Minimal subvarieties of involutive residuated lattices

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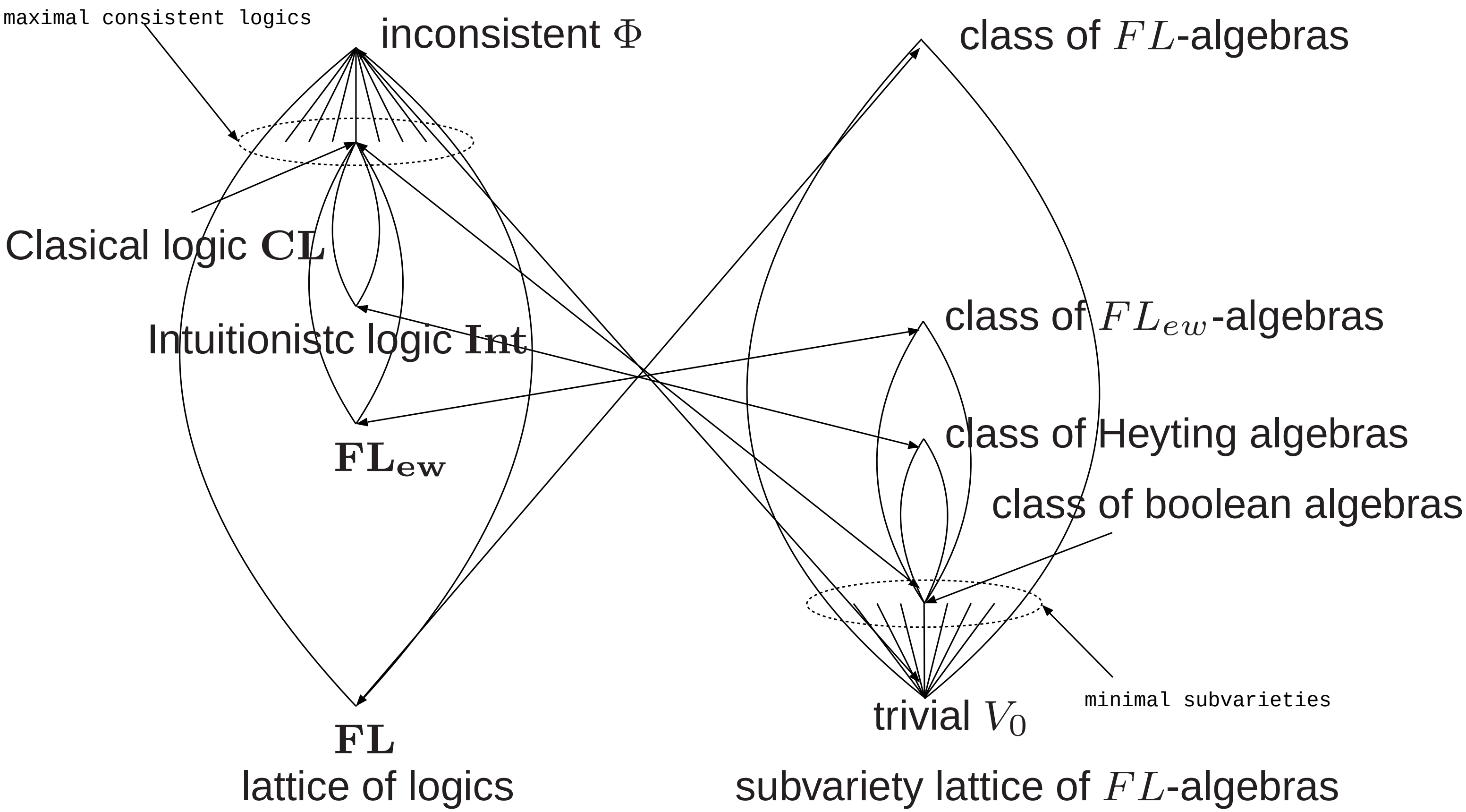
