Constructive logic with strong negation is a substructural logic over $\boldsymbol{FL}_{ew}$

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A Hilbert-style presentation of *IPC*

$p \to (q \to p)$

$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$p \wedge q \to p$

$p \wedge q \to q$

$(r \to p) \to ((r \to q) \to (r \to (p \wedge q)))$

$p \to p \vee q$

$q \to p \vee q$

$(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

$\neg p \to (p \to \mathbf{0})$

$(p \to \mathbf{0}) \to \neg p$

$\mathbf{0} \to p$

$p \to \mathbf{1}$

$p, p \to q \vdash q.$

Constructive logic with strong negation

- Constructive logic with strong negation, in symbols ***CLSN***, is the axiomatic expansion of ***IPC*** by a unary connective $\sim$ and axioms:

$$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q) \qquad \sim p \rightarrow (p \rightarrow q) \qquad \sim\neg p \leftrightarrow p$$

$$\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q) \qquad \sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q) \qquad \sim\sim p \leftrightarrow p$$

  - The unary connective $\sim$ is known as the **strong negation**.
- Milestones:
  - 1949 ***CLSN*** introduced by Nelson.
  - 1958 Algebraic semantics introduced by Rasiowa.
  - 1977 Counterexample semantics developed by Vakarelov.
  - 1990s Proof theoretic treatments of logics with strong negation.

CLSN is usually studied relative (in some sense) to ***IPC***.

A Hilbert-style presentation of *CLSN*

$p \to (q \to p)$
$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$

$p \wedge q \to p$
$p \wedge q \to q$
$(r \to p) \to ((r \to q) \to (r \to (p \wedge q)))$

$p \to p \vee q$
$q \to p \vee q$
$(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$

$\neg p \to (p \to \mathbf{0})$
$(p \to \mathbf{0}) \to \neg p$

$\mathbf{0} \to p$
$p \to \mathbf{1}$

$p, p \to q \mid\!- q$

$\sim p \to (p \to q)$
$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
$\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
$\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
$\sim\neg p \leftrightarrow p$
$\sim\sim p \leftrightarrow p.$

The algebraic counterpart of *CLSN*

- ***CLSN*** is **regularly algebraisable** in the sense of Blok and Pigozzi.
  - This means ∃ a class of algebras **K** that is to ***CLSN*** as **BA** is to ***CPC***.
- The **equivalent quasivariety** of ***CLSN*** is the class **N** of all Nelson algebras.
  - **N** is the algebraic counterpart of ***CLSN*** in the same way **BA** is the algebraic counterpart of ***CPC***.

Nelson algebras are De Morgan algebras enriched with a certain weak implication operation $\rightarrow$ generalising relative pseudocomplementation.

Nelson algebras

- A **Nelson algebra** is an algebra $\mathbf{A} := \langle A; \wedge, \vee, \rightarrow, \sim, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 1, 0, 0\rangle$ such that
  1. $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra with lattice order $\leq$.
  2. The relation $<<$ given by $a << b$ iff $a \rightarrow b = 1$ $(a, b \in A)$ is a preorder on A.
  3. The relation $\Xi := << \cap <<^{-1}$ is a congruence on $\mathbf{A}' := \langle A; \wedge, \vee, \rightarrow, \neg, 0, 1\rangle$, and $\mathbf{A}'/\Xi$ is a Heyting algebra.
  4. $\mathbf{A} \models \neg x \approx x \rightarrow \mathbf{0}$.
  5. $\forall\, a, b \in A$,
     1. $\sim(a \rightarrow b)\ \Xi\ a \wedge \sim b$
     2. $a \wedge \sim a << 0$
     3. $a \Rightarrow b = 1$ iff $a \leq b$.

