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A NEW LOOK AT TEMPORAL LOGIC:

ADDING (welcome) ACCEPTANCE TO OBTAIN CONSTRUCTIVE COMPLETENESS

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based on joint work with
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cfr.
Constructive satisfiability,
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Key idea of duality theory
and of standard completeness proofs:

represent $\Box$ as r^*

for some relation r

Here $r^* D \equiv \{x : rx \subseteq D\}$

$rx \equiv \{y : xry\}$

It extends classically to temporal logic:

$\Box$ corresponds to r

$\blacksquare$ corresponds to s

We must have something in the language telling that $s = r^-$

$\vdash \Box P \vee Q$ iff $\vdash P \vee \blacksquare Q$

which, assuming $\Box = \neg\Diamond\neg$ $\blacksquare = \neg\blacklozenge\neg$,
is equivalent to:

$\Diamond P \,\&\, Q \vdash$ iff $P \,\&\, \blacklozenge Q \vdash$

def. formal topology

S set observables

$a \lhd U$ prop $(a \in S,\ U \subseteq S)$ formal cover

$$\frac{a \in U}{a \lhd U} \quad \text{reflexivity}$$

$$\frac{a \lhd U \quad U \lhd V}{a \lhd V} \quad \text{transitivity}$$

$U \lhd V \equiv (\forall b \in U)\, b \lhd V$

$$\frac{a \lhd U \quad a \lhd V}{a \lhd U \downarrow V} \quad \text{convergence (distributivity)}$$

$U \downarrow V \equiv \{c :\ c \lhd a$ for some $a \in U$
$c \lhd b$ " " $b \in V\}$

U, V give the same formal open if $U \lhd V$ & $V \lhd U$
or $a \lhd U \leftrightarrow a \lhd V$

$\mathcal{A}U \equiv \{a :\ a \lhd U\}$

$\lhd$ is a formal cover iff $\mathcal{A}$ closure operator
$+\ \mathcal{A}(U \downarrow V) = \mathcal{A}U \cap \mathcal{A}V$

$Sat(\mathcal{A})$ is a locale

also a cHa with:

$U \to_{\mathcal{A}} V \equiv \{a :\ a \downarrow U \lhd V\}$

Completeness proof (sketch)

see GS, JSL '95

Frm = set of formulae

$\varphi, \psi \in$ Frm

$V : \mathrm{Frm} \longrightarrow \mathrm{Sat}(\mathcal{A})$

$$V(\varphi \,\&\, \psi) \equiv V\varphi \cap V\psi$$

$$V(\forall x\, \varphi x) \equiv \bigwedge_{d \in D} V(\varphi(d))$$

$$V(\varphi \vee \psi) \equiv V\varphi \vee V\varphi \equiv \mathcal{A}(V\varphi \cup V\psi)$$

$$V(\exists x\, \varphi x) \equiv \bigvee_{d \in D} V(\varphi(d))$$

$$V(\varphi \to \psi) \equiv V\varphi \to_{\mathcal{A}} V\psi$$

Canonical topology on Frm

$\lhd$ inductively generated by:

$$\frac{\varphi \in \Sigma}{\varphi \lhd \Sigma} \qquad \frac{\varphi \lhd \Sigma \quad \psi \lhd \Sigma}{\varphi \vee \psi \lhd \Sigma} \qquad \frac{\{\varphi t : t \in \mathrm{Trm}\} \lhd \Sigma}{\exists x\, \varphi x \lhd \Sigma}$$

$$\bot \lhd \Sigma \qquad \frac{\varphi \vdash \psi \quad \psi \lhd \Sigma}{\varphi \lhd \Sigma}$$

alternatively:

$$\varphi \lhd \Sigma \equiv \forall \psi \left(\Sigma \subseteq {\downarrow}\psi \to \varphi \in {\downarrow}\psi \right)$$

Dedekind-McNeille completion

lemma on canonical valuation

put $\quad V(P) \equiv \downarrow P \equiv \{\varphi : \varphi \vdash P\}$

for atomic P

then $\quad \lhd V\varphi = \downarrow\varphi \quad$ for every φ

$\psi \lhd V\varphi$ iff $\psi \vdash \varphi$

Completeness

$\Gamma \models \varphi \quad \Rightarrow \quad V(\Gamma) \lhd V(\varphi)$

in the canonical topology

$\gamma_1 \& \cdots \& \gamma_n \lhd V(\gamma_1 \& \cdots \& \gamma_n) \lhd V\varphi$