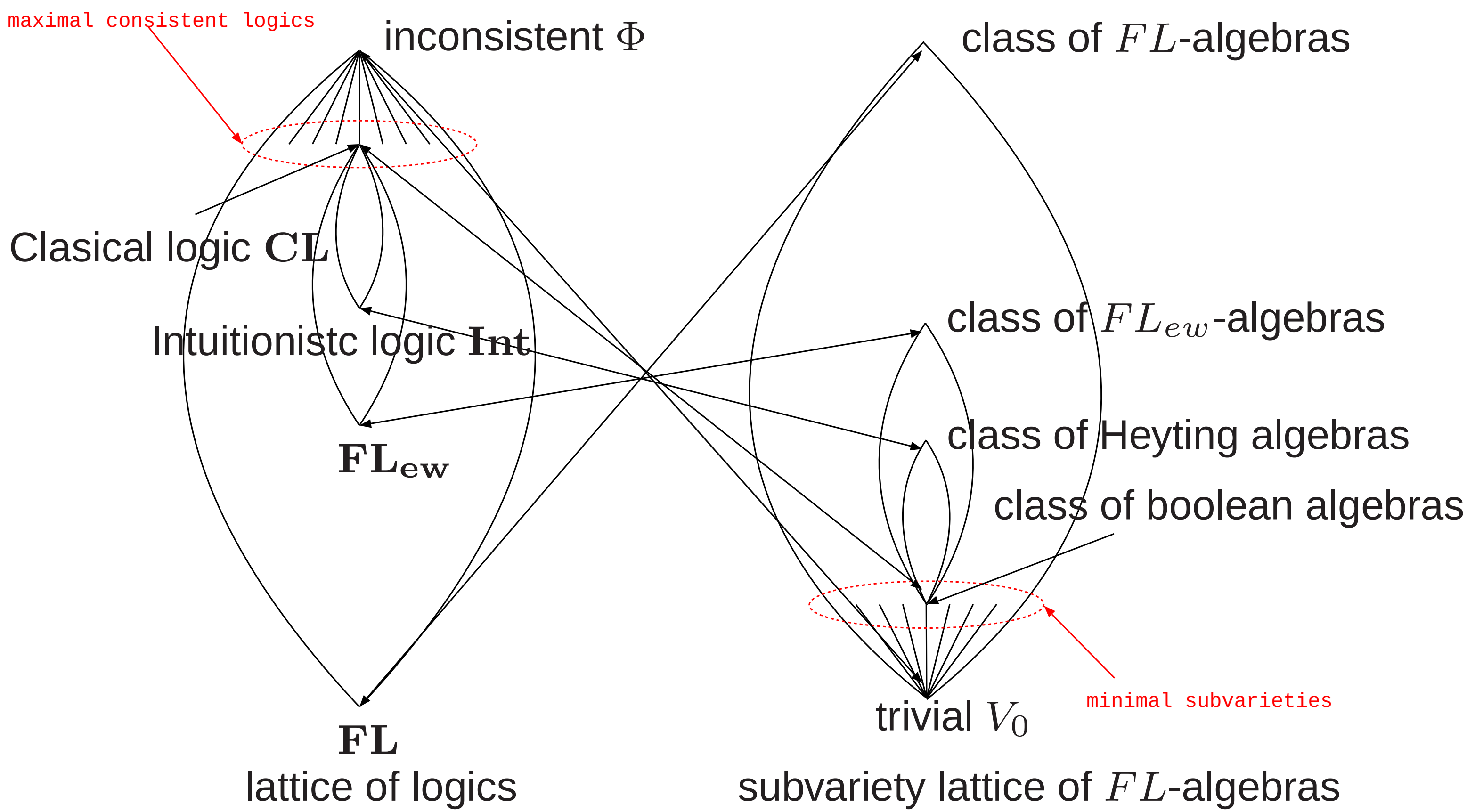
Introduction

lattice of logics $\overset{dually\ isomorphic}{\longleftrightarrow}$ subvariety lattice of algebras



