Modal operators on bounded residuated ℓ-monoids

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TANCL '07

August 2007, Oxford, UK

Bounded $R\ell$-monoids.

An algebra $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ of type $\langle 2, 2, 2, 2, 2, 0, 0 \rangle$

- $(M;\ \odot, 1)$ is a monoid;
- $(M;\ \vee, \wedge, 0, 1)$ is a bounded lattice;
- $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$;
- $(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x)$.

Additional operations:

$$x^{-} := x \to 0$$

$$x^{\sim} := x \rightsquigarrow 0$$

Examples of $R\ell$-monoids.

An $R\ell$-monoid M is

a) a pseudo BL-algebra iff

$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x);$$

b) a GMV-algebra (pseudo MV-algebra) iff

$$x^{-\sim} = x = x^{\sim -};$$

c) a Heyting algebra iff $" \odot " = " \wedge "$.

References.

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Modal operators. Definition.

Let M be an $R\ell$-monoid. A mapping $f : M \longrightarrow M$ is called a *modal operator* on M if, for any $x, y \in M$,

- $x \leq f(x)$;
- $f\,(f(x)) = f(x)$;
- $f(x \odot y) = f(x) \odot f(y)$.

Modal operators. Properties.

Proposition 1.

If f is a modal operator on an $R\ell$-monoid M, $x,\, y \in M$, then

(1) $x \leq y \Longrightarrow f(x) \leq f(y)$;

(2) $f(x \to y) \leq f(x) \to f(y) = f\,(f(x) \to f(y)) =$
$= x \to f(y) = f\,(x \to f(y))$,

$f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y) = f\,(f(x) \rightsquigarrow f(y)) =$
$= x \rightsquigarrow f(y) = f\,(x \rightsquigarrow f(y))$;

(3) $f(x) \leq (x \to f(0)) \rightsquigarrow f(0)$,
$f(x) \leq (x \rightsquigarrow f(0)) \to f(0)$,

(4) $x^- \odot f(x) \leq f(0)$,
$f(x) \odot x^\sim \leq f(0)$;

(5) $f(x \vee y) \leq f(x \vee f(y)) = f(f(x) \vee f(y))$.

Modal operators – criterion.

Theorem 2.

Let M be an $R\ell$-monoid and $f : M \longrightarrow M$ be a mapping. Then f is a modal operator on M if and only if for any $x, y \in M$ it is satisfied:

(a) $x \to f(y) = f(x) \to f(y)$;

(b) $x \rightsquigarrow f(y) = f(x) \rightsquigarrow f(y)$;

(c) $f(x) \odot f(y) \geq f(x \odot y)$.

Modal operators – the example.

For an $R\ell$-monoid M:

$$I(M) = \{a \in M : \ a \odot a = a\},$$

$$a \odot x = a \wedge x, \ \ a \in I(M), \ x \in M.$$

$$\psi_a^1 : \ M \longrightarrow M, \qquad \psi_a^1(x) := a \to x$$

$$\psi_a^2 : \ M \longrightarrow M, \qquad \psi_a^2(x) := a \rightsquigarrow x$$

Proposition 3.

For $a \in I(M)$ and $x, y \in M$,

$$x \to \psi_a^1(y) \ = \ \psi_a^1(x) \to \psi_a^1(y),$$

$$x \rightsquigarrow \psi_a^2(y) \ = \ \psi_a^2(x) \rightsquigarrow \psi_a^2(y).$$

Corollary 4.

Let M be an $R\ell$-monoid and $a \in I(M)$. Then ψ_a^1 is a modal operator on M if and only if for any $x, y \in M$

$$x \rightsquigarrow \psi_a^1(y) \ = \ \psi_a^1(x) \rightsquigarrow \psi_a^1(y),$$

$$\psi_a^1(x) \odot \psi_a^1(y) \ \geq \ \psi_a^1(x \odot y).$$

The set of fixed elements.

For an $R\ell$-monoid M and a modal operator f:

$$\mathsf{Fix}(f) = \{x \in M : f(x) = x\},$$

$$\mathsf{Fix}(f) = \mathsf{Im}(f).$$

$(\mathsf{Fix}(f); \vee_F, \wedge)$, where $x \vee_F y = f(x \vee y)$, is a lattice.

Theorem 5.
If f is a modal operator on an $R\ell$-monoid M then $\mathsf{Fix}(f)$ *is closed under the operations "$\odot$", "$\to$" and "$\rightsquigarrow$", and* $\mathsf{Fix}(f) = (\mathsf{Fix}(f); \odot, \vee_F, \wedge, \to, \rightsquigarrow, f(0), 1)$ *is an $R\ell$-monoid.*

Good $R\ell$-monoids.

