Semisimplicity, EDPC and discriminator varieties of bounded commutative residuated lattices with S4-like modal operator

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Outline of my talk

- Substructural logics & Residuated lattices
  - Substructural logics
  - Residuated lattices
  - Extensions: +modality
- Main result :

V⊆□BCRL, semisimple = discriminator

Substructural logics
&
Residuated lattices

Substructural logics

- Substructural logics :

LJ (or LK) – structural rules,

(e: exchange, w: weakening, c: contraction)

- rules

  – linear logic, relevant logic, fuzzy logic

Basic substructural logic : FL

No structural rules

FL = LJ – {e, w, c}

(CFL = LK – {e, w, c})

$$\frac{\Gamma, A, B \vdash C}{\Gamma, B, A \vdash C} \qquad \frac{\Gamma \vdash C}{\Gamma, A \vdash C} \qquad \frac{\Gamma, A, A \vdash C}{\Gamma, A \vdash C}$$

Sequent system : FL

$$a \vdash a, \quad \vdash 1, \quad 0 \vdash$$

$$\frac{\Gamma \vdash A \quad \Delta, A, \Sigma \vdash C}{\Delta, \Gamma, \Sigma \vdash C} \qquad \frac{\Gamma \vdash}{\Gamma \vdash C} \qquad \frac{\Gamma, \Delta \vdash C}{\Gamma, 1, \Delta \vdash C}$$

$$\frac{\Gamma \vdash A \quad \Delta, B, \Sigma \vdash C}{\Delta, \Gamma, A \rightarrow B, \Sigma \vdash C} \qquad \frac{A, \Gamma \vdash C}{\Gamma \vdash A \rightarrow C}$$

$$\frac{\Gamma \vdash A \quad \Delta, B, \Sigma \vdash C}{\Delta, B \leftarrow A, \Gamma, \Sigma \vdash C} \qquad \frac{\Gamma, A \vdash C}{\Gamma \vdash C \leftarrow A}$$

Sequent system : FL

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$$

$$\frac{\Gamma, A(B), \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C} \qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma, A, \Delta \vdash C \quad \Gamma, B, \Delta \vdash C}{\Gamma, A \vee B, \Delta \vdash C} \qquad \frac{\Gamma \vdash A\ (B)}{\Gamma \vdash A \vee B}$$

Basic substructural logics

- FL, FLe, FLw, FLew, …
- FLew = FL + {e, w} = LJ – {c}

  Monoidal logic (Fuzzy logic)
- FLe = ILL – {!,?}

Basic results

- Cut elimination theorem :

  FL, FLe, FLw, FLew, FLec, FLecw(= LJ)

  (CFLe, CFLew, CFLec, CFLecw(= LK))

Residuated lattices

- Definition : A= $(A, \bullet, \to, \leftarrow, \wedge, \vee, 1)$
  - $(A, \bullet, 1)$: monoid
  - $(A, \wedge, \vee)$: lattice
  - $x \bullet y \leqq z \Leftrightarrow x \leqq z \leftarrow y \Leftrightarrow y \leqq x \to z$

- Pointed residuated lattice = FL-algebra
  - A= $(A, \bullet, \to, \leftarrow, \wedge, \vee, 1, 0)$
  - 0 : arbitrary but fixed element of A

Basic facts

- The class of residuated lattices forms a variety : RL
- Subvarieties :
  - FL, CRL, IRL, …
  - Commutativity, integrality, increasing-idenpotency

Substructural logics & Residuated lattices

- Completeness theorem :

  Algebras for FLx is FLx-algebras

  (x = e, w, ew, …)

- Lindenbaum construction :

  Frm /∼ $\quad A \sim B \equiv A \vdash B$ and $B \vdash A$

Algebra - Logic

- commutativity ⇔ exchange
- integrality ⇔ weakening
- increasing-idempotency ⇔ contraction

- FLe, FLw-, FLew-algebra, …
- FLe, FLw, FLew, …

Book

- Residuated Lattices: an algebraic glimpse at substructural logics, P. Jipsen, T. Kowalski, N. Galatos and H. Ono
  - Residuated Lattices: an algebraic glimpse at logics without contraction, T. Kowalski and H. Ono (starting point for the book)

Extensions

- Substructural logics + modalities
  - What is natural modalities in substructural logics?
  - H. Ono, Modalities in substructural logics, a preliminary report
- Algebras for modal substructural logics = Residuated lattices + operators

  (cf. BAO's)

□FLe (□BCRL)

- □FLe = FLe + S4-like modality

$$\frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A} \qquad \frac{A, \Gamma \vdash B}{\Box A, \Gamma \vdash B}$$

Cut elimination theorem holds for □FLe

□FLe-algebras (□BCRL)

- A = (A, ·, →, ∧ ,∨ ,1, 0, T, ⊥,□)
  - (A, ·, →, ∧ ,∨ ,1, 0, T, ⊥) : FLe-algebra
  - S4-like modality
  - 1≦□1,
  - □x · □y ≦ □(x · y)
  - □x ≦ x
  - □x ≦ □ □x
  - x ≦ y ⇒□x ≦□y
- The class of □FLe-algebras forms a variety

□ FLe & Modal FLe-algebras

- Completeness theorem :
  - Algebras for □FLe is □FLe-algebras

Congruence filter of □FLe-algebra

- F is a congruence filter :
  - $1 \in F$
  - $x, y \in F \Rightarrow x \wedge y \in F$
  - $x, x \to y \in F \Rightarrow y \in F$
  - $x \in F \Rightarrow \Box x \in F$
- $\langle S \rangle = \{x \in A: x \geq \Box(s_1 \wedge 1) \ldots \Box(s_k \wedge 1), s_i \in S\}$