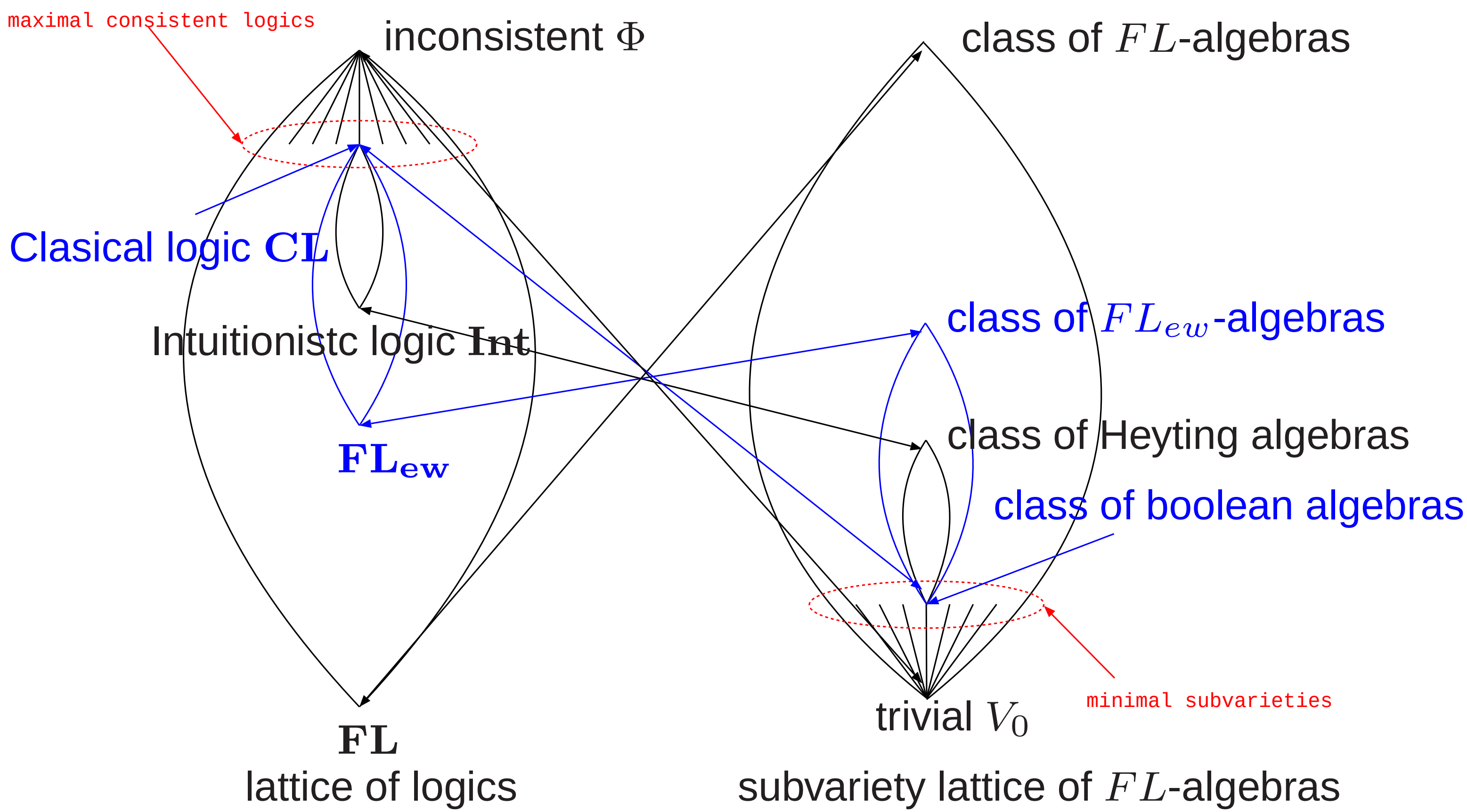
Introduction

maximal consistent logics $\leftrightarrow$ minimal subvarieties



Introduction

$\mathbf{CL}$ is the only one maximal consistent logic over $\mathbf{FL_{ew}}$



Minimal subvarieties of InRL

variety	minimal subvarieties
$\mathcal{I}n\mathcal{RL}$	uncountably many (Tsinakis-Wille)
$\mathcal{I}n\mathcal{RRL}_\perp + (x^2 \leq x)$	?
$\mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$	?