Nelson algebras are a variety

- **Theorem** (Brignole, 1969). A Nelson algebra is an algebra $\mathbf{A} := \langle A; \wedge, \vee, \to, \sim, \neg, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 1, 0, 0\rangle$ where:

  1. $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra.
  2. $\mathbf{A}$ satisfies the following identities:

$$(x \wedge \sim x) \wedge (y \vee \sim y) \approx x \wedge \sim x$$
$$x \to x \approx \mathbf{1}$$
$$(x \to y) \wedge (\sim x \vee y) \approx \sim x \vee y$$
$$x \wedge (\sim x \vee y) \approx x \wedge (x \to y)$$
$$(x \to y) \wedge (x \to z) \approx x \to (y \wedge z)$$
$$(x \wedge y) \to z \approx x \to (y \to z)$$
$$\neg x \approx x \to \mathbf{0}.$$

Substructural logics over ***FL***

- Informally, a **substructural logic** is a logic that lacks some or all of the structural rules when presented as a sequent system.
- Let **FL** denote the sequent system obtained from **LJ** by deleting the structural rules:

  (e) Exchange, (c) Contraction, (w) Weakening

  and by adding rules for the fusion connective * and the residuals.
- Let ***FL*** denote the deductive system determined by **FL**.
- Let $\boldsymbol{FL}_{e[c]w}$ denote the extension of ***FL*** by (e), [(c)], and (w).

The language type of $\boldsymbol{FL}_{e[c]w}$ is $\{\wedge, \vee, *, \Rightarrow, 0, 1\}$.

Substructural logics over *FL*

- A deductive system ***S*** is **non-Fregean** if ∃ a theory *T* of ***S*** for which the *T*-theory interderivability relation $-||-^T$ is not a congruence on the formula algebra.
- **Theorem** (S., Galatos, 2005). An extension of ***FL*** is Fregean iff it is an axiomatic extension of $\boldsymbol{FL}_{ecw}$ iff it is definitionally equivalent to an axiomatic extension of ***IPC***.
- A **substructural logic over *FL*** is a deductive system ***S*** that is definitionally equivalent to a non-Fregean extension of ***FL***.
  - Thus ***IPC*** is *not* a substructural logic over ***FL***.

We are interested in substructural logics over $\boldsymbol{FL}_{ew}$.

A Hilbert-style presentation of FL_{ew}

$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$
$(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r))$
$p \Rightarrow (q \Rightarrow p)$

$p \Rightarrow (q \Rightarrow (p * q))$
$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p * q) \Rightarrow r)$

$(p \wedge q) \Rightarrow p$
$(p \wedge q) \Rightarrow q$
$(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$

$p \Rightarrow (p \vee q)$
$q \Rightarrow (p \vee q)$
$(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r))$

$p \Rightarrow \mathbf{1}$
$\mathbf{0} \Rightarrow p$

$p, p \Rightarrow q \vdash q.$

The algebraic counterpart of $\boldsymbol{FL}_{ew}$

- ***FL***$_{ew}$ is regularly algebraisable in the sense of Blok and Pigozzi.
  - This means ∃ a class of algebras **K** that is to ***FL***$_{ew}$ as **BA** is to ***CPC***.
- The equivalent quasivariety of ***FL***$_{ew}$ is the class **FL**$_{ew}$ of all ***FL***$_{ew}$-algebras.
  - **FL**$_{ew}$ is the algebraic counterpart of ***FL***$_{ew}$ in the same way **BA** is the algebraic counterpart of **CPC**.

FL$_{ew}$-algebras are bounded, commutative, integral residuated lattices.

FL_{ew}-algebras

- A **commutative, integral residuated lattice** is an algebra $\langle A; \wedge, \vee, *, \Rightarrow, 1\rangle$ of type $\langle 2, 2, 2, 2, 0\rangle$ where:
  1. $\langle A; \wedge, \vee\rangle$ is a lattice with lattice ordering $\leq$.
  2. $\langle A; *, 1\rangle$ is a commutative monoid.
  3. $\forall\, a, b, c \in A$, $a * b \leq c$ iff $a \leq b \Rightarrow c$.
  4. $\forall\, a \in A$, $a \leq 1$.
- An **FL_{ew}-algebra** $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a commutative, integral residuated lattice with distinguished least element $0 \in A$.

