Comparing the profinite completion and MacNeille completion of a modal algebra

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Outline

Completions of lattice expansions

- Three distinct flavours
- Some general comparisons

A special case: modal algebras

- A characterization theorem for profinite $\cong$ lower MacNeille

Applications

- An alternative proof for Bezhanishvili & Bezhanishvili
- Master modality and finitely generated algebras

Three completions

Lattice expansion: $\mathbb{A} = \langle A; \wedge, \vee, 0, 1, (\omega^{\mathbb{A}})_{\omega \in \Omega} \rangle$ s.t. $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice.

- Profinite completion $\hat{\mathbb{A}}$ is a compactification if $\mathbb{A}$ is residually finite.
- MacNeille completion $\bar{\mathbb{A}}$ fills in the gaps in the order structure $\mathbb{A}$.
- Canonical extension $\mathbb{A}^{\sigma}$ arises naturally through Stone / Priestley / Urquhart / . . . duality.

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Connections between completions

- Profinite completion $\hat{\mathbb{A}}$ and Stone-ish duality mix well (cf. Bezhanishvili e.a.).
- Canonical extension $\mathbb{A}^\sigma$ and $\hat{\mathbb{A}}$ are (consequently) strongly related (sometimes even $\mathbb{A}^\sigma \cong \hat{\mathbb{A}}$, cf. Harding).
- $\mathbb{A}^\sigma$ and MacNeille completion $\bar{\mathbb{A}}$ are less compatible because
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- $\langle A; \wedge, \vee, \neg, 0, 1 \rangle$ Boolean algebra,
- $\Diamond \colon A \to A$ s.t. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$ and $\Diamond 0 = 0$.

Properties:

- Stone-ish duality: Jónsson-Tarski duality for Kripke frames,
- Lower MacNeille completion $\bar{\mathbb{A}}$ of $\mathbb{A}$ is not always a modal algebra,
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- $\langle W, R, \tau \rangle$ is the topological dual of $\mathbb{A}$,
- $W_{\mathrm{fin}} \subseteq W$ is the generating set of $\hat{\mathbb{A}} = \langle \mathcal{P}(W_{\mathrm{fin}}), \cap, \cup, \ldots \rangle$.

Theorem

Let $\mathbb{A}$ be a modal algebra. Then the following are equivalent:

1. $\bar{\mathbb{A}} \cong \hat{\mathbb{A}}$ *(isomorphic over $\mathbb{A}$),*
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3. *W_{fin} dense in $\langle W, \tau \rangle$ and $w \in W_{\mathrm{fin}} \Rightarrow w$ isolated (cf. [B&B2007]),*
4. *$\mathbb{A}$ residually finite and $\theta \in \mathrm{Con}\,\mathbb{A}$ s.t. $\mathbb{A}/\theta$ is finite $\Rightarrow 1/\theta$ is principal lattice filter (cf. [B&B2007]).*

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When is profinite completion a trivial extension?

The theorem below was originally about Heyting algebras.

Theorem (Bezhanishvili & Bezhanishvili 2007)

Let $\mathbb{A}$ be a modal algebra. $\mathbb{A} \cong \hat{\mathbb{A}}$ iff $\mathbb{A}$ is complete, residually finite and $\mathbb{A}/\theta$ finite implies that $1/\theta$ is a principal lattice filter.

Alternative proof ($\Leftarrow$).

$\mathbb{A}$ is complete iff $\mathbb{A} \cong \bar{\mathbb{A}}$. By the other assumptions above and the previous theorem, $\bar{\mathbb{A}} \cong \hat{\mathbb{A}}$. The statement follows. □

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Master modality

$\mathcal{V}$ variety of (poly-) modal algebras.

- Compound diamond: $\blacklozenge ::= x | \blacklozenge \vee \blacklozenge | \Diamond_i \blacklozenge$.
- $\blacklozenge \sqsubseteq \blacklozenge'$ if $\mathcal{V} \models \blacklozenge(x) \leq \blacklozenge'(x)$.
- $\mathcal{V}$ has master modality if there exists a $\sqsubseteq$-maximal $\blacklozenge$.

For Kripke frames $\langle W, (R_i)_{i \in I} \rangle$. this means $\left(\bigcup R_i \right)^*$ is modally definable.

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Lemma (Kracht 1999)

Let $\mathcal{V}$ be a variety of modal algebras. Then $\mathcal{V}$ admits a master modality iff $\mathbb{A}$ in $\mathcal{V}$ and $\theta \in \mathrm{Con}\,\mathbb{A}$ principal $\Rightarrow 1/\theta$ is a principle lattice filter.

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