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METHODS IN NON-CLASSICAL LOGICS***
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Topology and logic of decision problem solving

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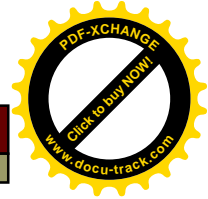


Contents

- I. Involutive Brouwerian D-algebras

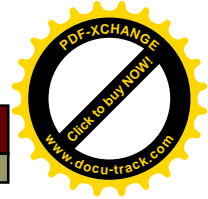
- II. Propositional symbolic logic of
decision problem solving

- III. Concluding remarks



Part I. Notations 1

- $\tau = (2, 2, 2, 2, 0, 0)$ is a type of algebras with four binary operations denoted by $\vee, \wedge, \rightarrow, \leftarrow$, and two constants $0, 1$.
- $\mu = (2, 2, 2, 2, 1, 1, 1, 0, 0)$ is a type of algebras with four binary operations denoted by $\vee, \wedge, \rightarrow, \leftarrow$, three unary operations denoted by $\bar{\vee}, \bar{f}, \neg$ and two constants $0, 1$.



Part I. Notations 2

- $\mathcal{A}(\tau)$ is the class of all algebras $\mathbf{A} = (A, \vee, \wedge, \multimap, \dashv, 0, 1)$ of type τ ,
- $\mathcal{A}(\mu)$ is the class of all algebras $\mathbf{A} = (A, \vee, \wedge, \multimap, \dashv, \nu, \bar{f}, \neg, 0, 1)$ of type μ ,
- $\mathcal{A}^{\wedge}(\mu)$ is the class of all algebras $\mathbf{A} \in \mathcal{A}(\mu)$ such that its underlying algebra $\tau[\mathbf{A}] = (A, \vee, \wedge, \multimap, \dashv, 0, 1) \in \mathcal{A}(\tau)$ and the three new unary operations $\nu, \bar{f}, \neg: A \rightarrow A$ satisfy the following equations:

$$\nu x = 1 \multimap x, \quad \bar{f}x = x \dashv 0,$$

$$x \multimap 0 = \neg x = 1 \dashv x.$$

Part I. Heyting algebras

A *Heyting algebra* is a system $H = (H, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ such that $(H, \vee, \wedge, 0, 1)$ is a bounded lattice and for all $x, y \in H$, $z \leq x \rightarrow y$ if and only if $z \wedge x \leq y$.

We will denote by \mathbf{Y} the class of Heyting algebras.

An *embedded Heyting algebra* in $\wedge (\mu)$ is a system

$\ddot{H} = (H, \vee, \wedge, \multimap, \uparrow, \nu, \bar{f}, \neg, 0, 1) \in \wedge (\mu)$ such that

$H = (H, \vee, \wedge, \multimap, 0, 1) \in \mathbf{Y}$ and \ddot{H} satisfies:

- $x \uparrow y = (x \multimap y) \multimap 0$, $\neg x = x \multimap 0 = 1 \uparrow x$,
- $\nu x = 1 \multimap x = x$, $\bar{f}x = x \uparrow 0 = \neg \neg x$.

Part I. Brouwer algebras

A *Brouwer algebra* is a system $\mathbf{B} = (B, \vee, \wedge, \mathbf{t}, 0, 1)$ of type $(2, 2, 2, 0, 0)$ such that $(B, \vee, \wedge, 0, 1)$ is a bounded lattice and for all $x, y \in B$, $x \mathbf{t} y \leq z$ if and only if $x \leq y \vee z$.

We will denote by \mathbf{Sr} the class of Brouwer algebras.

An *embedded Brouwer algebra* in $\wedge(\mu)$ is a system

$\hat{\mathbf{B}} = (B, \vee, \wedge, \mathbf{b}, \mathbf{t}, \nu, \bar{f}, \neg, 0, 1) \in \wedge(\mu)$ such that

$\mathbf{B} = (B, \vee, \wedge, \mathbf{t}, 0, 1) \in \mathbf{Sr}$ and $\hat{\mathbf{B}}$ satisfies:

- $x \mathbf{b} y = 1 \mathbf{t} (x \mathbf{t} y), \quad \neg x = x \mathbf{b} 0 = 1 \mathbf{t} x,$
- $\nu x = 1 \mathbf{b} x = \neg \neg x, \quad \bar{f} x = x \mathbf{t} 0 = x.$



Part I. Boolean algebras

A *Boolean algebra* is a system $\mathbf{B} = (B, \vee, \wedge, {}^c, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and for all $x \in B$, $x \vee x^c = 1$ and $x \wedge x^c = 0$.

We will denote by \mathbf{S} the class of Boolean algebras.

