Monadic *GMV*-algebras

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- monadic structures = algebras with quantifiers = algebraic models for one-variable fragments of predicate calculi of logics
- Halmos, 1955: monadic Boolean algebras = algebraic model of the predicate calculus of classical two-valued logic in which only one variable occurs
- Rutledge, 1959: monadic MV-algebras (MMV-algebras) = algebraic model for one-variable fragment of the Łukasiewicz many-valued predicate calculus
- Georgescu,Iorgulescu, Leustean, 1998
- Di Nola, Grigolia, 2004
- Belluce, Grigolia, Lettieri, 2005

GMV-algebras (generalized *MV*-algebras) = non-commutative generalizations of *MV*-algebras
(Georgescu, Iorgulescu, 2001; Rachůnek, 2002)

the non-commutative Łukasiewicz infinite valued propositional logic $\mathcal{PL}$, *GMV*-algebras = an algebraic semantics of $\mathcal{PL}$ (I. Leuştean, 2006)

$\mathcal{PL}$ - based on connectives $\neg$, $\sim$, $\rightarrow$ and $\rightsquigarrow$, and two deductive rules modus ponens

We define the monadic non-commutative Łukasiewicz propositional calculus $\mathcal{MPL}$ and introduce and investigate monadic *GMV*-algebras (*MGMV*-algebras).

The monadic non-commutative Łukasiewicz propositional calculus $\mathcal{MPL}$ is the logic containing $\mathcal{PL}$ in which the following formulas are axioms for arbitrary formulas φ and ψ:

(M1) $\varphi \to \exists\varphi, \ \ \varphi \rightsquigarrow \exists\varphi$;

(M2) $\exists(\varphi \vee \psi) \ \equiv \ \exists\varphi \vee \exists\psi$;

(M3) $\exists(\neg\exists\varphi) \ \equiv \ \neg\exists\varphi, \ \ \exists(\sim \exists\varphi) \ \equiv \sim \exists\varphi$;

(M4) $\exists(\exists\varphi \oplus \exists\psi) \ \equiv \ \exists\varphi \oplus \exists\psi$;

(M5) $\exists(\varphi \oplus \varphi) \ \equiv \ \exists\varphi \oplus \exists\varphi$;

(M6) $\exists(\varphi \odot \varphi) \ \equiv \ \exists\varphi \odot \exists\varphi$.

Let $\forall\varphi$ mean $\sim (\exists(\neg\varphi))$. Then the deductive rules in $\mathcal{MPL}$ are two modus ponens (MP$\to$) $\dfrac{\varphi, \ \varphi \to \psi}{\psi}$ and (MP$\rightsquigarrow$) $\dfrac{\varphi, \ \varphi \rightsquigarrow \psi}{\psi}$, and the necessitation (Nec) $\dfrac{\varphi}{\forall\varphi}$.

Definition

Let $A = (A; \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $\langle 2, 1, 1, 0, 0\rangle$. Set $x \odot y := (x^- \oplus y^-)^\sim$ for any $x, y \in A$. Then A is called a generalized MV-algebra (briefly: GMV-algebra) if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = x = 0 \oplus x$;

(A3) $x \oplus 1 = 1 = 1 \oplus x$;

(A4) $1^- = 0 = 1^\sim$;

(A5) $(x^\sim \oplus y^\sim)^- = (x^- \oplus y^-)^\sim$;

(A6)
$$x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x;$$

(A7) $(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim)$;

(A8) $x^{-\sim} = x$.

If we put $x \leq y$ if and only if $x^- \oplus y = 1$ then $L(A) = (A; \leq)$ is a bounded distributive lattice (0 is the least and 1 is the greatest element) with $x \vee y = x \oplus (y \odot x^\sim)$ and $x \wedge y = x \odot (y \oplus x^\sim)$.

GMV-algebras are in a close connection with unital ℓ-groups. (A unital ℓ-group is a pair (G, u) where G is an ℓ-group and u is a strong order unit of G.)

If G is an ℓ-group, and $0 \leq u \in G$ then
$\Gamma(G, u) = ([0, u]; \oplus, ^-, ^\sim, 0, u)$, where
$[0, u] = \{x \in G : \ 0 \leq x \leq u\}$, and for any
$x, y \in [0, u], \ x \oplus y = (x + y) \wedge u, \ x^- = u - x, \ x^\sim = -x + u$, is a *GMV*-algebra.

Conversely (A. Dvurečenskij), every *GMV*-algebra is isomorphic to $\Gamma(G, u)$ for an appropriate unital ℓ-group (G, u), and, moreover, the categories of *GMV*-algebras and unital ℓ-groups are equivalent.

