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Topology and logic of decision problem solving

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Part I. Notations 1

- $\tau = (2, 2, 2, 2, 0, 0)$ *is a type of algebras with four binary operations denoted by* $\vee, \wedge, \overset{B}{\rightarrow}, \overset{B}{-},$ *and two constants* $0, 1$.

- $\mu = (2, 2, 2, 2, 1, 1, 1, 0, 0)$ *is a type of algebras with four binary operations denoted by* $\vee, \wedge, \overset{B}{\rightarrow}, \overset{B}{-},$ *three unary operations denoted by* $v, \overline{f}, \neg$ *and two constants* $0, 1$.

Part I. Notations 2

- $\mathrm{h}(\tau)$ *is the class of all algebras* $\boldsymbol{A}=(A,\vee,\wedge,\to,\odot,0,1)$ *of type* τ,
- $\mathrm{h}(\mu)$ *is the class of all algebras* $\boldsymbol{A}=(A,\vee,\wedge,\to,\odot,v,\bar{f},\neg,0,1)$ *of type* μ,
- $\mathrm{j}(\mu)$ *is the class of all algebras* $\boldsymbol{A}\in\mathrm{h}(\mu)$ *such that its underlying algebra* $\tau[\boldsymbol{A}]=(A,\vee,\wedge,\to,\odot,0,1)\in\mathrm{h}(\tau)$ *and the three new unary operations* $v,\bar{f},\neg: A\to A$ *satisfy the following equations:*

$$vx=1\to x,\ \bar{f}x=x\odot 0,$$

$$x\to 0=\neg x=1\odot x.$$

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Part I. Heyting algebras

A Heyting algebra is a system $\boldsymbol{H} = (H, \vee, \wedge, \to, 0, 1)$ *of type* $(2, 2, 2, 0, 0)$ *such that* $(H, \vee, \wedge, 0, 1)$ *is a bounded lattice and for all* $x, y \in H$, $z \leq x \to y$ *if and only if* $z \wedge x \leq y$.

We will denote by e *the class of Heyting algebras.*

An ***embedded Heyting algebra*** *in* j (μ) *is a system*

$\ddot{\boldsymbol{H}} = (H, \vee, \wedge, \text{฿}\!\!\to, \text{฿}, v, \overline{f}, \neg, 0, 1) \in$ j (μ) *such that*

$\boldsymbol{H} = (H, \vee, \wedge, \text{฿}\!\!\to, 0, 1) \in$ e *and* $\ddot{\boldsymbol{H}}$ *satisfies:*

- $x \text{ ฿ } y = (x \text{ ฿}\!\!\to y) \text{ ฿}\!\!\to 0, \quad \neg x = x \text{ ฿}\!\!\to 0 = 1 \text{ ฿ } x,$
- $vx = 1 \text{ ฿}\!\!\to x = x, \qquad \overline{f}x = x \text{ ฿ } 0 = \neg\neg x.$

Part I. Brouwer algebras

A Brouwer algebra is a system $\boldsymbol{B} = (B, \vee, \wedge, \text{Ƀ}, 0, 1)$ *of type* $(2, 2, 2, 0, 0)$ *such that* $(B, \vee, \wedge, 0, 1)$ *is a bounded lattice and for all* $x, y \in B$, $x \,\text{Ƀ}\, y \le z$ *if and only if* $x \le y \vee z$.

We will denote by $\_r$ *the class of Brouwer algebras.*

An ***embedded Brouwer algebra*** *in* $\text{j}\,(\mu)$ *is a system*

$\hat{\boldsymbol{B}} = (B, \vee, \wedge, \text{Ƀ}\!\!\rightarrow, \text{Ƀ}, v, \overline{f}, \neg, 0, 1) \in \text{j}\,(\mu)$ *such that*

$\boldsymbol{B} = (B, \vee, \wedge, \text{Ƀ}, 0, 1) \in \_r$ *and* $\hat{\boldsymbol{B}}$ *satisfies:*

- $x \,\text{Ƀ}\!\!\rightarrow y = 1 \,\text{Ƀ}\, (x \,\text{Ƀ}\, y)$, $\quad \neg x = x \,\text{Ƀ}\!\!\rightarrow 0 = 1 \,\text{Ƀ}\, x$,
- $vx = 1 \,\text{Ƀ}\!\!\rightarrow x = \neg\neg x$, $\quad \overline{f}x = x \,\text{Ƀ}\, 0 = x$.

