Modal Fixpoint Logics

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(results based on joint work with Luigi Santocanale)

Example

- Add connective $\langle * \rangle$ to the language ML of modal logic
- $\langle * \rangle p := \bigvee_{n \in \omega} \Diamond^n p$

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  (a fixpoint of a map $f : S \to S$ is an $s \in S$ with $fs = s$)
- Notation: $\langle * \rangle p \equiv \mu x.p \vee \Diamond x$.

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  - new fixpoint connectives such as $\langle * \rangle$
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- Many applications in process theory, epistemic logic, . . .
- Interesting mathematical theory:
  - connections with theory of automata on infinite objects
  - game-theoretical semantics

General Program

Achieve a better understanding of modal fixpoint logics by studying the interaction between

- combinatorial
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Here: consider algebraic aspects

Knaster-Tarski Theorem

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Then f has both a least fixpoint LFP.f and a greatest fixpoint GFP.f.

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Hence $q = fq$ and so $\bigwedge \mathrm{PRE}(f)$ is the least fixpoint of f.

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This definition applies to non-complete lattices too!

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  (arity of $\sharp_\gamma$ depends on γ but notation hides this)
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- Obtain language ML_Γ:

$$\varphi \ ::= \ p \mid \neg p \mid \bot \mid \top \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \Diamond_i \varphi \mid \Box_i \varphi \mid \sharp_\gamma(\boldsymbol{\varphi})$$

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For simplification assume ML has only one diamond $\Diamond$, and Γ is singleton.

Flat Modal Fixpoint Logics: Kripke Semantics

- Kripke frame $\mathbb{S} = \langle S, R \rangle$ with $R \subseteq S \times S$.
- Complex algebra: $\mathbb{S}^+ := \langle \wp(S), \varnothing, S, \sim_S, \cup, \cap, \langle R \rangle \rangle$,

  $\langle R \rangle : \wp(S) \to \wp(S)$ given by $\langle R \rangle(X) := \{ s \in S \mid Rst \text{ for some } t \in X \}$

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- Every modal formula $\varphi(p_1, \ldots, p_n)$ corresponds to a term function

$$\varphi^{\mathbb{S}} : \wp(S)^n \to \wp(S).$$

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- Kripke $\sharp$-algebra $\mathbb{S}^\sharp := \langle \wp(S), \varnothing, S, \sim_S, \cup, \cap, \langle R \rangle, \sharp^{\mathbb{S}} \rangle$.

Question

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Warning Limits imposed by high undecidability results!

Overview

- Introduction
- Completeness
- Constructiveness
- Free modal $\sharp$-algebras
- Systems of equations
- A general completeness result
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Axiomatization results for modal fixpoint logics

- LTL: Gabbay et alii (1980)
- PDL: Kozen & Parikh (1981)
- μML (aconjunctive fragment): Kozen (1983)
- CTL: Emerson & Halpern (1985)
- μML: Walukiewicz (1993/2000)
- CTL*: Reynolds (2000)
- LTL/CTL uniformly: Lange & Stirling (2001)
- common knowledge logics: various
- ...

Candidate Axiomatization

Definition

Let $\mathbf{K}_\Gamma$ be the basic modal logic $\mathbf{K}$, extended with the following axiom and derivation rule, for each $\gamma \in \Gamma$:

Axiom ("$\sharp(\boldsymbol{p})$ is a prefixpoint of $\gamma_{\boldsymbol{p}}$")

$$\vdash \gamma(\sharp(\boldsymbol{\varphi}), \boldsymbol{\varphi}) \to \sharp(\boldsymbol{\varphi}).$$

Rule ("$\sharp(\boldsymbol{p})$ is the least prefixpoint of $\gamma_{\boldsymbol{p}}$")

$$\text{from } \vdash \gamma(\psi, \boldsymbol{\varphi}) \to \psi \text{ infer } \vdash \sharp_\gamma(\boldsymbol{\varphi}) \to \psi.$$

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- Interpret any formula $\varphi(p_1, \ldots, p_n)$ by its term function $\varphi^A : A^n \to A$.

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- Interpret any formula $\varphi(p_1, \ldots, p_n)$ by its term function $\varphi^{\mathbb{A}} : A^n \to A$.
- Modal $\sharp$-algebra: $\mathbb{A} = \langle A, \bot, \top, \neg, \wedge, \vee, \Diamond, \sharp \rangle$ with $\sharp : A^n \to A$ satisfying

$$\sharp(\boldsymbol{b}) = \mathsf{LFP}.\gamma_{\boldsymbol{b}}^{\mathbb{A}},$$

where $\gamma_{\boldsymbol{b}}^{\mathbb{A}} : A \to A$ is given by $\gamma_{\boldsymbol{b}}^{\mathbb{A}}(a) := \gamma^{\mathbb{A}}(a, \boldsymbol{b})$.