An *embedded Boolean algebra* in $\wedge(\mu)$ is a system

$\hat{\mathbf{B}} = (B, \vee, \wedge, \multimap, \dashv, \nu, \bar{f}, \neg, 0, 1) \in \wedge(\mu)$ such that

$\mathbf{B} = (B, \vee, \wedge, \neg, 0, 1)$ is a Boolean algebra and $\hat{\mathbf{B}}$ satisfies:

- $x \multimap y = \neg x \vee y$, $x \dashv y = x \wedge \neg y$,
- $\nu x = 1 \multimap x = x$, $\bar{f}x = x \dashv 0 = x$, $\neg x = x \multimap 0 = 1 \dashv x$.

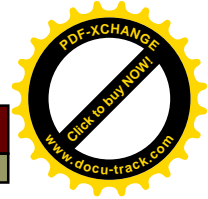


Part I. Properties 1

Notations

- 1) $Y(\mu)$ is the class of *embedded Heyting algebras* in $\wedge(\mu)$.
- 2) $Sr(\mu)$ is the class of *embedded Brouwer algebras* in $\wedge(\mu)$.
- 3) $S(\mu)$ is the class of *embedded Boolean algebras* in $\wedge(\mu)$.

Proposition 1. $S(\mu) = Y(\mu) \cap Sr(\mu)$



Part I. Properties 2

Proposition 2. *Every algebra $\mathbf{A} \in \mathbf{Y}(\mu) \cup \mathbf{Sr}(\mu)$, satisfies the following relations:*

(i) $x \multimap 0 = \neg x = 1 \multimap x$,

(ii) $v(x \multimap y) = \neg(x \multimap y)$,

(iii) $\bar{f}(x \multimap y) = \neg(x \multimap y)$

(iv) $v(v(x)) = v(x) \leq x \leq \bar{f}(x) = \bar{f}(\bar{f}(x))$,

(v) $v(x \vee y) = v(x) \vee v(y)$,

(vi) $\bar{f}(x \wedge y) = \bar{f}(x) \wedge \bar{f}(y)$,

(vii) $v(\bar{f}(x)) = \neg\neg x = \bar{f}(v(x))$.



Part I. A mathematical problem regarding the previous structures

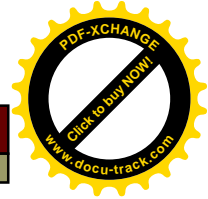
Problem formulation

Determine a minimal class of algebras T such that the following relation holds:

$$Y(\mu) \cup Sr(\mu) \subseteq T \subseteq \wedge (\mu).$$

Remark

New classes of structures must be considered with respect to the above mathematical problem.



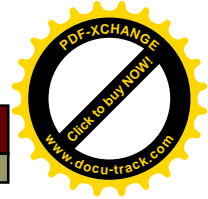
Part I. Heyting-Brouwer algebras 1

A Heyting-Brouwer algebra (double Heyting algebra) is a system

$\mathbf{A} = (A, \vee, \wedge, \multimap, \mathbf{t}, 0, 1) \in \mathcal{L}(\tau)$ such that

$(A, \vee, \wedge, \multimap, 0, 1) \in \mathbf{Y}$ and $(A, \vee, \wedge, \mathbf{t}, 0, 1) \in \mathbf{Sr}$.

Let $\mathbf{YSr}(\tau)$ be the class of all Heyting-Brouwer algebras.



Part I. Heyting-Brouwer algebras 2

Proposition 3. Let $\mathbf{A} = (A, \vee, \wedge, \multimap, \dashv, 0, 1) \in \mathbf{YSr}(\tau)$.

The following conditions are equivalent :

- (i) For all $x, y \in A$, $x \multimap 0 = 1 \dashv x$.*
- (ii) The underlying poset of \mathbf{A} is a Boolean algebra.*

If \mathbf{A} satisfies (i) or (ii) then the corresponding unary operation of complementation $^c: A \rightarrow A$ satisfies :

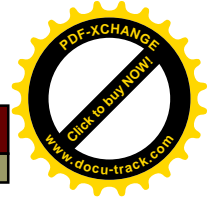
$x \multimap 0 = x^c = 1 \dashv x$, $x^c \vee y = x \multimap y$ and $x \wedge y^c = x \dashv y$.



Part I. New structure required

In a Heyting-Brouwer algebra A the two unary operations of Heyting negation and of Brouwer negation coincide if and only if A is a Boolean algebra.

Thus a new structure must be introduced.



Part I. Definition of the notion of Brouwerian D-algebras

A Brouwerian D-algebra is a system $A \in \setminus(\mu)$ such that there exist $\hat{H} \in Y(\mu)$, $\hat{B} \in Sr(\mu)$ and an isomorphism from A onto a subdirect product of the pair (\hat{H}, \hat{B}) in $\setminus(\mu)$.