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Part I. Boolean algebras

A Boolean algebra is a system $\boldsymbol{B} = (B,\vee,\wedge,{}^c,0,1)$ *of type* $(2,2,1,0,0)$ *such that* $(B,\vee,\wedge,0,1)$ *is a bounded distributive lattice and for all* $x\in B,\ x\vee x^c = 1$ *and* $x\wedge x^c = 0.$

We will denote by _ *the class of Boolean algebras.*

An ***embedded Boolean algebra*** *in* j (μ) *is a system*

$\hat{\boldsymbol{B}} = (B,\vee,\wedge,\to,Ƀ,v,\overline{f},\neg,0,1)\in$ j (μ) *such that*

$\boldsymbol{B} = (B,\vee,\wedge,\neg,0,1)$ *is a Boolean algebra and* $\hat{\boldsymbol{B}}$ *satisfies:*

- $x\to y = \neg x\vee y,\quad x\,Ƀ\,y = x\wedge\neg y,$
- $vx = 1\to x = x,\quad \overline{f}x = x\,Ƀ\,0 = x,\quad \neg x = x\to 0 = 1\,Ƀ\,x.$

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Part I. Properties 1

Notations

1) e(μ) *is the class of* ***embedded Heyting algebras*** *in* j (μ).

2) _$r(\mu)$ *is the class of* ***embedded Brouwer algebras*** *in* j (μ).

3) _(μ) *is the class of* ***embedded Boolean algebras*** *in* j (μ).

Proposition 1. _$(\mu) =$ e$(\mu) \cap$ _$r(\mu)$

Part I. Properties 2

Proposition 2. *Every algebra* $\boldsymbol{A} \in \mathsf{e}(\mu) \cup \_r(\mu)$, *satisfies the following relations*:

(*i*) $x \overset{B}{\rightarrow} 0 = \neg x = 1 \text{฿}\, x$,

(*ii*) $v(x \text{฿}\, y) = \neg(x \overset{B}{\rightarrow} y)$,

(*iii*) $\overline{f}(x \overset{B}{\rightarrow} y) = \neg(x \text{฿}\, y)$

(*iv*) $v(v(x)) = v(x) \leq x \leq \overline{f}(x) = \overline{f}(\overline{f}(x))$,

(*v*) $v(x \vee y) = v(x) \vee v(y)$,

(*vi*) $\overline{f}(x \wedge y) = \overline{f}(x) \wedge \overline{f}(y)$,

(*vii*) $v(\overline{f}(x)) = \neg\neg x = \overline{f}(v(x))$.

Part I. A mathematical problem regarding the previous structures

Problem formulation

Determine a ***minimal class of algebras*** ` *such that the following relation holds:*

$$e(\mu) \cup \_r(\mu) \subseteq ` \subseteq j(\mu).$$

Remark

New classes of structures must be considered with respect to the above mathematical problem.

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Part I. Heyting-Brouwer algebras 1

A Heyting-Brouwer algebra (double Heyting algebra) is a system

$\mathbf{A} = (A, \vee, \wedge,$ Ƀ→ $,$ Ƀ $, 0, 1) \in$ h(τ) *such that*

$(A, \vee, \wedge,$ Ƀ→ $, 0, 1) \in$ e *and* $(A, \vee, \wedge,$ Ƀ $, 0, 1) \in$ _*r*.

Let e_*r*(τ) *be the class of all Heyting-Brouwer algebras.*

Part I. Heyting-Brouwer algebras 2

Proposition 3. Let $\boldsymbol{A} = (A, \vee, \wedge, \to, \div, 0, 1) \in \mathsf{e}\_r(\tau)$.

The following conditions are equivalent:

(i) *For all* $x, y \in A$, $x \to 0 = 1 \div x$.

(ii) *The underlying poset of* $\boldsymbol{A}$ *is a Boolean algebra.*

If $\boldsymbol{A}$ *satisfies* (i) *or* (ii) *then the corresponding unary operation of complementation* ${}^{c}: A \to A$ *satisfies:*

$x \to 0 = x^c = 1 \div x$, $x^c \vee y = x \to y$ *and* $x \wedge y^c = x \div y$.

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Part I. New structure required

In a Heyting-Brouwer algebra **A** *the two unary operations of Heyting negation and of Brouwer negation coincide if and only if* **A** *is a Boolean algebra*.

Thus a new structure must be introduced.

Part I. Definition of the notion of Brouwerian D-algebras

A ***Brouwerian D-algebra*** *is a system* $\mathbf{A} \in$ h(μ) *such that there exist* $\ddot{H} \in$ e(μ), $\hat{B} \in$ _$r(\mu)$ *and an isomorphism from* $\mathbf{A}$ *onto a subdirect product of the pair* $(\ddot{H}, \hat{B})$ *in* h(μ).