$\mathcal{I}n\mathcal{RL}$: the class of all involutive residuated lattices
$\mathcal{I}n\mathcal{RRL}_\perp$: the class of all bounded representable involutive residuated lattices

Residuated lattices

An algebra $\mathbf{A} = \langle \mathrm{A}, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$ is a residuated lattice (RL) if it satisfies the following conditions.

(R1) $\langle \mathrm{A}, \wedge, \vee, 1 \rangle$ is a lattice,

(R2) $\langle \mathrm{A}, \cdot, 1 \rangle$ is a monoid with the unit 1,

(R3) for $x, y, z \in \mathrm{A}$, $x \cdot y \leq z \Leftrightarrow y \leq x \backslash z \Leftrightarrow x \leq z / y$.

$\mathcal{RL}$ is the variety of all residuated lattices.

Involutive and Representable RL

- A RL $\mathbf{A}$ is bounded if it has the greatest element $\top$ and least element $\bot$. ($\mathcal{RL}_\bot$)
- A RL $\mathbf{A}$ is representable if it can be represented as subdirect products of totally ordered algebras. ($\mathcal{RRL}$)
- A RL $\mathbf{A}$ is involutive (InRL) if it has a fundamental unary operation $'$ called involution which satisfies the following conditions.($\mathcal{I}n\mathcal{RL}$)
  1. $x'' = x$,
  2. $x \backslash y' = x' / y$.

Strictly simple RL

A non-trivial RL $\mathbf{A}$ is strictly simple, if it has neither non-trivial proper subalgebras nor non-trivial congruences.

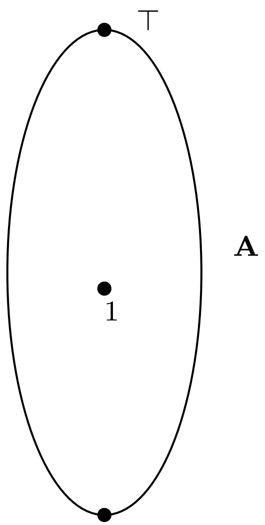
The bottom element $\bot \in \mathbf{A}$ is nearly term-definable, if there is an n-ary term-operation $t(\bar{x})$ such that for any n-tuple $\bar{a} \neq (\underbrace{1, \ldots, 1}_{n\text{-times}})$ of elements of A $t(\bar{a}) = \bot$ holds.

A condition for a minimal subvariety

Lemmma 1 *Let* $\mathbf{A}$ *be a strictly simple* RL *with the nearly term definable bottom element* $\bot$*. Then, the variety* $V(\mathbf{A})$ *generated by* $\mathbf{A}$ *is minimal.*

Thus, it suffices to find such a RL $\mathbf{A}$.

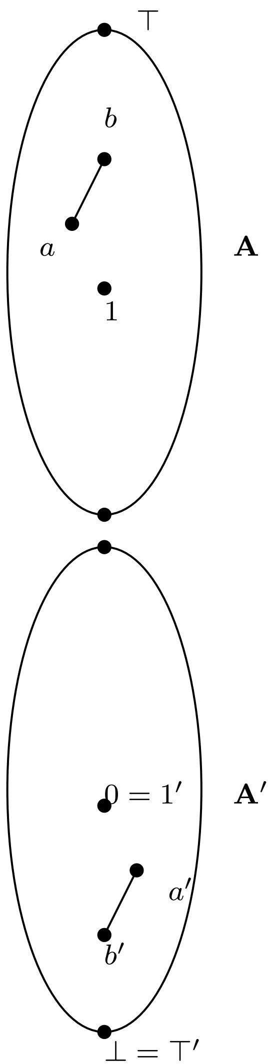
A construction of InRL



This construction is given by N. Galatos and J. G. Raftery

- An upper-bounded RL $\mathbf{A}$ is given.