Let SrU(μ) be the class of Brouwerian D-algebras.

We present now basic properties derived from this definition, in order to establish that a solution of the problem previously mentioned is defined by the class T = SrU(μ).



Part I. Properties 3

Proposition 4. *Let $A \in \setminus(\mu)$. Then $A \in \text{SrU}(\mu)$ if and only if there exist $\ddot{H} \in Y(\mu)$, $\hat{B} \in \text{Sr}(\mu)$ and an injective homomorphism $i: A \rightarrow \ddot{H} \times \hat{B}$ such that $\pi_H(i(A)) = \ddot{H}$ and $\pi_B(i(A)) = \hat{B}$.*



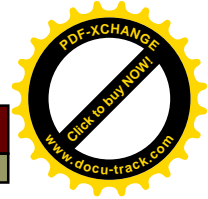
Part I. Properties 4

Proposition 5. *The following conditions hold :*

(i) $Y(\mu) \cup Sr(\mu) \subseteq SrU(\mu),$

(ii) *If* $H \in Y$ *and* $B \in Sr$ *then* $\ddot{H} \times \hat{B} \in SrU(\mu) \cap \wedge (\mu),$

(iii) $SrU(\mu) \subseteq \wedge (\mu).$



Part I. Consequences

Specific examples. *From the properties previously presented one obtains different examples of algebras of the class $SrU(\mu)$ associated with topological structures.*

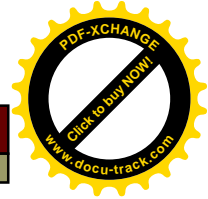
Part I. Specific examples of Brouwerian D-algebras

□ *Direct product of two chains*

$A = \mathbb{E}_H \times \mathbb{E}_{Br}$, where \mathbb{E}_H is an embedded Heyting chain and \mathbb{E}_{Br} is an embedded Brouwer chain in $\setminus(\mu)$.

○ *Topological examples*

$A = \mathbb{O}_p(U) \times \mathbb{C}l(V)$, where $\mathbb{O}_p(U)$ is an embedded Heyting algebra of open sets of a topological space U and $\mathbb{C}l(V)$ is an embedded Brouwer algebra of closed sets of a topological space V .

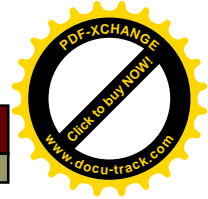


Part I. Properties 5

Proposition 6. *The unary operation $v:A \rightarrow A$ is a join semilattice homomorphism and it is an interior operator of the poset $(A, \leq, 0, 1)$.*

Proposition 7. *The unary operation $\bar{f}:A \rightarrow A$ is a meet semilattice homomorphism and it is a closure operator of the poset $(A, \leq, 0, 1)$.*

Proposition 8. *If $A \in \text{SrU}(\mu)$ then A is isomorphic to a subdirect product of the pair of algebras $(\bar{\vee}A, \bar{f}A) \in Y(\mu) \times \text{Sr}(\mu)$.*



Part I. Properties 6

Proposition 9. *If $A \in \text{SrU}(\mathfrak{b})$ then $\overline{\neg}A \in \text{S}(\mu)$ is an embedded Boolean algebra with the support set verifying $\neg A = \vee A \cap \overline{f}A$, where the Boolean complement of x is $x^c = \neg x$, for all $x \in \neg A$.*

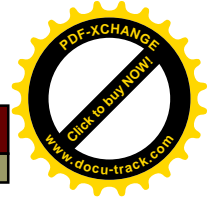
Theorem 1. *$\text{SrU}(\mu)$ is the variety of algebras generated by $Y(\mu) \cup \text{Sr}(\mu)$.*

Theorem 2. *The dual space of any Brouwerian D -algebra A is a gluing of two Stone spaces of the pair $(\overline{\vee}A, \overline{f}A)$ defined by Priestley duality theory.*



Part I. An informal description of involutive Brouwerian D-algebras

- An involutive Brouwerian D-algebra is a Brouwerian D-algebra equipped with an involution satisfying some natural conditions.
- A specific model and the definition of the structure of involutive Brouwerian D-algebra are derived from the following particular topological example:



Part I. A topological model

Let X be a topological space. Then $\underline{Sq}(X) = \underline{Op}(X) \times \underline{Cl}(X)$ is a particular Brouwerian D -algebra with the property that $\underline{Cl}(X)$ is isomorphic with the dual of $\underline{Op}(X)$.