Let _r a(μ) *be* ***the class of Brouwerian D-algebras***.

We present now basic properties derived from this definition, in order to establish that ***a solution of the problem previously mentioned is defined by the class*** ` = _r a(μ).

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Part I. Properties 3

Proposition 4. *Let* $\boldsymbol{A} \in \mathrm{h}(\mu)$. *Then* $\boldsymbol{A} \in \_r\,\mathrm{a}(\mu)$ *if and only if there exist* $\ddot{\boldsymbol{H}} \in \mathrm{e}(\mu)$, $\hat{\boldsymbol{B}} \in \_r(\mu)$ *and an injective homomorphism* $i : \boldsymbol{A} \to \ddot{\boldsymbol{H}} \times \hat{\boldsymbol{B}}$ *such that* $\pi_{\mathrm{H}}(i(\boldsymbol{A})) = \ddot{\boldsymbol{H}}$ *and* $\pi_{\mathrm{B}}(i(\boldsymbol{A})) = \hat{\boldsymbol{B}}$.

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Part I. Properties 4

Proposition 5. *The following conditions hold :*

(*i*) $\mathsf{e}(\mu) \cup \_r(\mu) \subseteq \_r\,\mathsf{a}(\mu)$,

(*ii*) *If* $\boldsymbol{H} \in \mathsf{e}$ *and* $\mathbf{B} \in \_r$ *then* $\ddot{\boldsymbol{H}} \times \hat{\boldsymbol{B}} \in \_r\,\mathsf{a}(\mu) \cap \mathsf{j}\,(\mu)$,

(*iii*) $\_r\,\mathsf{a}(\mu) \subseteq \mathsf{j}\,(\mu)$.

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Part I. Consequences

Specific examples. *From the properties previously presented one obtains different examples of algebras of the class* $\_r\,\mathrm{a}(\mu)$ *associated with topological structures.*

<u>Part I.</u> Specific examples of Brouwerian D-algebras

- ***Direct product of two chains***

  $\mathbf{A} = \mathbb{C}_H \times \mathbb{C}_{Br}$, *where* $\mathbb{C}_H$ *is an embedded Heyting chain and* $\mathbb{C}_{Br}$ *is an embedded Brouwer chain in* $\mathrm{h}(\mu)$.

- ***Topological examples***

  $\mathbf{A} = \mathbf{O}p(U) \times \mathbf{C}l(V)$, *where* $\mathbf{O}p(U)$ *is an embedded Heyting algebra of open sets of a topological space* U *and* $\mathbf{C}l(V)$ *is an embedded Brouwer algebra of closed sets of a topological space* V.

Part I. Properties 5

Proposition 6. *The unary operation $v: A \rightarrow A$ is a* ***join semilattice homomorphism*** *and it is an* ***interior operator of the poset*** $(A, \leq, 0, 1)$.

Proposition 7. *The unary operation $\bar{f}: A \rightarrow A$ is a* ***meet semilattice homomorphism*** *and it is a* ***closure operator of the poset*** $(A, \leq, 0, 1)$.

Proposition 8. *If* $\mathbf{A} \in \_r\, \mathrm{a}(\mu)$ *then* $\mathbf{A}$ is *isomorphic to a* ***subdirect product of the pair of algebras*** $(\check{v}\mathbf{A}, \vec{f}\mathbf{A}) \in \mathrm{e}(\mu) \times \_r(\mu)$.

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Part I. Properties 6

Proposition 9. *If* ***A***∈ _r a(ꞵ) *then* ¬ð***A***∈ _(μ) ***is an embedded Boolean algebra*** *with the support set verifying* ¬*A*=*vA*∩$\overline{f}$ *A*, *where the Boolean complement of x is* x^{C}=¬*x*, *for all x*∈¬*A*.

Theorem 1. _r a(μ) ***is the variety of algebras generated by*** e(μ)∪_r(μ).

Theorem 2. *The* ***dual space*** *of any Brouwerian D-algebra* ***A*** *is a* ***gluing of two Stone spaces*** *of the pair* (*v*ð***A***, $\overline{f}$ð***A***) ***defined by Priestley duality theory***.

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Part I. An informal description of involutive Brouwerian D-algebras

- *An* ***involutive Brouwerian D-algebra*** *is a Brouwerian D-algebra equipped with an involution satisfying some natural conditions.*

- *A* ***specific model*** *and the definition of the structure of involutive Brouwerian D-algebra are derived from the following* ***particular topological example****:*

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Part I. A topological model

Let X be a topological space. Then $\mathbf{Sq}(X)=\mathbf{Op}(X)\times\mathbf{Cl}(X)$ *is a particular Brouwerian D-algebra with the property that* $\mathbf{Cl}(X)$ *is isomorphic with the dual of* $\mathbf{Op}(X)$.