The logic $\boldsymbol{NFL}_{ew}$

- Let
  - $\sim p$ abbreviate $p \Rightarrow \mathbf{0}$.
  - $p \Rightarrow^2 q$ abbreviate $p \Rightarrow (p \Rightarrow q)$.
  - $p \Rightarrow^3 q$ abbreviate $p \Rightarrow (p \Rightarrow (p \Rightarrow q))$.
- **Nelson $\boldsymbol{FL}_{ew}$ logic**, in symbols $\boldsymbol{NFL}_{ew}$, is the axiomatic extension of $\boldsymbol{FL}_{ew}$ by the axioms:

$\sim\sim p \Rightarrow p$	(Double negation)
$(p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))$	(Distributivity)
$(p \Rightarrow^3 q) \Rightarrow (p \Rightarrow^2 q)$	(3-potency)
$((p \Rightarrow^2 q) \wedge (\sim q \Rightarrow^2 \sim p)) \Rightarrow (p \Rightarrow q)$.	(Nelson)

A Hilbert-style presentation of $\mathbf{\mathit{NFL}}_{\mathbf{ew}}$

$(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$
$(p \Rightarrow (q \Rightarrow r)) \Rightarrow (q \Rightarrow (p \Rightarrow r))$
$p \Rightarrow (q \Rightarrow p)$

$p \Rightarrow (q \Rightarrow (p * q))$
$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p * q) \Rightarrow r)$

$(p \wedge q) \Rightarrow p$
$(p \wedge q) \Rightarrow q$
$(p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow (q \wedge r)))$

$p \Rightarrow (p \vee q)$
$q \Rightarrow (p \vee q)$
$(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow ((p \vee q) \Rightarrow r))$

$p \Rightarrow \mathbf{1}$
$\mathbf{0} \Rightarrow p$

$p, p \Rightarrow q \mid\!- q$

$\sim\sim p \Rightarrow p$
$(p \wedge (q \vee r)) \Rightarrow ((p \wedge q) \vee (p \wedge r))$
$(p \Rightarrow^3 q) \Rightarrow (p \Rightarrow^2 q)$
$((p \Rightarrow^2 q) \wedge (\sim q \Rightarrow^2 \sim p)) \Rightarrow (p \Rightarrow q).$

Nelson FL_{ew}-algebras

- Let
  - $\sim x$ abbreviate $x \Rightarrow \mathbf{0}$.
  - $x \Rightarrow^2 y$ abbreviate $x \Rightarrow (x \Rightarrow y)$.
  - $x \Rightarrow^3 y$ abbreviate $x \Rightarrow (x \Rightarrow (x \Rightarrow y))$.
- An $\boldsymbol{FL}_{ew}$-algebra **A** is
  - **distributive** if $\langle A; \wedge, \vee \rangle$ is distributive.
  - **classical** if $\mathbf{A} \models \sim\sim x \approx x$.
  - **3-potent** if $\mathbf{A} \models x \Rightarrow^3 y \approx x \Rightarrow^2 y$.
- A **Nelson $\boldsymbol{FL}_{ew}$-algebra** is a 3-potent, classical, distributive $\boldsymbol{FL}_{ew}$-algebra that satisfies the **Nelson identity**:

$$(x \Rightarrow^2 y) \wedge (\sim y \Rightarrow^2 \sim x) \approx x \Rightarrow y.$$

A question of David Nelson

- **Question** (Nelson, 1969). Is the variety of Nelson algebras a class of residuated lattices?
- **Answer** (S., V., 2006). Yes!

An answer to Nelson's question

- **Theorem** (S., V., 2006).