Part I. Notations 3

- $\mu^d = (2, 2, 2, 2, 1, 1, 1, 1, 0, 0)$ is a type of algebras of $\setminus(\mu)$ extended with a new unary operation denoted by d .
- $\setminus(\mu^d)$ is the class of all algebras $\mathbf{A} = (A, \vee, \wedge, \rightarrow, \leftarrow, \bar{f}, \neg^d, 0, 1)$ of type μ^d .
- $\wedge(\mu^d)$ is the class of all algebras $\mathbf{A} \in \setminus(\mu^d)$ such that $\mu[\mathbf{A}] = (A, \vee, \wedge, \rightarrow, \leftarrow, \bar{f}, \neg, 0, 1) \in \wedge(\mu)$.



Part I. Definition

An involutive Brouwerian D-algebra is a system

$$\mathbf{A} = (A, \vee, \wedge, \multimap, \dashv, \nu, \bar{f}, \neg, {}^d, 0, 1) \in \mathcal{S}(\mu^d),$$

such that the following conditions hold :

IBrD1) The underlying μ -system of \mathbf{A} is a

Brouwerian D-algebra, namely

$$\mu[\mathbf{A}] = (A, \vee, \wedge, \multimap, \dashv, \nu, \bar{f}, \neg, 0, 1) \in \text{SrU}(\mu).$$

IBrD2) The system $(A, \vee, \wedge, {}^d, 0, 1)$ is a De Morgan algebra.

IBrD3) The following equation holds, for all $x, y \in A$,

$$\underline{(x \multimap y)^d = y^d \dashv x^d}.$$

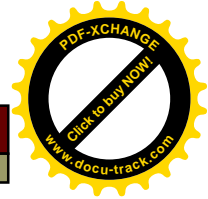
Part I. Examples

- Example 1
The Brouwerian D-algebra $Sq(X) = Op(X) \times Cl(X)$ associated with a topological space X together with the unary operation d defined by $(O, F)^d = (F^c, O^c)$, for all $O \in Op(X)$ and $F \in Cl(X)$, where for a subset $S \subseteq X$, $S^c = X \setminus S$ is the complement set of S in $a(X)$.
- Example 2
The real unit square $P = H[0,1] \times B[0,1]$ together with the unary operation d representing the symmetry with respect to the diagonal segment $[(0,1), (1,0)]$ defined by $(\alpha, \beta)^d = (1-\beta, 1-\alpha)$, for all $\alpha, \beta \in [0,1]$, where $H[0,1]$ is the real Heyting chain $[0,1]$ and $B[0,1]$ is the real Brouwer chain $[0,1]$.



Part I. Remark

The class $ZSrU(\mu^d)$ of all involutive Brouwerian D -algebras
is a variety of algebras in $\wedge(\mu^d)$ characterized by three sets of
equations expressing the conditions **IBrD1**), **IBrD2**) and **IBrD3**).



Part I. Properties 7: an equational characterization

Theorem 3. *Let $\underline{V}(\mu^d)$ be the class of all algebras*

$\underline{A} = (A, \vee, \wedge, \dashv, \vdash, \bar{f}, \neg^d, 0, 1) \in \underline{V}(\mu^d)$ such that

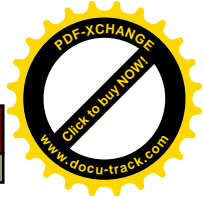
the following equations hold, for all $x, y \in A$:

$$e1) \underline{x \vdash y = (y^d \dashv x^d)^d},$$

$$e2) \underline{\vee x = 1 \dashv x},$$

$$e3) \underline{\bar{f}x = (0^d \dashv x^d)^d},$$

$$e4) \underline{\neg x = x \dashv 0}.$$



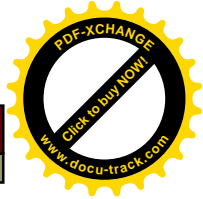
Part I. Properties 7: an equational characterization

A system $\mathbf{A} = (A, \vee, \wedge, \multimap, \dashv, \bar{}, \neg^d, 0, 1) \in \mathcal{V}(\mu^d)$

is an involutive Brouwerian D-algebra if and only if

\mathbf{A} satisfies the following axioms, for all $x, y, z \in A$:

- 1) $x \wedge (y \vee z) = (z \wedge (x \vee 0)) \vee (y \wedge (x \vee 0))$,
- 2) $x \wedge (x \vee y) = x$,
- 3) $0^d = 1$,
- 4) $(x^d \wedge y^d)^d = x \vee y$,
- 5) $1 \multimap (x \vee y) = (1 \multimap x) \vee (1 \multimap y)$,
- 6) $(1 \multimap x) \vee x = x$,
- 7) $(1 \multimap x) \vee ((0^d \multimap x^d)^d \wedge (x \multimap 0)) = x$,