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  where $\gamma^{\mathbb{A}}_{\boldsymbol{b}} : A \to A$ is given by $\gamma^{\mathbb{A}}_{\boldsymbol{b}}(a) := \gamma^{\mathbb{A}}(a, \boldsymbol{b})$.
- Axiomatically: modal $\sharp$-algebras satisfy
  - $\sharp(\boldsymbol{y}) \approx \gamma(\sharp(\boldsymbol{y}), \boldsymbol{y})$
  - if $x \approx \gamma(x, \boldsymbol{y})$ then $\sharp(\boldsymbol{y}) \preceq x$.

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  - Prove that all fixpoints on free algebras are constructive

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Proof
Via a step-by-step construction/generalized Lindenbaum Lemma.
Alternatively, use Rasiowa-Sikorski Lemma.

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Using continuity?

- A CPO is a poset with bottom in which $\bigvee D$ exists for every directed D.
- An order-preserving map $f : C \to C$ between CPO's is (Scott) continuous if it preserves all directed joins.

CPO Fixpoint Theorem
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Problem
Free modal $\sharp$-algebras are not CPOs!

Need to show not that $\bigvee_{n\in\omega} \gamma_{\boldsymbol{b}}^n(\bot) = \mathsf{LFP}.\gamma_{\boldsymbol{b}}$, but that $\bigvee_{n\in\omega} \gamma_{\boldsymbol{b}}^n(\bot)$ exists!

$\mathcal{O}$-adjoints

Let $f : (P, \leq) \to (Q, \leq)$ be an order-preserving map.

Definition f is a (left) adjoint or residuated if there is a residual $g : Q \to P$ with

$$fp \leq q \iff p \leq gq.$$

Definition f is a (left) $\mathcal{O}$-adjoint if it has an $\mathcal{O}$-residual $G_f : Q \to \wp_\omega(P)$ with

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- $\wedge$ is continuous but not an $\mathcal{O}$-adjoint.

Finitary $\mathcal{O}$-adjoints

Let $f : A^n \to A$ be an $\mathcal{O}$-adjoint with $\mathcal{O}$-residual G.

Define $G^n : A \to \wp(A)$ inductively by

$$\begin{aligned} G^0(a) &:= \{a\} \\ G^{n+1}(a) &:= G[G^n(a)] \end{aligned}$$

Call f finitary if $G^\omega(a) := \bigcup_{n\in\omega} G^n(a)$ is finite.

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$$\begin{aligned} G^0(a) &:= \{a\} \\ G^{n+1}(a) &:= G[G^n(a)] \end{aligned}$$

Call f finitary if $G^\omega(a) := \bigcup_{n\in\omega} G^n(a)$ is finite.

Theorem (Santocanale 2005)
If $f : A \to A$ is a finitary $\mathcal{O}$-adjoint, then LFP.f, if existing, is constructive.

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The Coalgebraic/Cover Modality ∇

- The language ML of standard modal logic is given by

$$\varphi \ ::= \ p \mid \neg p \mid \bot \mid \top \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond \varphi \mid \Box \varphi$$

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Semantics

Fix a Kripke model $\mathbb{S} = \langle S, R, V \rangle$.

$\mathbb{S}, s \Vdash \nabla\Phi$ iff for all $t \in R[s]$ there is a $\varphi \in \Phi$ with $\mathbb{S}, t \Vdash \varphi$
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Proposition $\mathbb{S}, s \Vdash \nabla\Phi$ iff the satisfaction relation $\Vdash$ is full on $R[s]$ and Φ

A relation Z is full on two sets A and B if
- $\forall a \in A \exists b \in B. Zab$ and
- $\forall b \in B \exists a \in A. Zab$.

Reorganizing Modal Logic

Conversely, express $\Box$ and $\Diamond$ in terms of ∇

$$\begin{aligned} \Diamond\varphi &\equiv \nabla\{\varphi, \top\} \\ \Box\varphi &\equiv \nabla\varnothing \vee \nabla\{\varphi\}. \end{aligned}$$

Reorganizing Modal Logic

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Proposition The languages ML and ML_∇ are effectively equi-expressive.

Properties of Free Modal $\sharp$-Algebras

Theorem (after Santocanale 2005)
Let F be a free modal $\sharp$-algebra. Then

- ∇^F is a finitary $\mathcal{O}$-adjoint,
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Systems of Equations

- A system of equations (SoE) is a set $\{x_i \approx t_i \mid i \in I\}$ of equations such that each x_i occurs only positively in every t_j.
- Widen setting to that of ULE (uniform lattice expansions)
  - $\mathcal{E}$ is similarity type of additional (non-lattice) operation symbols
  - each $f \in \mathcal{E}$ is interpreted as a map $f^A : A^n \to A$ that is uniform, i.e. f^A reverses/preserves order in each coordinate

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- F_S has a least fixpoint if A is complete, but not necessarily in general.
- Focus on language used in SoE with the aim of eliminating conjunctions.