Part I. Notations 3

- $\mu^d = (2,2,2,2,1,1,1,1,0,0)$ *is a type of algebras of* $\mathrm{h}(\mu)$ *extended with a new unary operation denoted by* ^{d}.

- $\mathrm{h}(\mu^d)$ *is the class of all algebras* $\boldsymbol{A} = (A, \vee, \wedge, \text{Ƀ}\!\to, \text{Ƀ}, v, \overline{f}, \neg, ^d, 0, 1)$ *of type* μ^d.

- $\mathrm{j}\,(\mu^d)$ *is the class of all algebras* $\boldsymbol{A} \in \mathrm{h}(\mu^d)$ *such that* $\mu[\boldsymbol{A}] = (A, \vee, \wedge, \text{Ƀ}\!\to, \text{Ƀ}, v, \overline{f}, \neg, 0, 1) \in \mathrm{j}\,(\mu)$.

Part I. Definition

An ***involutive Brouwerian D-algebra*** *is a system*

$$\boldsymbol{A} = (A, \vee, \wedge, \rightarrow, \text{฿}, v, \bar{f}, \neg, {}^d, 0, 1) \in \mathrm{h}(\mu^d),$$

such that the following conditions hold :

IBrD1) *The underlying μ-system of $\boldsymbol{A}$ is a* ***Brouwerian D-algebra****, namely*

$$\mu[\boldsymbol{A}] = (A, \vee, \wedge, \rightarrow, \text{฿}, v, \bar{f}, \neg, 0, 1) \in \_r\,\mathrm{a}(\mu).$$

IBrD2) *The system* $(A, \vee, \wedge, {}^d, 0, 1)$ *is a* ***De Morgan algebra***.

IBrD3) *The following equation holds,* ***for all $x, y \in A$,***

$$(x \rightarrow y)^d = y^d \,\text{฿}\, x^d.$$

Part I. Examples

- ***Example 1***

  The Brouwerian D-algebra $\mathbf{Sq}(X)=\check{\mathbf{O}}p(X)\times\check{\mathbf{C}}l(X)$ ***associated with a topological space*** X ***together with the unary operation*** ^{d} ***defined by*** $(O,F)^d=(F^c,O^c)$, ***for all*** $O\in Op(X)$ ***and*** $F\in Cl(X)$, ***where for a subset*** $S\subseteq X, S^c=X\setminus S$ ***is the complement set of*** S ***in*** m(X).

- ***Example 2***

  The real unit square $\mathbf{P}=\check{\mathbf{H}}[0,1]\times\ddot{\mathbf{B}}[0,1]$ ***together with the unary operation*** ^{d} ***representing the symmetry with respect to the diagonal segment*** $[(0,1),(1,0)]$ ***defined by*** $(\alpha,\beta)^d=(1-\beta,1-\alpha)$, ***for all*** $\alpha,\beta\in[0,1]$, ***where*** $\mathbf{H}[0,1]$ ***is the real Heyting chain*** $[0,1]$ ***and*** $\mathbf{B}[0,1]$ ***is the real Brouwer chain*** $[0,1]$.

Part I. Remark

The class $f\_r\,a(\mu^d)$ ***of all involutive Brouwerian D-algebras is a variety of algebras in*** $j\,(\mu^d)$ *characterized by three sets of equations expressing the conditions* **IBrD1)**, **IBrD2)** *and* **IBrD3)**.

Part I. Properties 7: an equational characterization

Theorem 3. *Let* $\mathcal{B}(\mu^d)$ *be the class of all algebras* $\mathbf{A} = (A, \vee, \wedge, \rightarrow, \odot, v, \overline{f}, \neg, {}^d, 0, 1) \in \mathcal{H}(\mu^d)$ *such that the following equations hold, for all* $x, y \in A$:

$$e1)\quad x \odot y = (y^d \rightarrow x^d)^d,$$

$$e2)\quad vx = 1 \rightarrow x,$$

$$e3)\quad \overline{f}x = (0^d \rightarrow x^d)^d,$$

$$e4)\quad \neg x = x \rightarrow 0.$$

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Part I. Properties 7: an equational characterization

A system $\boldsymbol{A} = (A, \vee, \wedge, \overset{B}{\rightarrow}, \not B, v, \overline{f}, \neg, {}^d, 0, 1) \in \mathrm{b}(\mu^d)$

is an ***involutive Brouwerian D-algebra*** *if and only if*

$\boldsymbol{A}$ *satisfies the following axioms, for all* $x, y, z \in A$:

1) $x \wedge (y \vee z) = (z \wedge (x \vee 0)) \vee (y \wedge (x \vee 0)),$

2) $x \wedge (x \vee y) = x,$

3) $0^d = 1,$

4) $(x^d \wedge y^d)^d = x \vee y,$

5) $1 \overset{B}{\rightarrow} (x \vee y) = (1 \overset{B}{\rightarrow} x) \vee (1 \overset{B}{\rightarrow} y),$

6) $(1 \overset{B}{\rightarrow} x) \vee x = x,$

7) $(1 \overset{B}{\rightarrow} x) \vee ((0^d \overset{B}{\rightarrow} x^d)^d \wedge (x \overset{B}{\rightarrow} 0)) = x,$

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Part I. Properties 7: an equational characterization

8) $x \rightarrow y = (1 \rightarrow x) \rightarrow y$,

9) $x \rightarrow (x \vee y) = 1$,

10) $1 \rightarrow (x \wedge (x \rightarrow y)) = 1 \rightarrow (x \wedge y)$,

11) $x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$,

12) $x \rightarrow (z^d \rightarrow y^d)^d = ((x \wedge (y \rightarrow z))^d \rightarrow 1^d)^d$.

Part I. 1. Strong filters, strong ideals and congruence relations

- *Suppose that* $\boldsymbol{A} = (A, \vee, \wedge, \rightarrow, \oplus, v, \overline{f}, \neg, {}^{d}, 0, 1) \in$ f_ra(μ^{d}).

- *A* ***strong filter*** *of* $\boldsymbol{A}$ *is a* ***nonempty subset F of A*** *verifying :*

  F1) $x \in F$ ***and*** $y \in A$ ***imply*** $x \vee y \in F$,

  F2) $x \in F$ ***and*** $y \in F$ ***imply*** $v(x \wedge y) \in F$,

  F3) $x \in F$ ***implies*** $\neg x^{d} \in F$.

- ***The set of strong filters of*** $\boldsymbol{A}$ ***is a complete lattice*** $c^{*}(\boldsymbol{A})$ *with respect to the set inclusion because it is closed under arbitrary set intersection.*

Part I. 2. Strong filters, strong ideals and congruence relations

- A ***strong ideal*** *of* A *is a* ***nonempty subset I of*** A *verifying :*

$$\text{I1) } x \in I \text{ and } y \in A \text{ imply } x \wedge y \in I,$$

$$\text{I2) } x \in I \text{ and } y \in I \text{ imply } \bar{f}(x \vee y) \in I,$$

$$\text{I3) } x \in I \text{ implies } \neg x^{d} \in I.$$

- ***The set of strong ideals of*** A ***is a complete lattice*** $\mathfrak{f}^{*}(A)$ *with respect to the set inclusion because it is closed to arbitrary set intersection.*

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Part I. 3. Strong filters, strong ideals and congruence relations

Theorem 4. *Let* $\boldsymbol{A}$ *be an involutive Brouwerian D - algebra. Then there exists a* ***complete lattice isomorphism from*** $\mathrm{c}^*(\boldsymbol{A})$ ***onto*** $\mathrm{f}^*(\boldsymbol{A})$ *defined by the following correspondence:*

$$\mathrm{c}^*(\boldsymbol{A}) \ni F \sim d_i[F] = F^d = \{x^d / x \in F\} \in \mathrm{f}^*(\boldsymbol{A}).$$

The inverse of this isomorphism from $\mathrm{f}^*(\boldsymbol{A})$ ***onto*** $\mathrm{c}^*(\boldsymbol{A})$ *is defined by the following correspondence:*

$$======\ \mathrm{f}^*(\boldsymbol{A}) \ni I \sim d_f[I] = I^d = \{x^d / x \in I\} \in \mathrm{c}^*(\boldsymbol{A}).$$

Part I. 4. Strong filters, strong ideals and congruence relations

Theorem 5. *Let* $`(\mathbf{A})$ *be* ***the complete lattice of congruence relations of*** $\mathbf{A}$. *Then* ***there exists an isomorphism from*** $`(\mathbf{A})$ ***onto*** $\mathrm{c}^{*}(\mathbf{A})$ *defined by the following correspondence:*

$$`(\mathbf{A}) \ni R \sim f(R) = F_R \in \mathrm{c}^{*}(\mathbf{A}),$$

where $F_R = \{x \in A / xR1\}$ *is the strong filter of* $\mathbf{A}$ *associated with the congruence relation* R.