Part I. Properties 7: an equational characterization

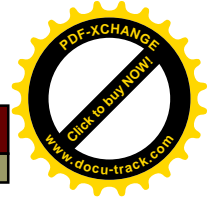
$$8) x \multimap y = (1 \multimap x) \multimap y,$$

$$9) x \multimap (x \vee y) = 1,$$

$$10) 1 \multimap (x \wedge (x \multimap y)) = 1 \multimap (x \wedge y),$$

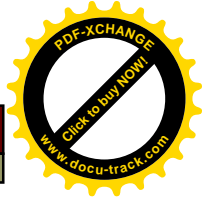
$$11) x \multimap (y \multimap z) = (x \wedge y) \multimap z,$$

$$12) x \multimap (z^d \multimap y^d)^d = ((x \wedge (y \multimap z))^d \multimap 1^d)^d.$$



Part I. 1. Strong filters, strong ideals and congruence relations

- Suppose that $A = (A, \vee, \wedge, \multimap, \text{t}, \nu, \bar{f}, \neg^d, 0, 1) \in \text{ZSrU}(\mu^d)$.
- A strong filter of A is a nonempty subset F of A verifying :
 - F1) $x \in F$ and $y \in A$ imply $x \vee y \in F$,
 - F2) $x \in F$ and $y \in F$ imply $\nu(x \wedge y) \in F$,
 - F3) $x \in F$ implies $\neg x^d \in F$.
- The set of strong filters of A is a complete lattice $W^*(A)$ with respect to the set inclusion because it is closed under arbitrary set intersection.



Part I. 2. Strong filters, strong ideals and congruence relations

- A strong ideal of A is a nonempty subset I of A verifying :
 - I1) $x \in I$ and $y \in A$ imply $x \wedge y \in I$,
 - I2) $x \in I$ and $y \in I$ imply $\bar{f}(x \vee y) \in I$,
 - I3) $x \in I$ implies $\neg x^d \in I$.
- The set of strong ideals of A is a complete lattice $Z^*(A)$ with respect to the set inclusion because it is closed to arbitrary set intersection.



Part I. 3. Strong filters, strong ideals and congruence relations

Theorem 4. *Let \mathbf{A} be an involutive Brouwerian D -algebra.*

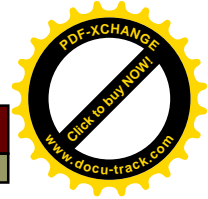
Then there exists a complete lattice isomorphism from

$W^(\mathbf{A})$ onto $Z^*(\mathbf{A})$ defined by the following correspondence :*

$$W^*(\mathbf{A}) \ni F \mapsto d_i[F] = F^d = \{x^d / x \in F\} \in Z^*(\mathbf{A}).$$

The inverse of this isomorphism from $Z^(\mathbf{A})$ onto $W^*(\mathbf{A})$ is defined by the following correspondence :*

$$Z^*(\mathbf{A}) \ni I \mapsto d_f[I] = I^d = \{x^d / x \in I\} \in W^*(\mathbf{A}).$$



Part I. 4. Strong filters, strong ideals and congruence relations

Theorem 5. *Let $\mathcal{T}(A)$ be the complete lattice of congruence relations of A . Then there exists an isomorphism from $\mathcal{T}(A)$ onto $W^*(A)$ defined by the following correspondence :*

$$\mathcal{T}(A) \ni R \mapsto f(R) = F_R \in W^*(A),$$

where $F_R = \{x \in A / xR1\}$ is the strong filter of A associated with the congruence relation R .



Part I. 5. Strong filters, strong ideals and congruence relations

The inverse of the previous correspondence is defined by

$$W^*(A) \ni F \text{ } \ulcorner \text{ } R_F \in T(A),$$

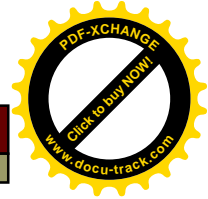
*with R_F representing the **congruence relation of A associated***

with F such that $xR_F y$ if and only if $e(x, y) \in F$, where

*$e : A \times A \rightarrow vA$ is a binary operation on A called **the equivalence***

function on A expressed by

$$e(x, y) = v[(x \overset{\leftarrow}{\rightarrow} y) \wedge (y \overset{\leftarrow}{\rightarrow} x) \wedge (x^d \overset{\leftarrow}{\rightarrow} y^d) \wedge (y^d \overset{\leftarrow}{\rightarrow} x^d)]$$

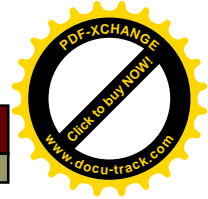


Part I. 6. Strong filters, strong ideals and congruence relations

A standard duality principle is true in the theory of involutive Brouwerian D -algebras. The two notions of strong filter and strong ideal are dual one to another.