$\gamma_1 \& \cdots \& \gamma_n \vdash \varphi \quad$ by lemma

$\Gamma \vdash \varphi$

$$X \xrightarrow[\Vdash]{r} S \qquad X, S \text{ sets}, \quad r \text{ relation}$$

$\overset{\Vdash}{r}$ induces four operators:

$$\mathrm{ext}\, a \equiv \{x : x \Vdash a\}$$
$$\Diamond x \equiv \{a : x \Vdash a\}$$

$$\mathcal{P}X \underset{r^-,\ r^*}{\overset{r,\ r^{-*}}{\rightleftarrows}} \mathcal{P}S$$

$$x \in \underset{\mathrm{ext}}{r^-} U \equiv \underset{\Diamond x}{rx} \between U$$

$$a \in \underset{\Diamond}{r} D \equiv \underset{\mathrm{ext}\, a}{r^- a} \between D$$

$$x \in \underset{\mathrm{rest}}{r^*} U \equiv \underset{\Diamond x}{rx} \subseteq U$$

$$a \in \underset{\Box}{r^{-*}} D \equiv \underset{\mathrm{ext}\, a}{r^- a} \subseteq D$$

Then:

$$\overset{r^- r^{-*}}{\mathrm{ext}\,\Box} = \mathrm{int} \qquad \overset{r\, r^*}{\Diamond\, \mathrm{rest}} = \mathcal{J}$$

interior / reduction / co-monad

$$\overset{r^* r}{\mathrm{rest}\,\Diamond} = \mathrm{cl} \qquad \overset{r^{-*} r^-}{\Box\, \mathrm{ext}} = \mathcal{A}$$

closure / saturation / monad

follows from adjunctions

$$r \dashv r^* \qquad r^- \dashv r^{-*}$$

Def. basic topology

S set

$\lhd$ basic cover — corresponds to $\mathcal{A}$

only reflexive and transitive

$\ltimes$ positivity relation

corresponds to $\mathcal{J}$ interior operator
reduction

compatibility
$$\frac{a \lhd U \qquad a \ltimes V}{U \ltimes V} \equiv (\exists b \in U)\, b \ltimes V$$

alternatively

$\mathcal{A}$ saturation, $\mathcal{J}$ reduction

$+ \quad \mathcal{A}U \between \mathcal{J}V \to U \between \mathcal{J}V$

this corresponds to: $\mathcal{A}, \mathcal{J}$ come from the same r

formal topology: add convergence

def. formal point $\qquad$ trace of a point x on S

$\alpha \subseteq S$

α inhabited $\qquad \exists a (a \in \alpha) \qquad \alpha \between \alpha$

convergent $\qquad U \between \alpha \ \& \ V \between \alpha \to U \downarrow V \between \alpha$

splits $\lhd$

$$\frac{\alpha \Vdash a \qquad a \lhd U}{\alpha \between U}$$

enters $\ltimes$

$$\frac{\alpha \Vdash a \qquad \alpha \subseteq U}{\alpha}$$

α formal point of the canonical top. on Frm

$= \alpha$ Henkin set

$= \alpha$ model

def. overlap algebra

$\mathcal{P}$ collection e.g. $\mathcal{P}X$

$(\mathcal{P}, \leq, \wedge, \vee, \to, 0)$ cHa

$+ \quad p \between q$ p overlaps q $\quad D \between E$

$p \between q \to q \between p$
$p \between q \to p \between p \wedge q$
$p \between \bigvee_{i \in I} q_i \leftrightarrow (\exists i \in I)(p \between q_i)$

$\mathcal{P}X \xrightarrow{r} \mathcal{P}S$
$\mathcal{P} \underset{F', G}{\xrightarrow{F, G'}} Q$

Prop.: there is $X \xrightarrow{r} S$ s.t.
$F = r \quad F' = r^{-} \quad G = r^{*} \quad G' = r^{-*}$
iff

symmetric pair of adjunction $\begin{cases} F \dashv G \quad F' \dashv G' \quad F \cdot | \cdot F' \\ \qquad F(D) \between U \leftrightarrow D \between F'(U) \end{cases}$