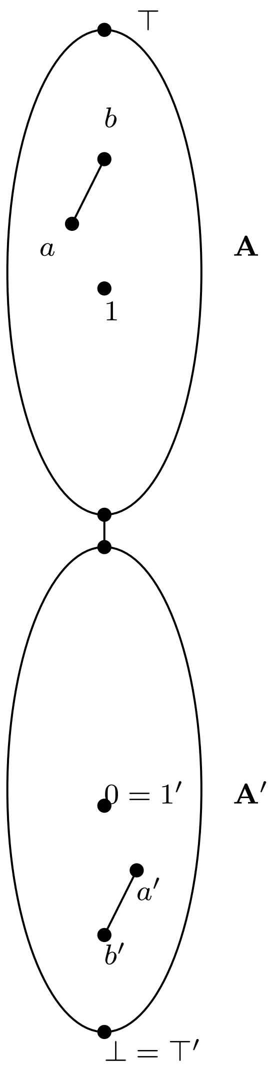
A construction of InRL



This construction is given by N. Galatos and J. G. Raftery

- $\mathbf{A}$ is an upper-bounded RL
- $\mathrm{A}' = \{a' | a \in \mathrm{A}\}$ is a disjoint copy of A, with the reverse order.

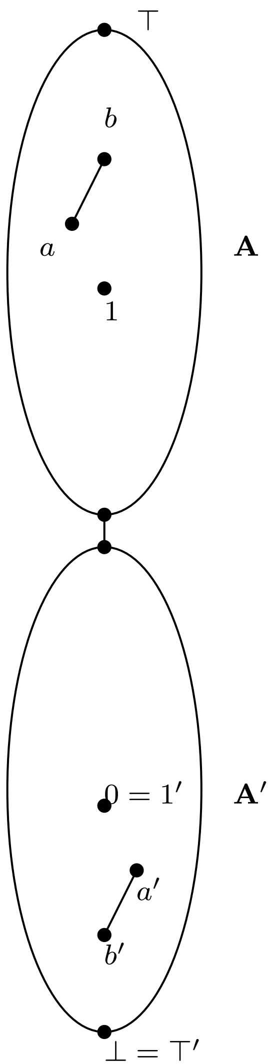
A construction of InRL



This construction is given by N. Galatos and J. G. Raftery

- $\mathbf{A}$ is an upper-bounded RL
- $\mathrm{A}' = \{a' | a \in \mathrm{A}\}$ is a copy of A.
- Take the union A^* of A and A'.
  - $a' < b$ and
  - $a' \leq b' \leftrightarrow b \leq a$.

A construction of InRL



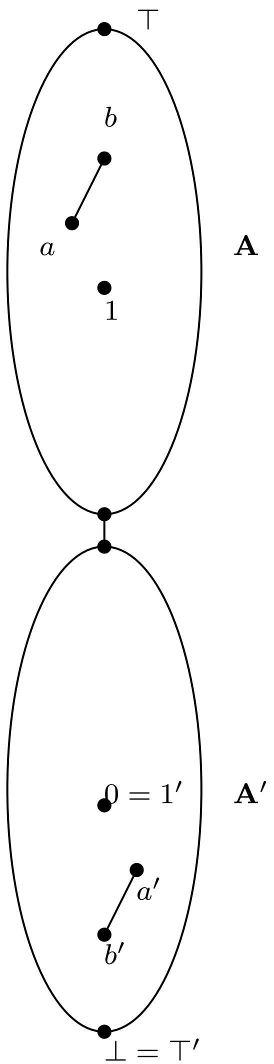
This construction is given by N. Galatos and J. G. Raftery

- $\mathbf{A}$ is an upper-bounded RL
- $\mathrm{A}' = \{a' | a \in \mathrm{A}\}$ is a copy of A.
- A^* is $\mathrm{A} \cup \mathrm{A}'$.
- Extend the monoid operation of $\mathbf{A}$ to $\mathbf{A}^*$.