Part I. 5. Strong filters, strong ideals and congruence relations

The inverse of the previous correspondence *is defined by*

$$c^{*}(\boldsymbol{A}) \ni F \sim R_F \in `(\boldsymbol{A}),$$

with R_F *representing the* ***congruence relation of A associated with*** *F such that* $xR_F y$ *if and only if* $e(x,y) \in F$, *where* $e : A \times A \rightarrow vA$ *is a binary operation on A called* ***the equivalence function on A*** *expressed by*

$$e(x,y) = v[(x \rightarrow y) \wedge (y \rightarrow x) \wedge (x^d \rightarrow y^d) \wedge (y^d \rightarrow x^d)]$$

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Part I. 6. Strong filters, strong ideals and congruence relations

*A standard **duality principle** is true in the theory of involutive Brouwerian D-algebras. **The two notions of strong filter and strong ideal are dual one to another**.*

*On this basis it follows that from the study of strong filters one obtains corresponding dual results for strong ideals. For example, we present **the dual of Theorem 5**.*

Part I. 7. Strong filters, strong ideals and congruence relations

Theorem 5°. *Let* $\boldsymbol{A}$ *be an involutive Brouwerian D-algebra. Then there exists an* **<u>*isomorphism from the complete lattice*</u>** **<u>$`(\boldsymbol{A})$ *of congruences relations of* $\boldsymbol{A}$ *onto* $f^*(\boldsymbol{A})$</u>** *defined by the following correspondence:*

$$`(\boldsymbol{A}) \ni R \sim i(R) = I_R \in f^*(\boldsymbol{A}),$$

where $I_R = \{x \in A / xR0\}$ *is the strong ideal of* $\boldsymbol{A}$ *associated with the congruence relation* R.

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Part I. 7. Strong filters, strong ideals and congruence relations

The inverse of the previous correspondence *is defined by*

$$f^{*}(\boldsymbol{A}) \ni I \sim R_I \in \grave{}(\boldsymbol{A}),$$

with R_I *representing* ***the congruence relation on A associated with*** *I such that* xR_Iy *if and only if* $d(x,y) \in I$*, where* $d: A \times A \to \bar{f}A$ *is a binary operation on A called* ***the distance function on A*** *expressed by*

$$d(x,y) = \bar{f}[(x \text{ ฿ } y) \vee (y \text{ ฿ } x) \vee (x^d \text{ ฿ } y^d) \vee (y^d \text{ ฿ } x^d)].$$

Part II. 1. Motivations

1. Algebraic study of the Kolmogorov logic of problem solving

2. Development of real graded membership spaces in fuzzy set theory

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Part II. 2. On the Kolmogorov logic of problem solving

- *Propositional intuitionistic logic* expresses the ***logical laws of positive solution*** (*Heyting algebra*)

$p \vee \neg p$ is not a logical law of positive solution

- *Dual propositional intuitionistic logic* expresses the ***logical laws of negative solution*** (*Brouwer algebra*)

$p \wedge \neg p$ is not a logical law of negative solution

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<u>Part II.</u> 3. Starting point: known real graded membership spaces

- ***Support set****: real unit interval* [0, 1]
- ***Basic operations***: *meet, join, algebraic product, bounded sum, implications (Gödel, probabilistic, Łukasiewicz)*
- ***Basic structure***: residuated commutative monoid

Part II. 4. To combine different specific commutative residuated structures

- ***Residuated commutative monoid:***

  $(A, \cdot, /, \leq, 1)$, *where*

  - $(A, \cdot, \leq, 1)$ *is an ordered commutative monoid,*
  - $y / x = \max\{z \in A / x \cdot z \leq y\}$.

- ***Dual residuated commutative monoid:***

  $(A, +, \backslash, \leq, 0)$, *where*

  - $(A, +, \leq, 0)$ *is an ordered commutative monoid,*
  - $x \backslash y = \min\{z \in A / x \leq z + y\}$.

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Part II. 5. Positive axioms schemata [*propositional intuitionistic logic*]

1) $a \to (b \to a)$
2) $(a \to (b \to c)) \to ((a \to b) \to (a \to c))$
3) $(a \to \neg a) \to \neg a$
4) $a \to (\neg a \to b)$
5) $a \to (b \to (a \wedge b))$
6) $(a \wedge b) \to a$
7) $(a \wedge b) \to b$
8) $a \to (a \vee b)$
9) $b \to (a \vee b)$
10) $(a \to c) \to ((b \to c) \to ((a \vee b) \to c))$

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Part II. 6. Negative axioms schemata [*dual propositional intuitionistic logic*]

1°) $(a \text{ ฿ } b) \text{ ฿ } a$

2°) $((c \text{ ฿ } a) \text{ ฿ } (b \text{ ฿ } a)) \text{ ฿ } ((c \text{ ฿ } b) \text{ ฿ } a)$

