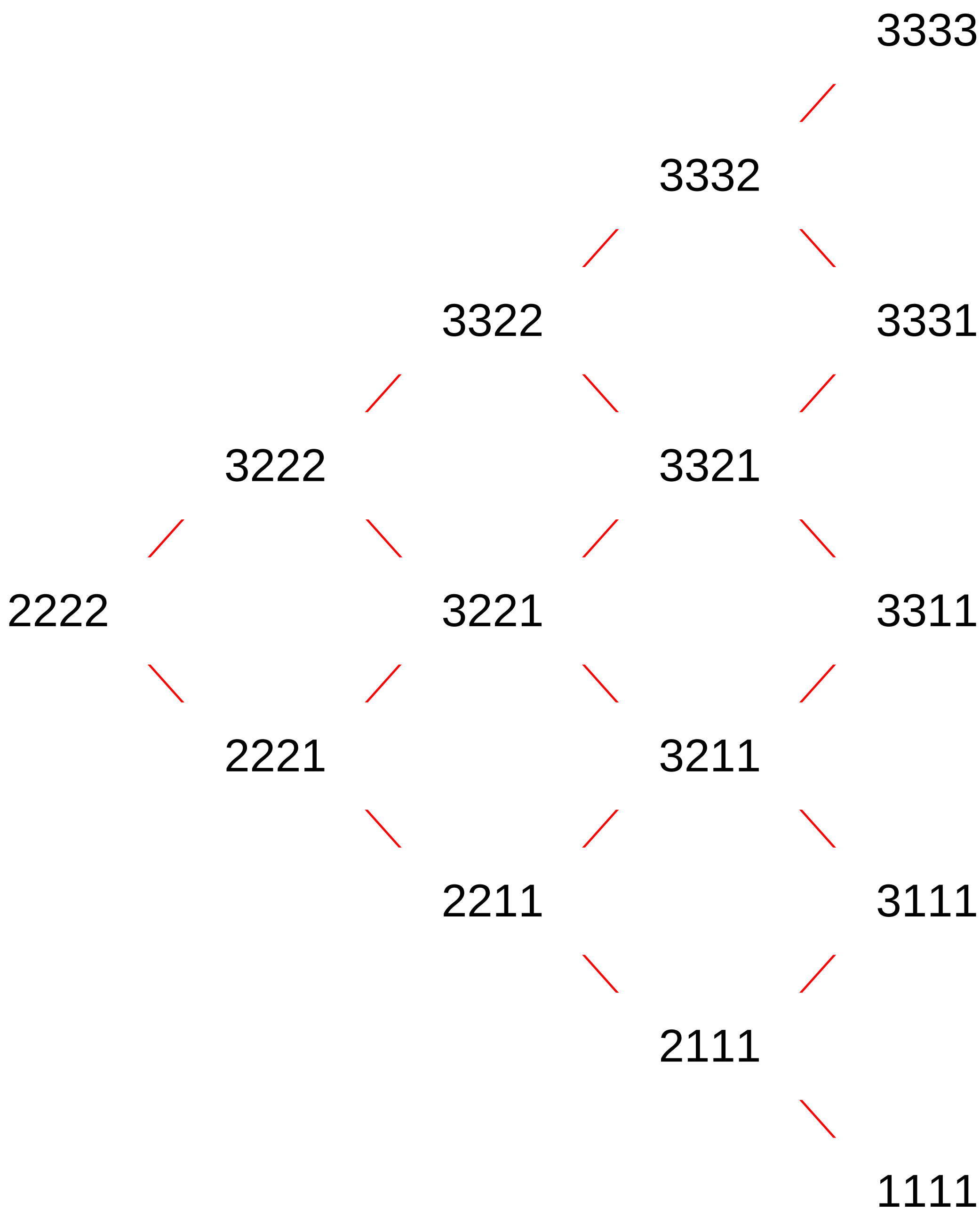
m	$\|\mathbf{H}(\mathbf{m}^6)\|$	$\|\mathbf{KH}(\mathbf{m}^6)\|$
2	7	7
3	28	19
4	84	39
5	210	69
6	462	111
7	924	167
8	1716	239
9	3003	329

m	$\|\mathbf{H}(\mathbf{m}^7)\|$	$\|\mathbf{KH}(\mathbf{m}^7)\|$
2	8	8
3	36	27
4	120	66
5	330	135
6	792	246
7	1716	413
8	3432	652
9	6435	981