· $a \cdot b' = (b/a)'$, $b' \cdot a = (a \backslash b)'$ and

· $a' \cdot b' = \bot$.

A construction of InRL

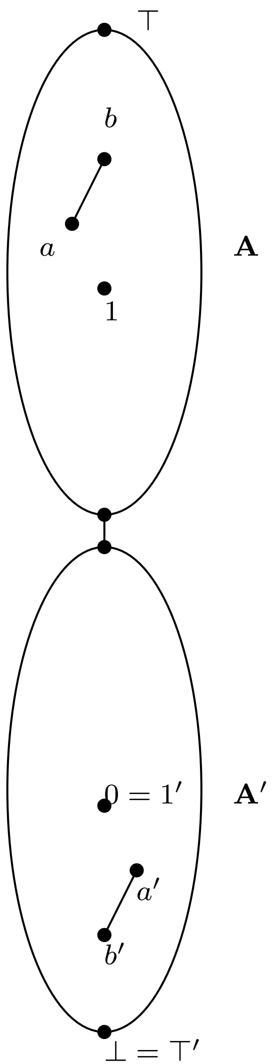


This construction is given by N. Galatos and J. G. Raftery

- $\mathbf{A}$ is an upper-bounded RL
- $\mathrm{A}' = \{a' | a \in \mathrm{A}\}$ is a copy of A.
- A^* is $\mathrm{A} \cup \mathrm{A}'$.
- Extend the monoid operation.
- Extend the division operation of $\mathbf{A}$ to $\mathbf{A}^*$.

  - $a \backslash b' = a'/b = (b \cdot a)'$,
  - $b' \backslash a = a/b' = \top$,
  - $a' \backslash b' = a/b$,
  - $b'/a' = b \backslash a$.

A construction of InRL

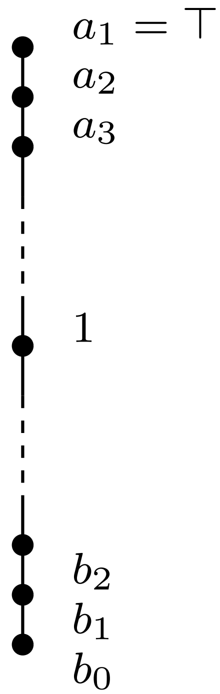


Facts

- The constructed algebra $\mathbf{A}^*$ is a bounded InRL.
- If $\mathbf{A}$ is totally orderd then so is $\mathbf{A}^*$.
- If $\mathbf{A}$ satisfies the mingle axiom $x^2 \leq x$ then so does $\mathbf{A}^*$

RL $\mathbf{D_S}$

Let D be the following bounded lattice



For each $S \subseteq \omega$, we define the monoid and division operations on D as follows.

Monoid operation of D_S

	a_1	a_2	a_3	$\cdots$	1	$\cdots$	b_2	b_1	b_0
a_1	a_1	a_1	a_1	$\cdots$	a_1	$\cdots$	a_1	y_1	b_0
a_2	a_1	a_2	a_2	$\cdots$	a_2	$\cdots$	y_2	b_1	b_0
a_3	a_1	a_2	a_3	$\cdots$	a_3	$\cdots$	b_2	b_1	b_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
1	a_1	a_2	a_3	$\cdots$	1	$\cdots$	b_2	b_1	b_0
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
b_2	a_1	x_2	b_2	$\cdots$	b_2	$\cdots$	b_2	b_1	b_0
b_1	x_1	b_1	b_1	$\cdots$	b_1	$\cdots$	b_1	b_1	b_0
b_0	b_0	b_0	b_0	$\cdots$	b_0	$\cdots$	b_0	b_0	b_0

$$x_i = \begin{cases} b_i & \text{if } i \in S \\ a_i & \text{if } i \notin S \end{cases}$$

$$y_i = \begin{cases} b_j & \text{if } i \notin S \\ a_i & \text{if } i \in S \end{cases}$$

Note that the operation $\cdot_S$ is almost commutative.