Definition

If A is a GMV-algebra and $\exists : A \longrightarrow A$ is a mapping then $\exists$ is called an existential quantifier on A if the following identities are satisfied:

(E1) $x \leq \exists x$;

(E2) $\exists(x \vee y) = \exists x \vee \exists y$;

(E3) $\exists((\exists x)^{-}) = (\exists x)^{-}, \quad \exists((\exists x)^{\sim}) = (\exists x)^{\sim}$;

(E4) $\exists(\exists x \oplus \exists y) = \exists x \oplus \exists y$;

(E5) $\exists(x \odot x) = \exists x \odot \exists x$;

(E6) $\exists(x \oplus x) = \exists x \oplus \exists x$.

Definition

If A is a GMV-algebra and $\forall : A \longrightarrow A$ is a mapping then $\forall$ is called a universal quantifier on A if the following identities are satisfied:

(U1) $x \geq \forall x$;

(U2) $\forall (x \wedge y) = \forall x \wedge \forall y$;

(U3) $\forall ((\forall x)^{-}) = (\forall x)^{-}, \quad \forall ((\forall x)^{\sim}) = (\forall x)^{\sim}$;

(U4) $\forall (\forall x \odot \forall y) = \forall x \odot \forall y$;

(U5) $\forall (x \odot x) = \forall x \odot \forall x$;

(U6) $\forall (x \oplus x) = \forall x \oplus \forall x$.

Lemma

Let A be a GMV-algebra.
(a) If $\exists$ is an existential quantifier on A then $(\exists x^-)^\sim = (\exists x^\sim)^-$ for each $x \in A$.
(b) If $\forall$ is a universal quantifier on A then $(\forall x^-)^\sim = (\forall x^\sim)^-$ for each $x \in A$.

Theorem

If A is a GMV-algebra then there is a one-to-one correspondence between existential and universal quantifiers on A. Namely, if $\exists$ is an existential quantifier and $\forall$ is a universal one on A, then the mapping $\forall_\exists : A \longrightarrow A$ and $\exists_\forall : A \longrightarrow A$ such that for each $x \in A$,

$$\forall_\exists x := (\exists x^-)^\sim = (\exists x^\sim)^-$$

and

$$\exists_\forall x := (\forall x^-)^\sim = (\forall x^\sim)^-,$$

is a universal and an existential quantifier on A, respectively, and, moreover,

$$\exists_{(\forall_\exists)} = \exists \quad \text{and} \quad \forall_{(\exists_\forall)} = \forall.$$

Definition

If A is a GMV-algebra and $\exists$ is an existential quantifier on A then the couple $(A, \exists)$ is called a monadic GMV-algebra (an $MGMV$-algebra, in brief).

Every existential quantifier on an $MGMV$-algebra A is a closure operator on A (and every universal quantifier on A is an interior operator on A).

Let M be a GMV-algebra and X be a non-empty set.
M^X forms, with respect to the pointwise operations, also a GMV-algebra.
For any $p \in M^X$, put $R(p) := \{p(x) : \ x \in X\}$, the range of p.

Definition

A subalgebra A of M^X is called a functional monadic GMV-algebra if A satisfies the following conditions:

(i) for every $p \in A$ there exist
$\sup_M R(p) = \bigvee R(p), \ \inf_M R(p) = \bigwedge R(p)$;

(ii) for every $p \in A$, the constant functions $\exists p$ and $\forall p$ defined such that

$$\exists p(x) := \bigvee R(p), \quad \forall p(x) := \bigwedge R(p),$$

for any $x \in X$, belong to A.

Theorem

If M is a GMV-algebra, X is a non-empty set and $A \subseteq M^X$ is a functional monadic GMV-algebra, then $(A, \exists)$ is a monadic GMV-algebra.

If $(A, \exists)$ is an *MGMV*-algebra, put $\exists A := \{x \in A : x = \exists x\}$.

Lemma

If $(A, \exists)$ is an MGMV-algebra then $\exists A$ is a subalgebra of the GMV-algebra A.

Definition

Let A be a GMV-algebra and B be its subalgebra. Then B is called relatively complete if for each element $a \in A$, the set $\{b \in B : \ a \leq b\}$ has a least element.
A subalgebra B of a GMV-algebra A is called m-relatively complete if it is relatively complete and satisfies the following conditions:
(MRC1) For every $a \in A$ and $x \in B$ such that $x \geq a \odot a$ there is an element $v \in B$ such that $v \geq a$ and $v \odot v \leq x$.
(MRC2) For every $a \in A$ and $x \in B$ such that $x \geq a \oplus a$ there is an element $v \in B$ such that $v \geq a$ and $v \oplus v \leq x$.

Theorem

If $(A, \exists)$ is an $MGMV$-algebra then $\exists A$ is an m-relatively complete subalgebra of the GMV-algebra A.