- There are many, many combinatorial identities between the various entities above. For example $|\mathbf{H}(\mathbf{m}^7)| = \left|\mathbf{H}((\mathbf{m}-\mathbf{1})^7)\right| + |\mathbf{H}(\mathbf{m}^6)|$, and the same holds for $\mathbf{KH}(\mathbf{m}^7)$. This is not surprising since $|\mathbf{H}(\mathbf{m}^\mathbf{n})|$ is a binomial coefficient and $|\mathbf{KH}(\mathbf{m}^\mathbf{n})|$ a closely related quantity.

- Although $\mathbf{H}(\mathbf{m}^\mathbf{n}) \approx \mathbf{H}(\mathbf{p}^\mathbf{q})$ if and only if $m = p$ and $n = q$ or $m = q + 1$ and $n = p - 1$, it is not necessarily true that $\mathbf{KH}(\mathbf{m}^\mathbf{n}) \approx \mathbf{KH}((\mathbf{n}+\mathbf{1})^{\mathbf{m}-\mathbf{1}})$.The two may not even be the same size. The diagrams below give an illustration.

$H(3^4)$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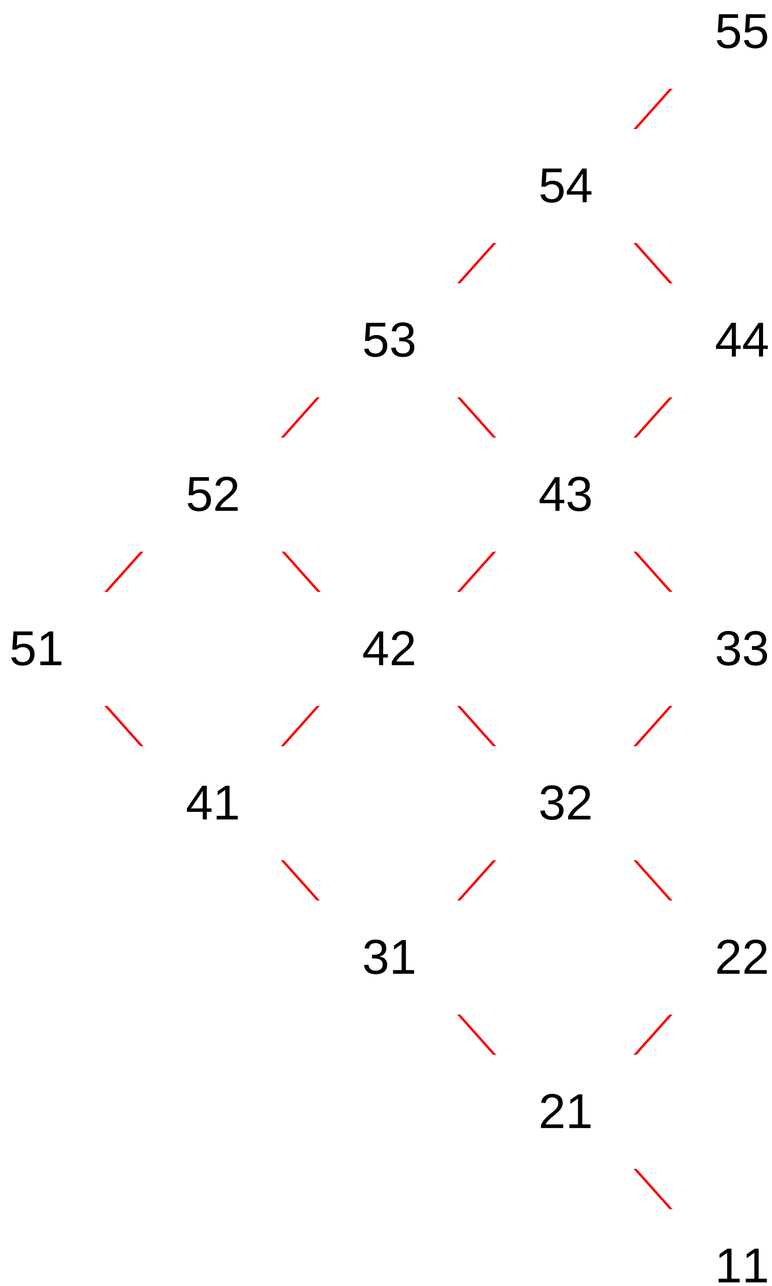
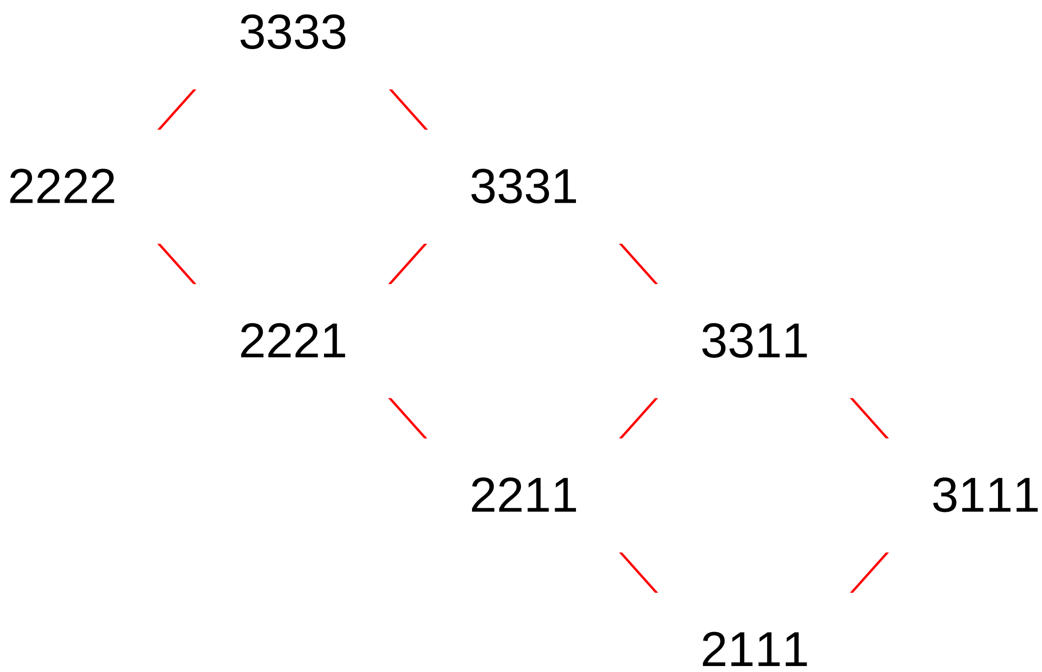
3211
2211
3111
2111
1111