On this basis it follows that from the study of strong filters one obtains corresponding dual results for strong ideals.

For example, we present the dual of Theorem 5.



Part I. 7. Strong filters, strong ideals and congruence relations

Theorem 5^o. *Let A be an involutive Brouwerian D -algebra. Then there exists an isomorphism from the complete lattice $T(A)$ of congruences relations of A onto $Z^*(A)$ defined by the following correspondence :*

$$T(A) \ni R \mapsto i(R) = I_R \in Z^*(A),$$

where $I_R = \{x \in A / xR0\}$ is the strong ideal of A associated with the congruence relation R .



Part I. 7. Strong filters, strong ideals and congruence relations

The inverse of the previous correspondence is defined by

$$Z^*(A) \ni I \vdash R_I \in T(A),$$

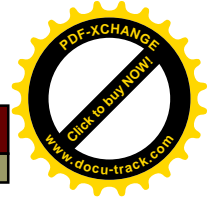
with R_I representing the congruence relation on A associated

with I such that $xR_I y$ if and only if $d(x, y) \in I$, where

$d : A \times A \rightarrow \bar{f}A$ is a binary operation on A called the distance

function on A expressed by

$$d(x, y) = \bar{f}[(x \text{ † } y) \vee (y \text{ † } x) \vee (x^d \text{ † } y^d) \vee (y^d \text{ † } x^d)].$$



Part II. 1. Motivations

1. Algebraic study of the Kolmogorov logic of problem solving
2. Development of real graded membership spaces in fuzzy set theory



Part II. 2. On the Kolmogorov logic of problem solving

- *Propositional intuitionistic logic expresses the logical laws of positive solution (Heyting algebra)*
- *Dual propositional intuitionistic logic expresses the logical laws of negative solution (Brouwer algebra)*

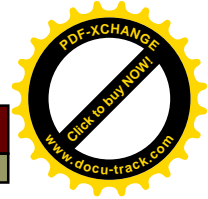
$p \vee \neg p$ is not a logical law of positive solution

$p \wedge \neg p$ is not a logical law of negative solution



Part II. 3. Starting point: known real graded membership spaces

- *Support set: real unit interval $[0, 1]$*
- *Basic operations: meet, join, algebraic product, bounded sum, implications (Gödel, probabilistic, Łukasiewicz)*
- *Basic structure: residuated commutative monoid*



Part II. 4. To combine different specific commutative residuated structures

○ *Residuated commutative monoid:*

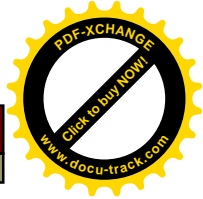
$(A, \cdot, /, \leq, 1)$, where

- $(A, \cdot, \leq, 1)$ is an ordered commutative monoid,
- $y / x = \max\{z \in A / x \cdot z \leq y\}$.

○ *Dual residuated commutative monoid:*

$(A, +, \backslash, \leq, 0)$, where

- $(A, +, \leq, 0)$ is an ordered commutative monoid,
- $x \backslash y = \min\{z \in A / x \leq z + y\}$.



Part II. 5. Positive axioms schemata *[propositional intuitionistic logic]*

$$1) a \multimap (b \multimap a)$$

$$2) (a \multimap (b \multimap c)) \multimap ((a \multimap b) \multimap (a \multimap c))$$

$$3) (a \multimap \neg a) \multimap \neg a$$

$$4) a \multimap (\neg a \multimap b)$$

$$5) a \multimap (b \multimap (a \wedge b))$$

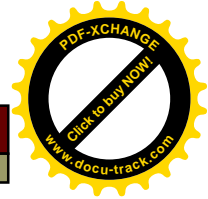
$$6) (a \wedge b) \multimap a$$

$$7) (a \wedge b) \multimap b$$

$$8) a \multimap (a \vee b)$$

$$9) b \multimap (a \vee b)$$

$$10) (a \multimap c) \multimap ((b \multimap c) \multimap ((a \vee b) \multimap c))$$



Part II. 6. Negative axioms schemata *[dual propositional intuitionistic logic]*

$$1^{\circ}) (a \multimap b) \multimap a$$

$$2^{\circ}) ((c \multimap a) \multimap (b \multimap a)) \multimap ((c \multimap b) \multimap a)$$

$$3^{\circ}) \neg a \multimap (\neg a \multimap a)$$

$$4^{\circ}) (b \multimap \neg a) \multimap a$$

$$5^{\circ}) ((a \vee b) \multimap b) \multimap a$$

$$6^{\circ}) a \multimap (a \vee b)$$

$$7^{\circ}) b \multimap (a \vee b)$$

$$8^{\circ}) (a \wedge b) \multimap a$$

$$9^{\circ}) (a \wedge b) \multimap b$$

$$10^{\circ}) ((c \multimap (a \wedge b)) \multimap (c \multimap b)) \multimap (c \multimap a)$$