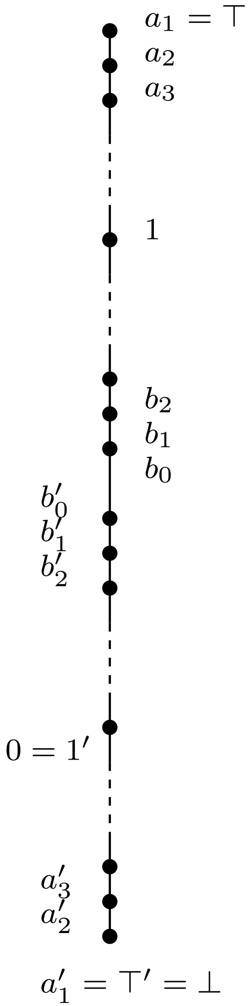
Division operations of $\mathrm{D_S}$

Define two division operations by

$$x \backslash y = \bigvee \{z | x \cdot_{\mathrm{S}} z \leq y\}$$
$$y / x = \bigvee \{z | z \cdot_{\mathrm{S}} x \leq y\}$$

$\mathbf{D_S} = \langle \mathrm{D}, \wedge, \vee, \cdot_{\mathrm{S}}, \backslash, /, 1, \bot, \top \rangle$ is a bounded RL, where a_1 is the top and b_0 is the bottom element. Moreover $x \cdot_{\mathrm{S}} x = x$ holds for any $x \in \mathrm{D}$.

Constructing $\mathrm{D}_{\mathrm{S}}^{*}$

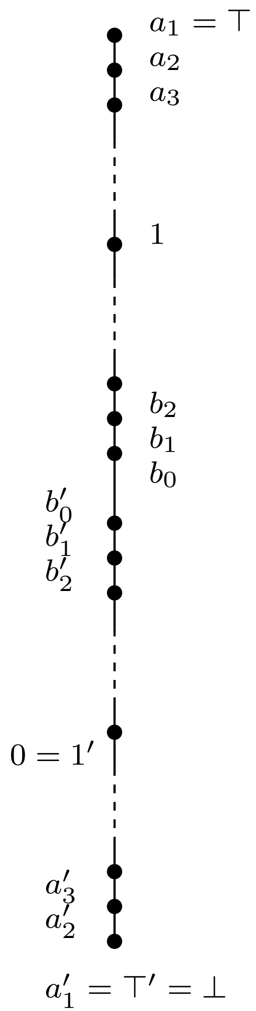


Let $\mathbf{D}_{\mathbf{S}}^{*}$ be the bounded representable $InRL$ obtained from $\mathbf{D}_{\mathbf{S}}$ by the Galatos-Raftery construction.

Then the $\mathbf{D}_{\mathbf{S}}^{*}$ satisfies mingle axiom.

Note that $x \cdot_{\mathrm{S}} x = \bot \leq x$ for $x \in \mathrm{D}'$.

Constructing $\mathbf{D}^*_\mathbf{S}$



Moreover we can show that

- $\mathbf{D}^*_\mathbf{S}$ is strictly simple,
- $\mathbf{D}^*_\mathbf{S}$ has nearly term-definable bottom element.