3°) $\neg a \text{ ฿ } (\neg a \text{ ฿ } a)$

4°) $(b \text{ ฿ } \neg a) \text{ ฿ } a$

5°) $((a \vee b) \text{ ฿ } b) \text{ ฿ } a$

6°) $a \text{ ฿ } (a \vee b)$

7°) $b \text{ ฿ } (a \vee b)$

8°) $(a \wedge b) \text{ ฿ } a$

9°) $(a \wedge b) \text{ ฿ } b$

10°) $((c \text{ ฿ } (a \wedge b)) \text{ ฿ } (c \text{ ฿ } b)) \text{ ฿ } (c \text{ ฿ } a)$

Part II. 7. Substitution rules

$$[rs1]\ \frac{va}{(b \rightarrow b) \rightarrow a} \qquad [rs2]\ \frac{v(a \text{ ฿ } b)}{\neg(a \rightarrow b)} \qquad [rs3]\ \frac{\neg a}{a \rightarrow (b \text{ ฿ } b)}$$

$$[rs4]\ \frac{\overline{f}a}{a \text{ ฿ } (b \text{ ฿ } b)} \qquad [rs5]\ \frac{\overline{f}(a \rightarrow b)}{\neg(a \text{ ฿ } b)} \qquad [rs6]\ \frac{\neg a}{(b \rightarrow b) \text{ ฿ } a}$$

$$[rs7]\ \frac{a}{a^{dd}} \qquad [rs8]\ \frac{(a \rightarrow b)^{d}}{b^{d} \text{ ฿ } a^{d}} \qquad [rs9]\ \frac{(a \wedge b)^{d}}{b^{d} \vee a^{d}}$$

$$[rs10]\ \frac{a \rightarrow b}{va \rightarrow b}$$

Part II. 8. Inference rules

Positive inference rule

$$[MP] \quad \frac{a, a \to b}{b}$$

Negative inference rule

$$[MT] \quad \frac{b \to a, a}{b}$$

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Part II. 9. Provability relations

A formula p is called a positive logical law (negative logical law) if and only if there exists a syntactical proof of p from positive (negative) axioms schemata using the substitution rules [rs1]-[rs10] and the positive (negative) inference rule [MP] ([MT]).

We will denote by Î$^{+}$ ***p the affirmation that the formula p is a positive logical law and by*** Î$^{-}$ ***p the affirmation that the formula p is a negative logical law.***

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Part II. 10. Equivalence of formulas

Define the following two binary relation $\equiv^{+}$ *and* $\equiv^{-}$ *on the set* m *of propositional formulas, for all* $a, b \in$ m,

$$a \equiv^{+} b \text{ if and only if } \text{Î}^{+}\, e(a,b),$$

$$a \equiv^{-} b \text{ if and only if } \text{Î}^{-}\, d(a,b),$$

where the formulas $e(a,b), d(a,b) \in$ m *are defined by*

$$e(a,b) = v((a \text{ ฿→ } b) \wedge (b \text{ ฿→ } a) \wedge (a^{d} \text{ ฿→ } b^{d}) \wedge (b^{d} \text{ ฿→ } a^{d})),$$

$$d(a,b) = \overline{f}((a \text{ ฿ } b) \vee (b \text{ ฿ } a) \vee (a^{d} \text{ ฿ } b^{d}) \vee (b^{d} \text{ ฿ } a^{d})).$$

Part II. 11. Tarski-Lindenbaum algebras

Theorem 6. *The relation* $\equiv^{+}$ *is an equivalence relation on the set* m *and the quotient set* $\mathrm{m}/\equiv^{+}$ *is an involutive Brouwerian D-algebra satisfying* $[p]_{\equiv^{+}} = 1$ *if and only if* $\hat{\text{I}}^{+}\ p$.

Theorem 6^{o}**.** *The relation* $\equiv^{-}$ *is an equivalence relation on the set* m *and the quotient set* $\mathrm{m}/\equiv^{-}$ *is an involutive Brouwerian D-algebra satisfying* $[p]_{\equiv^{-}} = 0$ *if and only if* $\hat{\text{I}}^{-}\ p$.

$\mathrm{m}/\equiv^{+}$ *and* $\mathrm{m}/\equiv^{-}$ *are called* *Tarski-Lindenbaum algebras*.

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Part II. 12. A completeness theorem

Using the two previouos properties one obtains:

Theorem 7. *For any formula* $p \in$ m, $\text{Î}^{+}\ p$ *if and only if* $\text{ï}^{+}\ p$, *where* ï^{+} *is the relation of universal validity of propositional formulas of* m *over the class* f_$r(\mu^{d})$.