Part II. 7. Substitution rules

$$[rs1] \frac{va}{(b \leftrightarrow b) \leftrightarrow a} \quad [rs2] \frac{v(a \leftrightarrow b)}{\neg(a \leftrightarrow b)} \quad [rs3] \frac{\neg a}{a \leftrightarrow (b \leftrightarrow b)}$$

$$[rs4] \frac{\bar{f}a}{a \leftrightarrow (b \leftrightarrow b)} \quad [rs5] \frac{\bar{f}(a \leftrightarrow b)}{\neg(a \leftrightarrow b)} \quad [rs6] \frac{\neg a}{(b \leftrightarrow b) \leftrightarrow a}$$

$$[rs7] \frac{a}{a^{dd}} \quad [rs8] \frac{(a \leftrightarrow b)^d}{b^d \leftrightarrow a^d} \quad [rs9] \frac{(a \wedge b)^d}{b^d \vee a^d}$$

$$[rs10] \frac{a \leftrightarrow b}{va \leftrightarrow b}$$



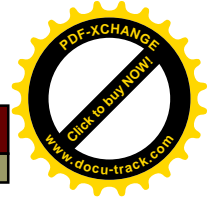
Part II. 8. Inference rules

Positive inference rule

$$[MP] \frac{a, a \rightarrow b}{b}$$

Negative inference rule

$$[MT] \frac{b \leftarrow a, a}{b}$$



Part II. 9. Provability relations

A formula p is called a positive logical law (negative logical law) if and only if there exists a syntactical proof of p from positive (negative) axioms schemata using the substitution rules [rs1]-[rs10] and the positive (negative) inference rule [MP] ([MT]).

We will denote by $a^+ p$ the affirmation that the formula p is a positive logical law and by $a^- p$ the affirmation that the formula p is a negative logical law.



Part II. 10. Equivalence of formulas

Define the following two binary relation \equiv^+ and \equiv^- on the set \mathfrak{a} of propositional formulas, for all $a, b \in \mathfrak{a}$,

$a \equiv^+ b$ if and only if $a^+ \vDash e(a, b)$,

$a \equiv^- b$ if and only if $a^- \vDash d(a, b)$,

where the formulas $e(a, b), d(a, b) \in \mathfrak{a}$ are defined by

$$e(a, b) = \nu((a \dashv\rightarrow b) \wedge (b \dashv\rightarrow a) \wedge (a^d \dashv\rightarrow b^d) \wedge (b^d \dashv\rightarrow a^d)),$$

$$d(a, b) = \bar{\nu}((a \dashv\leftarrow b) \vee (b \dashv\leftarrow a) \vee (a^d \dashv\leftarrow b^d) \vee (b^d \dashv\leftarrow a^d)).$$

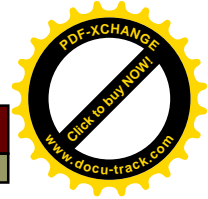


Part II. 11. Tarski-Lindenbaum algebras

Theorem 6. *The relation \equiv^+ is an equivalence relation on the set \mathfrak{a} and the quotient set \mathfrak{a}/\equiv^+ is an involutive Brouwerian D-algebra satisfying $[p]_{\equiv^+} = 1$ if and only if $\mathfrak{a}^+ p$.*

Theorem 6°. *The relation \equiv^- is an equivalence relation on the set \mathfrak{a} and the quotient set \mathfrak{a}/\equiv^- is an involutive Brouwerian D-algebra satisfying $[p]_{\equiv^-} = 0$ if and only if $\mathfrak{a}^- p$.*

\mathfrak{a}/\equiv^+ and \mathfrak{a}/\equiv^- are called Tarski-Lindenbaum algebras.



Part II. 12. A completeness theorem

Using the two previous properties one obtains:

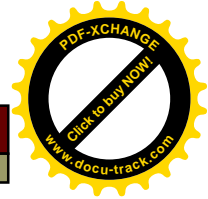
Theorem 7. *For any formula $p \in \mathfrak{a}$, $\mathfrak{a}^+ p$ if and only if $\ll^+ p$, where \ll^+ is the relation of universal validity of propositional formulas of \mathfrak{a} over the class $ZSr(\mu^d)$.*

Theorem 7°. *For any formula $p \in \mathfrak{a}$, $\mathfrak{a}^- p$ if and only if $\ll^- p$, where \ll^- is the relation of universal invalidity of propositional formulas of \mathfrak{a} over the class $ZSr(\mu^d)$.*



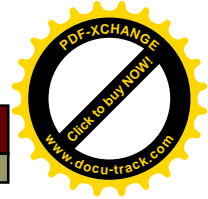
Part III. 1. Concluding remarks

- *Involutive Brouwerian D-algebras represent the algebraic counterpart of a propositional symbolic logic of decision problem solving.*
- *A modal interpretation can be considered using the three unary operators.*
 - $\nu : A \rightarrow A$ (*necessity operator*),
 - $\bar{f} : A \rightarrow A$ (*possibility operator*),
 - $\neg : A \rightarrow A$ (*impossibility operator*).