Algebra basics

- V: variety is semisimple
  - All its algebras are semisimple
- A in □BCRL, $x \in Rad_A \Leftrightarrow \forall n \geq 1\ \exists m$ s.t.,

  $(\Box\neg(\Box(x \wedge 1))^n)^m = \bot, \neg x = x \rightarrow \bot$
- A is semisimple :

$\forall x \in A$, not greater than 1, $\exists\ n \geq 1$, s.t.,

$(\Box\neg(\Box x \wedge 1))^n)^m \neq \bot$ for any m

Algebra basics

- V: variety is discriminator
  - The ternary discriominator is a term operation on every si algebra in V

    t(x,y,z) = x   if x=y

    z   otherwise
  - Algebra with discriminator term is simple

Algebra basics

- Discriminator variety ⇒ semisimple variety
- Discriminator variety V ⇒ V has the CEP
- DPC (definable principle congruence)
  - A first order formula Φ, a,b,c,d in A
  - (c,d) in $\Theta(a,b) \Leftrightarrow A \models \Phi(a,b,c,d)$
- EDPC (equational definable principle congruence)
  - If Φ can be taken a finite set of equations

Facts

- V is congruence-permutative ⇒ discriminator = semisimple + EDPC

If semisimple ⇒ EDPC

then discriminator = semisimple

Some historical remarks

- Every free classical FLew-algebras is semisimple (Grishin)
- The variety of FLew-algebras is generated by its finite simple members (Kowalski & Ono)
- Every free FLw-algebras is semisimple
- The variety of □FLew-algebras is generated by its finite simple members

Some historical remarks

- $V \subseteq \mathrm{FLew}$, V is discriminator
  = V is semisimple
  = V satisfies that $x \vee \neg(x^n) = 1$
  $x^n = x \bullet \ldots \bullet x$, n-times
  (Kowalski2005)

Goal of my talk

- $V \subseteq \Box$BCRL, V is discriminator

  = V is semisimple

  = $V \models \Box(x \wedge 1) \vee \neg (\Box(x \wedge 1))^n$

  for some natural number n

□E(1,n) & □EM(1,n)

- □E(1,n) :

  $(\Box(x \wedge 1))^{n} = \Box(x \wedge 1))^{n+1}$

  for any natural number n

- □EM(1,n) :

  $\Box(x \wedge 1) \vee \neg(\Box(x \wedge 1))^{n} = 1$

  for any natural number n

Proposition

- V ⊆ □BCRL, V has EDPC

= V has DPC

= V ⊆ □ E(1,n)

for some natural number n

= V |= $(\Box(x \wedge 1))^n = (\Box(x \wedge 1))^{n+1}$

for some natural number n

Set up congruence

- A in V s.t. $(\Box(a\wedge 1))^{n} > \bot$, a an element not greater than 1
- $\alpha = Cg(a, 1)$; nonzero, nonfull, principal

  $\Rightarrow \exists \beta$ subcover

Lemma $\exists$ m s.t.,

$$(\Box(a\wedge 1))^{m+1} \equiv_{\beta} (\Box(a\wedge 1))^{m}$$

$$\neg(\Box(a\wedge 1))^{m} \equiv_{\beta} (\neg(\Box(a\wedge 1)^{m}))^{2}$$

$$(\Box(a\wedge 1))^{m} \equiv_{\beta} \neg\neg(\Box(a\wedge 1))^{m}$$

A necessary condition for semisimplicity

- V is semisimple subvariety of □BCRL,

V|= ¶ ?

¶ ≡ □(x∧1) ≧(¬(¬□(x∧1)^{r})^{k})^{l}

Suppose V falsifies ¶, Put Θ ≡ ∨ θ_r ,

θ_r = Cg(¬(¬□(x∧1)^{r})^{K}, 1)

K is the smallest number V falsifies ¶

Some lemmas

- $0 < \Theta < \alpha$
- V is semisimple subvariety of □BCRL,

V|= ¶？ YES！

$$V \models (\Box x \wedge 1) \geqq (\neg(\neg\Box(x\wedge 1)^r)^k)^l$$

for any k there exist r & l

Function r

- Suppose

$$V \models (\Box x \wedge 1) \geqq (\neg(\neg \Box (x \wedge 1)^{r})^{k})^{l}$$

- $r : N \rightarrow N$,

  $r(i)$ the smallest number s.t., $\exists l \in N$ with

$$V \models (\Box (x \wedge 1)) \geqq (\neg(\neg \Box (x \wedge 1)^{r(i)})^{i})^{l}$$

- Lemma: r is non-decreasing function

Semisimple forces $\Box$EM(1,n)

- Lemma

$V \subseteq \Box$BCRL, semisimple,

$V \models (\Box(x \wedge 1))^{n+1} = (\Box(x \wedge 1))^{n}$

for some natural number n

Main theorem

- $V \subseteq$ □BCRL, V is discriminator
  = V is semisimple
  = $V \models \square(x \wedge 1) \vee \neg \square(x \wedge 1)^n$
  for some natural number n

Corollary 1

- V ⊆ □FLe, V is discriminator

  = V is semisimple

  = V |= □(x∧1)∨¬ □(x∧ 1) ^{n}

  for some natural number n

Corollary 2

- V ⊆ □FLew, V is discriminator

  = V is semisimple

  = $V \models \Box x \vee \neg (\Box x)^n$

  for some natural number n

Corollary 3

- $V \subseteq \mathrm{FL}_{ew}$, V is discriminator

  = V is semisimple

  = $V \models x \vee \neg x^{n}$

  for some natural number n