LJ usual sequent calculus

+ welcome $\Gamma \veebar \Delta$ classically $\Gamma, \Delta \nvdash \bot$

co-inductive rules

exch. weak. contr. for $\veebar$

transfer $$\frac{\Gamma, \varphi \veebar \Delta}{\Gamma \veebar \varphi, \Delta}$$

$\vee$-rule $$\frac{\varphi \vee \psi \veebar \Delta}{\varphi \veebar \Delta \qquad \psi \veebar \Delta}$$

$$\frac{\exists x\, \varphi x \veebar \Delta}{\varphi t \veebar \Delta \quad \text{some } t \in \mathrm{Trm}}$$

$$\frac{\bot \veebar \Gamma}{}$$ $\bot$ cannot welcome any Γ

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \veebar \Delta}{\varphi \veebar \Delta}$$

by the rules can show Γ does <u>not</u> welcome Δ

$$\dfrac{\dfrac{\varphi \veebar \neg\varphi}{\veebar \varphi, \neg\varphi} \qquad \varphi, \neg\varphi \vdash \bot}{\dfrac{\veebar \bot}{}}$$

in meta-language, can show $\Gamma \veebar \Delta$

Minimal constructive temporal logic

$LJ + \veebar$

$\Diamond \varphi \vdash \psi$ iff $\varphi \vdash \blacksquare \psi$

$\blacklozenge \varphi \vdash \psi$ iff $\varphi \vdash \Box \psi$

$\Diamond \varphi \veebar \psi$ iff $\varphi \veebar \blacklozenge \psi$

Models: o-Kripke frames

$\mathcal{P}$ overlap algebra

+ a relation on $\mathcal{P}$ i.e.

$F \dashv G \quad F' \dashv G' \quad F \cdot | \cdot F'$

Every Kripke frame (X, r)

gives an o-Kripke frame $(\mathcal{P}X, r, r^-, r^*, r^{-*})$

$V(\Diamond \varphi) \cong r^- V(\varphi) \qquad V(\blacklozenge \varphi) \cong r V(\varphi)$

$V(\Box \varphi) \cong r^* V(\varphi) \qquad V(\blacksquare \varphi) \cong r^{-*} V(\varphi)$

φ valid in $\wp$ $\equiv$ $V(\varphi) = 1$

$\Gamma \vdash \varphi$ " " $\equiv$ $V(\Gamma) \lhd V(\varphi)$

$\Gamma \nvdash \Delta$ " " $\equiv$ $V(\Gamma) \not\bowtie V(\Delta)$

a rule is valid if valid premises

$\Rightarrow$ one conclusion is valid

φ valid $\equiv$ φ valid in all $\wp$

$\Gamma \vdash \varphi$ " $\equiv$ " " " "

$\Gamma \nvdash \Delta$ valid $\equiv$ $\Gamma \nvdash \Delta$ valid in some $\wp$

Validity theorem

$\Gamma \vdash \varphi \quad \underset{\Leftarrow}{\Rightarrow} \quad V(\Gamma) \lhd V(\varphi)$ in all models

$\Gamma \nvdash \Delta \quad \underset{\Rightarrow}{\Leftarrow} \quad V(\Gamma) \not\bowtie V(\Delta)$ in some model

canonical model: as before

$+ \quad \Sigma \not\bowtie \Phi \equiv \exists \varphi \in \Sigma \; \exists \psi \in \Phi \;\; \varphi \nvdash \psi$

completeness: red arrows

$$F'(\Sigma) \equiv \{\Diamond\varphi : \varphi \lhd \Sigma\}$$

$$G'(\Sigma) \equiv \{\varphi : \Diamond\varphi \lhd \Sigma\}$$

$$F(\Sigma) \equiv \{\blacklozenge\varphi : \varphi \lhd \Sigma\}$$

$$G(\Sigma) \equiv \{\varphi : \blacklozenge\varphi \lhd \Sigma\}$$

Then

$$F'(\varphi) =_{\mathcal{A}} \Diamond\varphi \qquad F(\varphi) =_{\mathcal{A}} \blacklozenge\varphi$$

$$G\varphi =_{\mathcal{A}} \Box\varphi \qquad G'\varphi =_{\mathcal{A}} \blacksquare\varphi$$

F, G, F', G' form a symmetric pair of adjunctions