  (1) Let **A** be a Nelson algebra. $\forall\, a, b \in A$, let

  $$a * b := {\sim}(a \to {\sim}b) \vee {\sim}(b \to {\sim}a)$$

  $$a \Rightarrow b := (a \to b) \wedge ({\sim}b \to {\sim}a).$$

  – Then $\mathbf{A}^F := \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a Nelson $\boldsymbol{FL}_{ew}$-algebra.

  (2) Let **B** be a Nelson $\boldsymbol{FL}_{ew}$-algebra. $\forall\, a, b \in B$, let

  $$a \to b := a \Rightarrow (a \Rightarrow b)$$

  $$\neg a := a \Rightarrow (a \Rightarrow 0)$$

  $${\sim}a := a \Rightarrow 0.$$

  – Then $\mathbf{B}^N := \langle B; \wedge, \vee, \to, \sim, \neg, 0, 1\rangle$ is a Nelson algebra.

  (3) $\mathbf{A}^{FN} = \mathbf{A}$ and $\mathbf{B}^{NF} = \mathbf{B}$.

  Hence Nelson and Nelson $\boldsymbol{FL}_{ew}$- algebras are term equivalent.

CLSN and NFL_{ew} are definitionally equivalent

- **Theorem** (S., V., 2006). Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be two regularly algebraisable deductive systems over language types Λ_1 and Λ_2. Let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the equivalent quasivarieties of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ respectively. If $\mathbf{K}_1$ and $\mathbf{K}_2$ are term equivalent with interpretations $\alpha: \Lambda_1 \to \mathrm{Fm}_{\Lambda 2}$ and $\beta: \Lambda_2 \to \mathrm{Fm}_{\Lambda 1}$, then $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are definitionally equivalent with the same mutually inverse interpretations.
- **Theorem** (S., V., 2006). The deductive systems ***CLSN*** and $\boldsymbol{NFL}_{ew}$ are definitionally equivalent.

An example: $L_3 \equiv N_3$

- 3-valued ***CLSN*** is determined by the matrix $\langle \mathbf{N}_3; \{1\}\rangle$, where $\mathbf{N}_3$ is

$\wedge$	0	*a*	1
0	0	0	0
a	0	*a*	*a*
1	0	*a*	1

$\vee$	0	*a*	1
0	0	a	1
a	*a*	*a*	1
1	1	1	1

$\rightarrow$	0	*a*	1
0	1	1	1
a	1	1	1
1	0	*a*	1

$\neg$	
0	1
a	0
1	0

$\sim$	
0	1
a	*a*
1	0

- ***L***$_3$ is determined by the matrix $\langle$**Ł**$_3$; $\{1\}\rangle$, where **Ł**$_3$ is

$\wedge$	0	*a*	1
0	0	0	0
a	0	*a*	*a*
1	0	*a*	1

$\vee$	0	*a*	1
0	0	a	1
a	*a*	*a*	1
1	1	1	1

$\Rightarrow$	0	*a*	1
0	1	1	1
a	*a*	1	1
1	0	*a*	1

$*$	0	*a*	1
0	1	1	1
a	*a*	1	1
1	0	*a*	1

$\sim$	
0	1
a	*a*
1	0

- **Theorem** (Vakarelov, 1977). $\langle \mathbf{N}_3; \{1\}\rangle$ and $\langle$**Ł**$_3$; $\{1\}\rangle$ are isomorphic.

Vakarelov's theorem is immediate by the term equivalence result.

Some insight into the proof (I)

- Let $\mathbf{A} := \langle A; \wedge, \vee, \rightarrow, \sim, \neg, 0, 1\rangle$ be a Nelson algebra.
- $\forall\, a, b \in A$, define:

$$a \Rightarrow b := (a \rightarrow b) \wedge (\sim b \rightarrow \sim a).$$

- **Lemma** (Monteiro, 1963). $\mathbf{A} \models x \rightarrow y \approx x \Rightarrow (x \Rightarrow y)$.
- Monteiro's lemma suggests $\langle A; \Rightarrow, 1\rangle$ is a 3-potent BCK-algebra, and this is indeed the case.
- The monoid operation can thus be recovered on setting

$$a * b := \sim(a \Rightarrow \sim b) = \sim(a \rightarrow \sim b) \vee \sim(b \rightarrow \sim a).$$

- Now it is easy to check that $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a Nelson $\boldsymbol{FL}_{ew}$-algebra.