Lemma 2 *For each $S \subseteq \omega$, $\mathbf{D}^*_\mathbf{S}$ is a minimal subvariety in $\mathcal{I}n\mathcal{RRL}_\bot + (x^2 \leq x)$*

Uncountably many minimal subvarieties

Now we show that for any pair of distinct sets $S_1, S_2 \subseteq \omega$, $\mathbf{D}^*_{\mathbf{S_1}}$ and $\mathbf{D}^*_{\mathbf{S_2}}$ generate distinct varieties.

For any $a_i, b_i \in \mathrm{D}$, we can find constant terms q_{a_i} and q_{b_i} such that

- $f(q_{a_i}) = a_i$
- $f(q_{b_i}) = b_i$

for any assignment f of $\mathbf{D}^*_{\mathbf{S}}$.

Suppose that $\mathrm{S}_1 \neq \mathrm{S}_2$. Without a loss of generality we can assume that $i \in S_1 \backslash S_2$ for some $i \in \omega$.

By the definition $b_i \cdot_1 a_i = b_i$ but $b_i \cdot_2 a_i = a_i$.
Then,

- $\mathbf{D}^*_{\mathbf{S_1}} \models q_{b_i} \cdot q_{a_i} \approx q_{b_i}$.
- $\mathbf{D}^*_{\mathbf{S_2}} \models q_{b_i} \cdot q_{a_i} \approx q_{a_i}$ and $\mathbf{D}^*_{\mathbf{S_2}} \not\models q_{b_i} \cdot q_{a_i} \approx q_{b_i}$.

Hence $V(\mathbf{D}^*_{\mathbf{S_1}}) \neq V(\mathbf{D}^*_{\mathbf{S_2}})$.

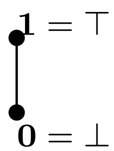
Theorem 3 *There are uncountably many minimal subvarieties of* $\mathcal{I}n\mathcal{RRL}_\perp + (x^2 \leq x)$.

Minimal subvarietites of InRL

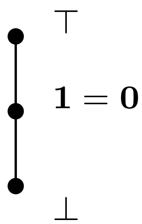
variety	minimal subvarieties
$\mathcal{InRL}$	uncountably many(Tsinakis-Wille)
$\mathcal{InRRL}_\perp + (x^2 \leq x)$	uncountably many
$\mathcal{InRRL}_\perp + (x = x^2)$	?

$InRL$ with idempotent axiom

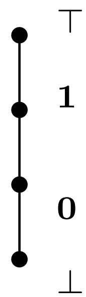
Let $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$ be the following bounded representable involutive residuated lattices with idempotent.



$\mathbf{2}$



$\mathbf{3}$



$\mathbf{4}$

where the monoid operations are defined as follows.

	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

	$\top$	$\mathbf{1}$	$\bot$
$\top$	$\top$	$\top$	$\bot$
$\mathbf{1}$	$\top$	$\mathbf{1}$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$

	$\top$	$\mathbf{1}$	$\mathbf{0}$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\bot$
$\mathbf{1}$	$\top$	$\mathbf{1}$	$\mathbf{0}$	$\bot$
$\mathbf{0}$	$\top$	$\mathbf{0}$	$\mathbf{0}$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Minimal subvarieties with $x = x^2$

Theorem 4 *There exists only two minimal subvarieties of* $\mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$.

Outline of the proof

- Every subdirect irreducible $\mathbf{A} \in \mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$ has a subalgebra which is isomorphic to one of $\mathbf{2}$, $\mathbf{3}$ and $\mathbf{4}$.
- $\mathbf{3}$ is a homomorphic image of $\mathbf{4}$.

Conclusion and future work

We have show that there are

- uncountably many minimal subvarieties in $\mathcal{I}n\mathcal{RRL}_\perp + (x^2 \leq x)$ (mingle)
- but only two in $\mathcal{I}n\mathcal{RRL}_\perp + (x = x^2)$ (idempotent)

How many minimal subvarieties are there in $\mathcal{I}n\mathcal{RRL}_\perp + (x \leq x^2)$ (contraction)?