Part III. 2. Concluding remarks

- *An interesting case for a modal interpretation can be considered using a subclass of involutive Brouwerian D-algebras A satisfying:*
- $(x \dashv\vdash y) \vee (y \dashv\vdash x) = 1, \forall x, y \in A.$
- *This relation is equivalent with the condition that v is an interior operator and \bar{f} is a closure operator of the underlying lattice $L(A) = (A, \vee, \wedge, 0, 1).$*



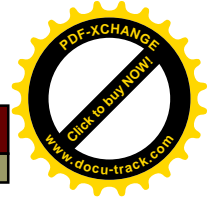
Part III. 3. Concluding remarks

- *The properties of the pair (e, d) are of interest in order to identify mathematical models for the development of first - order logic.*



Part III. 4. Concluding remarks

- *The function $e : A \times A \rightarrow \nu A$ satisfies for all $x, y, z \in A$:*
 - e1) $e(x, y) = 1$ if and only if $x = y$ (reflexivity),*
 - e2) $e(x, y) = e(y, x)$ (symmetry),*
 - e3) $\nu[e(x, y) \wedge e(y, z)] \leq e(x, z)$ (transitivity).*
- *If X is a nonempty set then a function $e : X \times X \rightarrow \nu A$ satisfying the relations e1), e2) and e3), for all $x, y, z \in X$, is called an equivalence νA - function on X .*



Part III. 5. Concluding remarks

- *The function $d : A \times A \rightarrow \bar{f}A$ satisfies for all $x, y, z \in A$:
d1) $d(x, y) = 0$ if and only if $x = y$ (dual reflexivity),
d2) $d(x, y) = d(y, x)$ (dual symmetry),
d3) $d(x, z) \leq \bar{f}[d(x, y) \vee d(y, z)]$ (dual transitivity).*
- *If X is a nonempty set then a function $d : X \times X \rightarrow \bar{f}A$ satisfying the relations d1), d2) and d3), for all $x, y, z \in X$, is called a distance $\bar{f}A$ -function on X .*

Part III. 6. Concluding remarks

- The functions e and d are A -complementary i.e. $\forall x, y \in A$,

$$c1) d(x, y) = [e(x, y)]^d,$$

$$c2) v[e(x, y) \wedge d(x, y)] = 0,$$

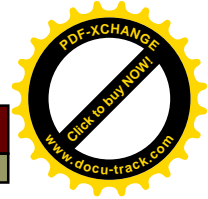
$$c3) \bar{f}[e(x, y) \vee d(x, y)] = 1.$$

- *If X is a nonempty set then the functions $e : X \times X \rightarrow vA$ and $d : X \times X \rightarrow \bar{f}A$ are called A -complementary if the relations $c1), c2)$ and $c3)$ hold, for all $x, y \in X$.*



Part III. 7. Concluding remarks

- *A development of first - order logic can be based on the following natural notion of generalized set over a complete involutive Brouwerian D - algebra A:*
- *An A-set is a triple $X = (X, e, d)$, such that*
 - 1) *$e : X \times X \rightarrow vA$ is an equivalence vA -function on X ,*
 - 2) *$d : X \times X \rightarrow \bar{f}A$ is a distance $\bar{f}A$ -function on X ,*
 - 3) *e and d are A -complementary.*



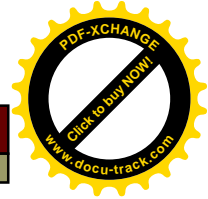
1. Future research

- *Development of the algebraic study in connection with the Logic of Tasks and the Computability Logic*
- *[Artemov, S. work and recent papers Japaridze, G., *Annals of Pure and Applied Logic* 117 (2002) 261-293; 123 (2003) 1-99].*



2. Future research

- *Elaboration of first-order double mathematical structures over the real unit square.*
- *[Possibilistic Logic using a combination of the two structures involutive Brouwerian D-algebra and MV-algebra].*



3. Future research

- *Physical interpretation of the structure of involutive Brouwerian D-algebras in connection with the Moisil view on involutive Boolean algebras in the study of circuits.*