Some insight into the proof (II)

- Let $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be an n-potent $\boldsymbol{FL}_{ew}$-algebra.
- $\forall\, a, b \in A$, define:

$$a \to b := a \Rightarrow^n b$$
$$\sim a := a \Rightarrow \mathbf{0}$$
$$\neg a := a \to \mathbf{0}.$$

- Then $\langle A; \wedge, \vee, \to, \sim, \neg, 0, 1\rangle$ is a "generalised" Nelson algebra.
  - This reflects the fact that any variety of n-potent $\boldsymbol{FL}_{ew}$-algebras is a **WBSO variety** in the sense of Blok and Pigozzi.
- The Nelson identity $(x \Rightarrow^2 y) \wedge (\sim y \Rightarrow^2 \sim x) \approx x \Rightarrow y$ ensures that $\langle A; \wedge, \vee, \to, \sim, \neg, 0, 1\rangle$ is a Nelson algebra.

… and some prospects for future work

- Apply the now well developed theories of
  - algebraisable logics (in the Blok-Pigozzi sense)
  - residuated lattices and $\boldsymbol{FL}_{ew}$-algebras

  to answer further questions about ***CLSN***.

- Extend the counterexample semantics of Vakarelov to varieties of n-potent $\boldsymbol{FL}_{ew}$-algebras.

- Explore varieties of n-potent $\boldsymbol{FL}_{ew}$-algebras satisfying the following n-potent analogue of the Nelson identity:

$$(x \Rightarrow^n y) \wedge ({\sim}y \Rightarrow^n {\sim}x) \approx x \Rightarrow y.$$

Ternary deductive terms

- $p(x, y, z)$ is a **ternary deductive** (TD) **term** on an algebra **A** if
  - $p(a, b, z) \equiv z \pmod{\Theta^{\mathbf{A}}(a, b)}$
  - $\{p(a, b, z): z \in A\}$ is a transversal of equivalence classes.
- $p(x, y, z)$ is **commutative** if $p(a, b, z)$ and $p(a', b', z)$ define the same transversal whenever $\Theta^{\mathbf{A}}(a, b) = \Theta^{\mathbf{A}}(a', b')$.
- $p(x, y, z)$ is **regular** if $\Theta^{\mathbf{A}}(p(x, y, z), \mathbf{1}^{\mathbf{A}}) = \Theta^{\mathbf{A}}(x, y)$ for some constant term **1**.

These definitions extend in the obvious way to varieties.

A question about TD terms

- **Question** (Blok, Pigozzi, 1994). Does the variety of Nelson algebras have a commutative, regular TD term, or even a TD term?
- **Answer** (S., 2004). Yes!
  Nelson algebras have a commutative TD term.
- **Answer** (S., V., 2006). Yes!
  Nelson algebras have a commutative, regular TD term.

TD terms for Nelson algebras

- **Theorem** (S., V., 2004-2006).

  (1) A commutative TD term for Nelson algebras is

  $$p(x, y, z) := (x \Rightarrow y) \to ((y \Rightarrow x) \to z).$$

  (2) A commutative, regular TD term with respect to **1** for Nelson algebras is

  $$p(x, y, z) := ((x \Rightarrow y) \wedge (y \Rightarrow x)) * ((x \Rightarrow y) \wedge (y \Rightarrow x)) * z.$$

(1) and (2) both follow immediately on observing that *n*-potent $\boldsymbol{FL}_{ew}$-algebras have both a commutative TD term and a commutative, regular TD term.