History of the question

The logics of chequered subsets were first introduced in the paper:

Johan van Benthem, Guram Bezhanishvili, Mai Gehrke
Euclidean Hierarchy in Modal Logic. *Studia Logica*, vol. **75**, pp. 327-344. Springer Netherlands, 2003.

A chequered subset in $\mathbb{R}^n$ is a finite union of hyper-rectangular convexes, i.e. products of convex subsets of $\mathbb{R}$.

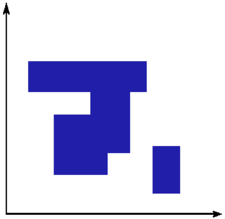


Figure: An example of chequered set in $\mathbb{R}^2$

Chequered subsets form a boolean algebra $CH(\mathbb{R}^n)$, closed under interior and closure operators of $\mathbb{R}^n$. The modal logic of such algebra is defined as follows:

$$L_n = \{\phi | \forall \nu : PV \to CH(\mathbb{R}^n) \quad \mathbb{R}^n, \nu \models \phi\}$$

Another modal logic, $L_\infty = \bigcap L_n$ corresponds to chequered sets in $\mathbb{R}^\infty$. All of the mentioned logics are normal extensions of $\mathbf{S4}$ and $\mathbf{Grz}$, and can be described in Kripke semantics as follows:

- $L_n = L(V^n)$, where $V^n = V^{n-1} \times V$ under standard product order and V is a "double fork" frame.
- $L_\infty = L(V^*)$, where $V^* = \bigsqcup V^n$

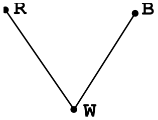


Figure: Frame V

Following the paper:
Tadeusz Litak
Some notes on superintuitionistic logic of chequered subsets of $\mathbb{R}^\infty$.
Bulletin of the Section of Logic, vol. **33**, pp. 81-86. University of Lodz, 2004.
we consider superintuitionistic analogs of those logics, which by Blok-Esakia isomorphism are determined by the same Kripke frames:

- $IL_n = \rho L_n = IL(V^n)$
- $\mathbf{Cheq} = \rho L_\infty = IL(V^*) = \bigcap IL_n$

These intermediate logics can also be described in topological semantics if we restrict valuations to open chequered subsets.
It can be shown that Medvedev's logic of finite frames $\mathbf{ML}$ is an extension of $\mathbf{Cheq}$. $\mathbf{ML}$ was proven to be not axiomatizable in finite number of variables in 1979 by Maksimova, Skvortsov and Shehtman. We use similar method to prove the same for $\mathbf{Cheq}$.

The results

The following results were obtained:

Theorem

For any natural number k, $\mathbf{Cheq}$ is not axiomatizable in k variables.

Corollary

$\mathbf{Cheq}$ is not finitely axiomatizable.

Corollary

L_∞ is not finitely axiomatizable.

Outline of the proof

For the proof we construct two families of finite Kripke frames $\Psi(m,n)$ and $\Psi^i(m,n)$, for which the following holds true:

1. There doesn't exist a p-morphism $V^* \twoheadrightarrow \Psi(2^{n+2}, n)$.
2. There exists a p-morphism $V^* \twoheadrightarrow \Psi^i(m,n)$ for any valid values of i, m, n.
3. If a formula A contains only k propositional variables then if $\Psi(m, k+2) \not\models A$ then $\Psi^i(m, k+2) \not\models A$ for some $1 \le i < k$.

Using Yankov's characteristic formulas $X(F)$ these statements can be rephrased as follows:

1. $X(\Psi(2^{n+2}, n)) \in \mathbf{Cheq}$
2. $X(\Psi^i(m,n)) \notin \mathbf{Cheq}$
3. If A contains only k propositional variables then if $X(\Psi(m, k+2)) \in (H + A)$ then $X(\Psi^i(m, k+2)) \in (H + A)$ for some i.

If $\mathbf{Cheq}$ is axiomatizable in k variables then $X(\Psi(2^{k+4}, k+2))$ is in some $(H + A_1 \wedge \ldots \wedge A_n) \subset \mathbf{Cheq}$, where A_i are axioms of $\mathbf{Cheq}$, which contradicts pt 2 and 3.

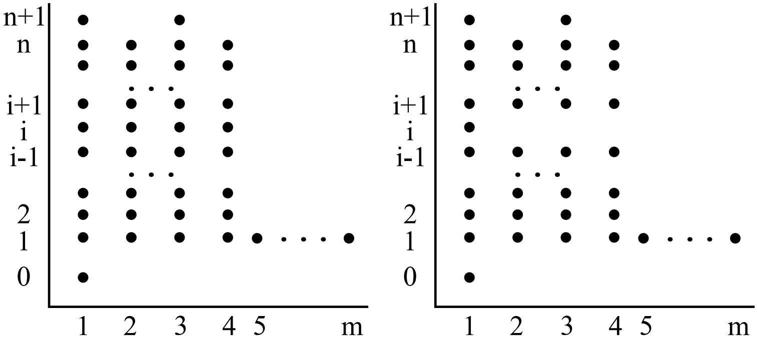


Figure: Frames $\Psi(m,n)$ and $\Psi^i(m,n)$

There doesn't exist a p-morphism $V^* \twoheadrightarrow \Psi(2^{n+2}, n)$.

In the next lemma $d(u)$ denotes the length of the longest chain of increasing elements starting from u. $br(u)$ is the number of immediate successors of u.

Lemma

If the frame F is such that $\forall u \in F$

- *If* $d(u) = 1$ *then* $br(u) \geq 2$.
- *If* $d(u) > 1$ *then* $br(u) \geq 4$.

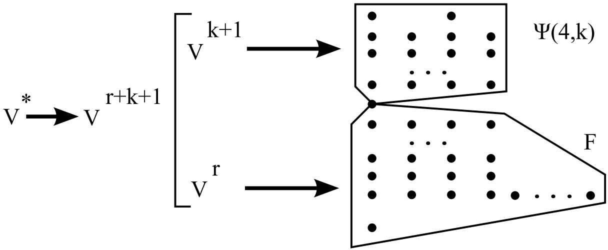
and there exists a p-morphism $V^* \to F$ *then*

$$\forall u \in F \quad br(u) \leq 2^{d(u)}$$

If this lemma is applied to the frame $\Psi(m, n)$ and u is its least element then $d(u) = n + 1$, $br(u) = m$ and there can't be a p-morphism $V^* \twoheadrightarrow \Psi(m, n)$ if $m > 2^{n+1}$.

There exists a p-morphism $V^* \twoheadrightarrow \Psi^i(m,n)$

The p-morphism is constructed as shown in the following figure:



The steps of the proof:

- There exists p-morphism $V^2 \twoheadrightarrow \Psi(4,1)$
- If $V^k \twoheadrightarrow \Psi(m,n)$ then $V^{k+1} \twoheadrightarrow \Psi(m,n+1)$
- If frame F has the greatest element then $\exists r \quad V^r \twoheadrightarrow F$.

Further questions

This is a list of some unresolved questions, which are related to the obtained result.

1. Can the same technique be used to prove non-finite axiomatizability of logics of Kripke frames $F* = \bigsqcup F^n$, where F is some other finite frame? Are such logics interesting from the geometric point of view?
2. Is $\mathbf{Cheq}$ decidable? This is a much harder question and perhaps related to the long-standing problem on whether $\mathbf{ML}$ is decidable.