Relational Semantics for Distributive Substructural Logics

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Distributive Substructural Logics

$$\text{Language} = \begin{cases} p, q, r, \ldots & \text{Propositional variables} \\ \mathbf{t}, \mathbf{f}, \top, \bot & \text{Constants} \\ \vee, \wedge, \circ, \backslash, / & \text{Logical connectives} \end{cases}$$

DFL = LJ - Structural rules + Distributivity

Contraction, Exchange, Left-(Right-)weakening

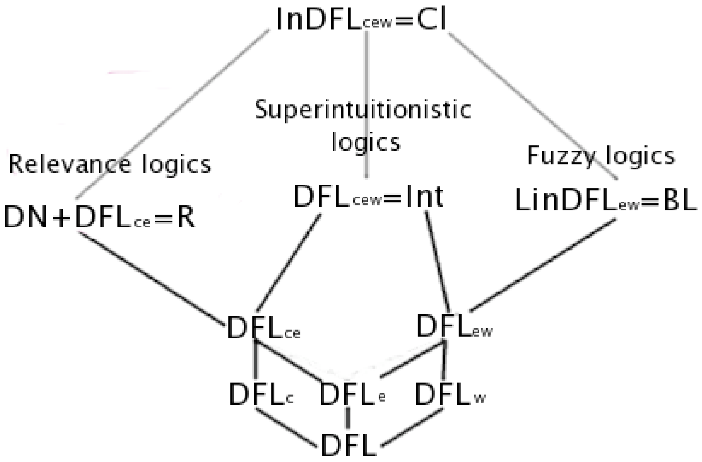
DFL : the set of provable formulas in DFL

L is a DFL logic, if **L** is an extension of **DFL**.

Basic DFL logics

$\mathbf{DFL}$, $\mathbf{DFL}_c$, $\mathbf{DFL}_e$, $\mathbf{DFL}_w$, $\mathbf{DFL}_{ce}$, $\mathbf{DFL}_{ew}$, $\mathbf{DFL}_{cew}$

Important classes of DFL logics



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Relational Semantics for DFL logics

Definition

A tuple $\mathfrak{F} = \langle W, W_t, W_f, R_\circ \rangle$ is a DFL-frame, if $\mathfrak{F}$ satisfies the following.

1. $R_\circ(w, t_1, w)$ and $R_\circ(w, w, t_2)$ for some $t_1, t_2 \in W_t$.
2. If $R_\circ(w, v, u)$, $w \preceq w'$, $v' \preceq v$ and $u' \preceq u$, then $R_\circ(w', v', u')$.
3. $R_\circ(w, x, s)$ and $R_\circ(x, v, u)$ for some $x \in W$, if and only if $R_\circ(w, v, y)$ and $R_\circ(y, u, s)$ for some $y \in W$.
4. If $w \in W_t$ and $w \preceq w'$, then $w' \in W_t$.
5. If $w \in W_f$ and $w \preceq w'$, then $w' \in W_f$.

Abbreviation: $w \preceq w' \iff R_\circ(w', t, w)$ or $R_\circ(w', w, t)$, for some $t \in W_t$.

$Up(W)$: the set of all subsets of W upward closed under $\preceq$.

Correspondence for structural rules

Proposition

- Contraction $\iff \begin{cases} \forall w \in W[R_\circ(w, w, w)] \\ \text{or equivalently} \\ \forall w, v \in W[w \preceq v \Rightarrow R_\circ(v, w, w)] \end{cases}$
- Exchange $\iff \forall w, v, u \in W[R_\circ(w, v, u) \Rightarrow R_\circ(w, u, v)]$
- Left weakening $\iff W_t = W$
- Right weakening $\iff W_f = \emptyset$

Kripke completeness for basic DFL logics

Theorem

All basic DFL logics are Kripke complete.

We have already proved several other DFL logics.

Contracts with other relational semantics

- Kripke frame for Intuitionistic logic $\Longleftrightarrow$ DFL_{cew}-frame
- Routley-Meyer semantics without negation $\Longleftrightarrow$ DFL_{ce}-frame without negation

Kripke frame for Intuitionistic logic

Definition (Int-frame)

A tuple $\mathfrak{F}_{Int} = \langle W_{Int}, R_{Int} \rangle$ is a Kripke frame for **Int**, if $\mathfrak{F}_{Int}$ satisfies the following.