Distributive Laws

- A distributive law for a pair $f, g \in \mathcal{E}$ consists of a term $t(s_1, \ldots, s_m)$ s.t.
  - $t(y_1, \ldots, y_m)$ is conjunction-free and
  - each s_i is a pure conjunction, i.e., a term of the form $\bigwedge_{j \in J_i} x_j$.
- Such a law holds in a class K of lattice expansions if

$$\mathsf{K} \models f(x_1, \ldots, x_n) \wedge g(x_{n+1}, \ldots, x_{n+k}) \approx t(s_1, \ldots, s_m).$$

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- Examples in modal logic:
  - $\Box x \wedge \Box y = \Box(x \wedge y)$
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- A class K of lattice expansions is fully distributive if every pair of symbols in $\mathcal{E}$ satisfies a distributive law.

Eliminating Conjunctions

Proposition Let K be a fully distributive class of lattice expansions.
With each term t we may effectively associate

- a conjunction-free term q, and
- pure conjunctions $s_1, \ldots, s_m$ such that

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Theorem (Arnold & Niwiński):
In a fully distributive setting of complete lattices,
every system of equations can be simulated by a conjunction-safe one.

Proof idea
For SoE $\{x_i \approx t_i \mid i \in I\}$, apply Proposition to all conjunctions of the t_i's.
Introduce a new fixpoint variable y_λ for each such conjunction.

Transforming Systems of Equations

- Fix System of Equations $S = \{x_i \approx t_i \mid i \in I\}$, write $X = \{x_i \mid i \in I\}$.
- Call a term
  - guarded if all variables x_i occur under scope of ≥ 1 symbol in $\mathcal{E}$,
  - shallow if all variables x_i occur under scope of ≤ 1 symbol in $\mathcal{E}$,
  - conjunction-safe if all conjunctions are of the form $z \wedge t$ with $z \notin X$.
- WLOG assume S is guarded and shallow.
- Let $\wp_+ I := \{\lambda \subseteq I \mid \lambda \neq \varnothing\}$.
- Fix set $Y := \{y_\lambda \mid \lambda \in \wp_+ I\}$ of fresh variables, and let σ be the substitution given by

$$\sigma(y_\lambda) := \bigwedge_{i \in \lambda} x_i.$$

Assume K is fully distributive.

Proposition
For each $\lambda \in \wp_+ I$ we may effectively obtain a conjunction-safe term $q_\lambda(y_\lambda \mid \lambda \in \wp_+ I)$ such that

$$\mathsf{K} \models \bigwedge_{i \in \lambda} t_i \approx \sigma(q_\lambda)$$

Definition
The simulation of S is the system $\breve{S}$ given by

$$\breve{S} := \{y_\lambda \approx q_\lambda \mid \lambda \in \wp_+ I\}.$$

Theorem (Arnold & Niwiński)
Let A be a complete distributive lattice expansion in K. Then

1. if LFP$(S) = \{a_i \mid i \in I\}$, then LFP$(\breve{S}) = \{\bigwedge_{i\in\lambda} a_i \mid \lambda \in \wp_+ I\}$.

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Definition A modal $\sharp$-algebra is regular (wrt γ) if it satisfies condition (1).

A modal distributive law

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Theorem For any sets Φ, Φ' of formulas,

$$\nabla\Phi \wedge \nabla\Phi' \equiv \bigvee_{Z \in \Phi \bowtie \Phi'} \nabla\{\varphi \wedge \varphi' \mid (\varphi, \varphi') \in Z\},$$

where $\Phi \bowtie \Phi' = \{Z \subseteq \Phi \times \Phi' \mid Z \text{ is full on } \Phi \text{ and } \Phi'\}$.

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The axiomatization

Key idea: ensure regularity of Lindenbaum-Tarski algebra.

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Concretely, obtain $\mathbf{K}_\Gamma$ by adding to $\mathbf{K}$, for each $\gamma \in \Gamma$, a finite (bounded) set of axioms and rules.

1 WLOG assume γ is a guarded $\{\nabla, \vee, \wedge\}$-formula

2 Turn γ into guarded and shallow SoE $S = \{x_i = t_i \mid i \in I\}$

- each x_i corresponds to a subformula ψ_i of γ
- one variable, say, x_0, corresponds to γ itself

3 Construct the simulation $\breve{S} = \{y_\lambda \mid \lambda \in \wp_+ I\}$ of S

4 From $\breve{S}$ read off axioms and rules, expressing that for all $\boldsymbol{y}$:

$$\{\textstyle\bigwedge_{i\in\lambda}\psi_i[\sharp(\boldsymbol{y})/x_0] \mid \lambda \in \wp_+ I\} \text{ is the LFP of } F_{\breve{S}}.$$

Proof sketch

Theorem (Santocanale & Venema)
Let Γ be a set of modal formulas $\gamma(x, \boldsymbol{p})$ in which x occurs only positively.
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- LFP.S is constructive.
- L, being countable and residuated, embeds in a Kripke $\sharp$-algebra.

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- a modal system of equations $\Sigma = \{x_s = t_s \mid s \in S\}$
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Examples in a more general setting of lattice expansions are also of interest!

Further work

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- What about automata?
  - coalgebraic aspects starting to be understood (coalgebra automata)
  - automata and algebra?