An $R\ell$-monoid is called *good* if it satisfies

$$x^{-\sim} = x^{\sim -}.$$

For a good $R\ell$-monoid:

$$(x^{-} \odot y^{-})^{\sim} = (x^{\sim} \odot y^{\sim})^{-}.$$

Define the binary operation "$\oplus$":

$$x \oplus y := (y^{-} \odot x^{-})^{\sim}.$$

Properties:

- $(x \oplus y)^{-\sim} = x^{-\sim} \oplus y^{-\sim} = x^{-\sim} \oplus y = x \oplus y^{-\sim} = x \oplus y$;
- $x \oplus y = (y^{\sim} \odot x^{\sim})^{-}$;
- $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
- $x,\, y \leq x \oplus y$;
- $x \oplus 0 = x^{-\sim} = 0 \oplus x$;
- $x \oplus 1 = 1 = 1 \oplus x$;
- $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$.

Modal operators. Definition.

Let M be an $R\ell$-monoid. A mapping $f : M \longrightarrow M$ is called a *modal operator* on M if, for any $x, y \in M$,

- $x \leq f(x)$;
- $f(f(x)) = f(x)$;
- $f(x \odot y) = f(x) \odot f(y)$.

Strong modal operators. Definition.

Let M be a good $R\ell$-monoid. A mapping $f : M \longrightarrow M$ is called a *strong modal operator* on M if, for any $x, y \in M$,

- $x \leq f(x)$;
- $f(f(x)) = f(x)$;
- $f(x \odot y) = f(x) \odot f(y)$;
- $f(x \oplus y) = f(x \oplus f(y)) = f(f(x) \oplus y)$.

Properties:

(6) $f(x \oplus y) = f(f(x) \oplus f(y))$;

(7) $x \oplus f(0) = f(x^{-\sim}) = f(0) \oplus x$.

Strong modal operators – examples.

An $R\ell$-monoid M is called *normal* if M satisfies

$$(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim},$$
$$(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}.$$

For a normal $R\ell$-monoid M:

$$a \in I(M) \implies a^{-\sim} \in I(M).$$

$\varphi_a : M \longrightarrow M, \quad \varphi_a(x) = a \oplus x$

Theorem 6.
If M is a good normal $R\ell$-monoid and $a \in M$ then φ_a is a strong modal operator on M if and only if $a^-, a^{-\sim} \in I(M)$.

Theorem 7.
Let M be a good normal $R\ell$-monoid and f be a modal operator on M such that $f(x) = f(x^{-\sim})$ for all $x \in M$. Then f is strong if and only if $f = \varphi_{f(0)}$ and $f(0)^- \in I(M)$.

On intervals of $R\ell$-monoids.

For an $R\ell$-monoid M and $a \in I(M)$:

$$I(a) := [0, a] = \{x \in M : 0 \le x \le a\}.$$

Set, for any $x, y \in I(a)$:

$x \odot_a y = x \odot y$, $x \rightarrow_a y := (x \rightarrow y) \wedge a$, $x \rightsquigarrow_a y := (x \rightsquigarrow y) \wedge a$.

Theorem 8.

$I(a) = (I(a); \odot_a, \vee, \wedge, \rightarrow_a, \rightsquigarrow_a, 0, a)$ is an $R\ell$-monoid.

Proposition 9.

a) If M is an $R\ell$-monoid, $a \in I(M)$ and $x \in I(a)$, then

$$x^{-_a} = x^- \wedge a, \quad x^{\sim_a} = x^\sim \wedge a.$$

b) Moreover, if M is good and satisfying the identities

$$(v \wedge w)^- = v^- \vee w^-, \ (v \wedge w)^\sim = v^\sim \vee w^\sim, \qquad (*)$$

then the $R\ell$-monoid $I(a)$ is good, too, and

$$x \oplus_a y = (x \oplus y) \wedge a.$$

Modal operators on intervals.

For an $R\ell$-monoid M, $a \in I(M)$ and modal operator f on M:

$$f^a : I(a) \longrightarrow I(a), \quad f^a(x) = f(x) \wedge a \; (= f(x) \odot a)$$

Theorem 10.
a) Let M be an $R\ell$-monoid, $a \in I(M)$ and f be a modal operator on M. Then f^a is a modal operator on the $R\ell$-monoid $I(a)$.

b) If M is good and it satisfies the identities $(*)$, and f is strong, then f^a is also a strong modal operator on $I(a)$.