1. For any $w \in W_{Int}$, $wR_{Int}w$.
2. If $wR_{Int}v$ and $vR_{Int}u$, then $wR_{Int}u$.

A valuation is a function from the set of propositional variables to $Up_R(W_{Int})$.

How can we consider DFL_{cew}-frame as Int-frame?

DFL_{cew}-frame: $\mathfrak{F} = \langle W, W_t, W_f, R_\circ \rangle$

$\Updownarrow$?

Int-frame: $\mathfrak{F}_{Int} = \langle W_{Int}, R_{Int} \rangle$

Int-frame vs. DFL_{cew}-frame

DFL_{cew}-frame $\Rightarrow$ Int-frame

$$R_\circ(w, v, u) \Rightarrow v \preceq w$$

- $R_\circ$-reflexivity $\Rightarrow$ R_{Int}-reflexivity
- $R_\circ$-transitivity $\Rightarrow$ R_{Int}-transitivity

Int-frame vs. DFL_{cew}-frame

DFL_{cew}-frame $\Rightarrow$ Int-frame

$$R_\circ(w, v, u) \Rightarrow v \preceq w$$

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- $R_\circ$-transitivity $\Rightarrow$ R_{Int}-transitivity

Int-frame vs. DFL_{cew}-frame

Int-frame $\Rightarrow$ DFL_{cew}-frame

$$R_\circ(v, w, u), \text{ if } wR_{Int}v \text{ and } uR_{Int}v.$$

- R_{Int}-reflexivity $\Rightarrow$ $R_\circ$-reflexivity and $R_\circ$-idempotency
- R_{Int}-transitivity $\Rightarrow$ $R_\circ$-transitivity

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General DFL-frame

Definition

A tuple $\mathfrak{G} = \langle \mathfrak{F}, A \rangle$ is a general DFL-frame, if $\mathfrak{F}$ is a DFL-frame and A satisfies the following.

1. A is a subset of $Up(W)$.
2. $W_t, W_f, W, \emptyset$ are included in A.
3. A is closed under $\cup$, $\cap$, $*$, $\searrow$ and $\swarrow$.

$X * Y := \{w \in W \mid R_\circ(w, v, u), v \in X \text{ and } u \in Y, \text{for some } v, u \in W\}$

$X \searrow Y := \{w \in W \mid R_\circ(u, v, w), v \in X \Rightarrow u \in Y, \text{for any } v, u \in W\}$

$Y \swarrow X := \{w \in W \mid R_\circ(u, w, v), v \in X \Rightarrow u \in Y, \text{for any } v, u \in W\}$

Anti-symmetry: $w \preceq v$ and $v \preceq w$ only if $w = v$.

Priestley-type Duality

Definition (Dual algebra)

Given a general L-frame $\mathfrak{G} = \langle W, W_t, W_f, R_\circ, A \rangle$, the tuple $\mathfrak{G}^* = \langle A, \cup, \cap, *, \setminus, /, W_t, W_f, W, \emptyset \rangle$ is *the dual algebra*.

Definition (Dual frame)

Given a dual algebra $\mathfrak{G}^* = \langle A, \cup, \cap, *, \setminus, /, W_t, W_f, W, \emptyset \rangle$, the tuple $(\mathfrak{G}^*)_* = \langle Pf(A), Pf_{W_t}(A), Pf_{W_f}(A), R_*, \widehat{A} \rangle$ is *the dual frame*

$$R_*(F_1, F_2, F_3) \iff \forall X, Y \in A[X \in F_2 \text{ and } Y \in F_3 \Rightarrow X * Y \in F_1]$$

$$\widehat{A} := \{\widehat{X} \mid X \in A\}$$

$$\widehat{X} := \{F \in Pf(A) \mid X \in F\}$$

Priestley-type Duality

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Topological Characterization of descriptive DFL-frame

Definition

A general L-frame $\mathfrak{G}$ is *descriptive*, if $\mathfrak{G}$ is isomorphic to $(\mathfrak{G}^*)_*$.

Theorem

A general L-frame $\mathfrak{G}$ is descriptive, if and only if $\mathfrak{G}$ satisfies $R_\circ$-tightness and Compactness.

$R_\circ$-tightness :
$R_\circ(w, v, u) \iff \forall X, Y \in A[v \in X, u \in Y \Rightarrow w \in X * Y]$

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Future Work

- Filtration or Finite model property
- Sahlqvist-type Theorems
- etc