Theorem 7°. *For any formula* $p \in$ m, $\text{Î}^{-}\ p$ *if and only if* $\text{ï}^{-}\ p$, *where* ï^{-} *is the relation of universal invalidity of propositional formulas of* m *over the class* f_$r(\mu^{d})$.

Part III. 1. Concluding remarks

- *Involutive Brouwerian D-algebras represent the algebraic counterpart of a propositional symbolic logic of decision problem solving.*

- *A modal interpretation can be considered using the three unary operators.*

$$v : A \to A\ (\textit{necessity operator}),$$

$$\overline{f} : A \to A\ (\textit{possibility operator}),$$

$$\neg : A \to A\ (\textit{impossibility operator}).$$

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Part III. 2. Concluding remarks

- *An interesting case for a modal interpretation can be considered using a subclass of involutive Brouwerian D-algebras* ***A*** *satisfying:*
- $(x \rightarrow y)\vee(y \rightarrow x)=1,\ \forall x,y\in A.$
- *This relation is equivalent with the condition that*

  v is an interior operator and $\overline{f}$ *is a closure operator*

  of the underlying lattice $L(\mathbf{A})=(A,\vee,\wedge,0,1).$

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Part III. 3. Concluding remarks

- $\textit{The properties of the pair } (e, d) \textit{ are of interest in order to identify mathematical models for the development of first-order logic.}$

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Part III. 4. Concluding remarks

- The function $e: A \times A \to vA$ satisfies for all $x, y, z \in A$:

  e1) $e(x, y) = 1$ if and only if $x = y$ (reflexivity),

  e2) $e(x, y) = e(y, x)$ (symmetry),

  e3) $v[e(x, y) \wedge e(y, z)] \leq e(x, z)$ (transitivity).

- If X is a nonempty set then a function $e: X \times X \to vA$ satisfying the relations e1), e2) and e3), for all $x, y, z \in X$, is called an <u>equivalence $v\boldsymbol{A}$- function</u> on X.

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Part III. 5. Concluding remarks

- *The function* $d : A \times A \to \overline{f}A$ *satisfies for all* $x, y, z \in A$:

  $d1)\ d(x,y)=0$ *if and only if* $x=y$ (*dual reflexivity*),

  $d2)\ d(x,y)=d(y,x)$ (*dual symmetry*),

  $d3)\ d(x,z) \le \overline{f}[d(x,y) \vee d(y,z)]$ (*dual transitivity*).

- *If* X *is a nonempty set then a function* $d : X \times X \to \overline{f}A$ *satisfying the relations* $d1)$, $d2)$ *and* $d3)$, *for all* $x, y, z \in X$, *is called a* <u>*distance* $\overline{f}$***A****-function*</u> *on* X.

Part III. 6. Concluding remarks

- *The functions e and d are **A**-complementary* i.e. $\forall x, y \in A$,

$$c1)\ d(x,y) = [e(x,y)]^d,$$

$$c2)\ v[e(x,y) \wedge d(x,y)] = 0,$$

$$c3)\ \overline{f}[e(x,y) \vee d(x,y)] = 1.$$

- *If X is a nonempty set then the functions $e : X \times X \to vA$ and $d : X \times X \to \overline{f}A$ are called **A**-complementary if the relations c1), c2) and c3) hold, for all $x, y \in X$.*

Part III. 7. Concluding remarks

- *A development of* *first-order logic* *can be based on the following natural notion of* *generalized set* *over a* *complete involutive Brouwerian D-algebra* $\mathcal{A}$:
- *An* $\mathcal{A}$*-set* *is a triple* $\mathcal{X} = (X, e, d)$, *such that*

  1) $e : X \times X \to vA$ *is an equivalence* $v\mathcal{A}$*-function on* X,

  2) $d : X \times X \to \bar{f}A$ *is a distance* $\bar{f}\,\mathcal{A}$*-function on* X,

  3) *e and d are* $\mathcal{A}$*-complementary*.

1. Future research

- *Development of the algebraic study in connection with the Logic of Tasks and the Computability Logic*

- **[Artemov, S. work and recent papers Japaridze, G., *Annals of Pure and Applied Logic 117 (2002) 261-293; 123 (2003) 1-99*].**

2. Future research

- *Elaboration of first-order double mathematical structures over the real unit square.*

- *[Possibilistic Logic using a combination of the two structures involutive Brouwerian D-algebra and MV-algebra].*

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3. Future research

- ***Physical interpretation* of the structure of involutive Brouwerian D-algebras in connection with the Moisil view on involutive Boolean algebras in the study of circuits.**