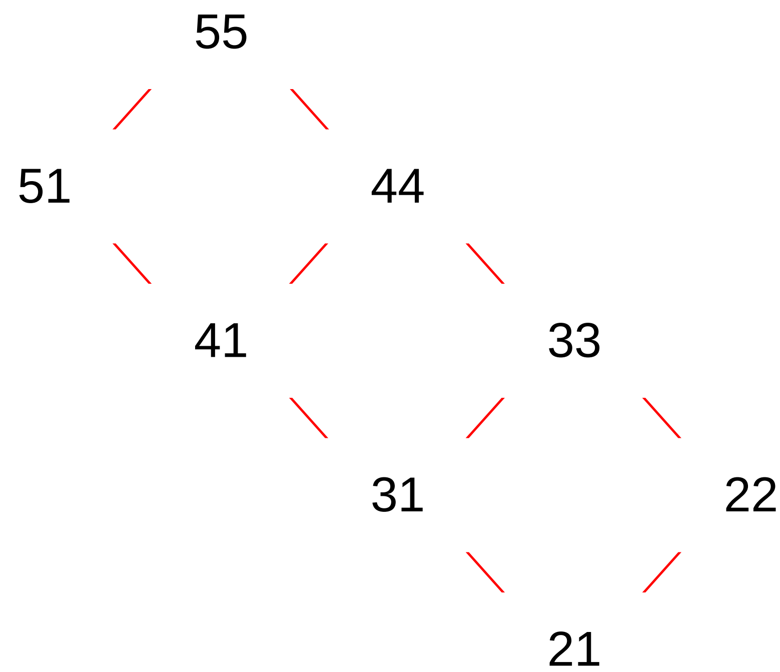
$H(5^2)$



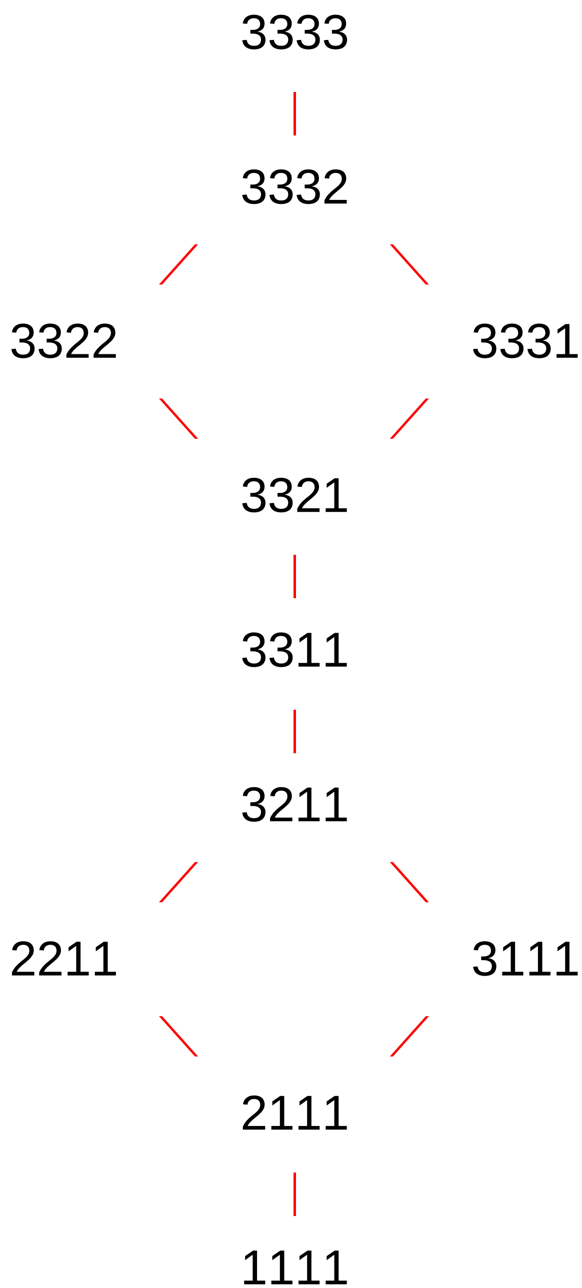
JIH(3^4)



JIH(5^2)



KH(3^4)


3333
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3321
3311
3211
2211
3111
2111
1111

KH(5^2)


55
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51
41
31
21
11

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