Definition

Let A be a GMV-algebra, B a subalgebra of A and $h : B \longrightarrow A$ a mapping. Then a mapping $\exists_h : A \longrightarrow B$ is called a left adjoint mapping to h if $\exists_h(a) \leq x \iff a \leq h(x)$ for each $a \in A$ and $x \in B$.
If $\exists_h$, moreover, satisfies the identities
$\exists_h(a \odot a) = \exists_h(a) \odot \exists_h(a)$, $\exists_h(a \oplus a) = \exists_h(a) \oplus \exists_h(a)$, then $\exists_h$ is called a left m-adjoint mapping to h.

Theorem

There are one-to-one correspondences among

1. *$MGMV$-algebras;*
2. *pairs (A, B), where B is an m-relatively complete subalgebra of a GMV-algebra A;*
3. *pairs (A, B), where B is a subalgebra of a GMV-algebra A such that the canonical embedding $h: B \hookrightarrow A$ has a left m-adjoint mapping.*

Theorem

Let L be a linearly ordered GMV-algebra, $n \in \mathbb{N}$ and $D = \{\langle a, \dots, a\rangle : \ a \in L\}$ be the diagonal subalgebra of a direct power L^n. Let A be a subalgebra of the GMV-algebra L^n containing D. Then there exists an existential quantifier $\exists$ on A such that $\exists A = D \cong L$ holds in the MGMV-algebra $(A, \exists)$.

Example

Let G be the group of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \text{ where } a, b \in \mathbb{R},\ a > 0,$$

and where the group binary operation is the common multiplication of matrices. Set

$$(a, b) := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

For any $(a, b),\ (c, d) \in G$ we put

$$(a, b) \leq (c, d) \ :\Longleftrightarrow\ a < c \ \text{ or } \ a = c,\ b \leq d.$$

Then $G = (G, \leq)$ is a linearly ordered (non-commutative) group and, e.g., $u = (2, 0)$ is its strong order unit.

Hence $A = \Gamma(G, u)$ is a linearly ordered non-commutative *GMV*-algebra in which

$(a, b) \oplus (c, d) = (\min(ac, 2),\ \min(ad + b,\ 0))$,

$(a, b)^{-} = \left(\frac{2}{a}, -\frac{2b}{a}\right)$,

$(a, b)^{\sim} = \left(\frac{2}{a}, -\frac{b}{a}\right)$.

Let us now consider the (non-commutative) *GMV*-algebra $M = A^2$. For any $((a, b),\ (c, d)) \in M$ we put $\exists((a, b),\ (c, d)) = \max\{(a, b),\ (c, d)\}$.

Then by the previous theorem, $\exists :\ M \longrightarrow\ M$ is an existential quantifier on the non-commutative *GMV*-algebra M and, moreover, $\exists M$ is isomorphic with A.

Definition

Let A be a GMV-algebra and $\emptyset \neq I \subseteq A$. Then I is called an ideal of A if the following conditions are satisfied:

(I1) if $x, y \in I$ then $x \oplus y \in I$;

(I2) if $x \in I,\ y \in A$ and $y \leq x$ then $y \in I$.

The set $\mathcal{I}(A)$ of all ideals in a GMV-algebra A ordered by set inclusion is a complete lattice (a Brouwerian lattice, moreover).

Definition

Let $(A, \exists)$ be an *MGMV*-algebra and let I be an ideal of the *GMV*-algebra A. Then I is called a monadic ideal (in short: m-ideal) of $(A, \exists)$ if the following condition is valid:
$x \in I \implies \exists x \in I$.

The set $\mathcal{I}(A, \exists)$ of m-ideals of any *MGMV*-algebra $(A, \exists)$ is a complete lattice with respect to the order by set inclusion.

Theorem

If $(A, \exists)$ is a MGMV-algebra then the lattice $\mathcal{I}(A, \exists)$ is isomorphic to the lattice $\mathcal{I}(\exists A)$ of ideals of the GMV-algebra $\exists A$.

Definition

a) If A is a GMV-algebra and $I \in \mathcal{I}(A)$ then I is called a normal ideal of A if

$$x^{-} \odot y \in I \iff y \odot x^{\sim} \in I,$$

for every $x, y \in A$.

b) If $(A, \exists)$ is an $MGMV$-algebra and θ is a congruence on A, then θ is called an *m*-congruence on $(A, \exists)$ provided

$$(x, y) \in \theta \implies (\exists x, \exists y) \in \theta,$$

for every $x, y \in A$.

Theorem

For any MGMV-algebra there is a one-to-one correspondence between its m-congruences and normal m-ideals.

The class $\mathcal{MGMV}$ of all *MGMV*-algebras is a variety of algebras of type $\langle 2, 1, 1, 0, 1 \rangle$.

Theorem

The variety $\mathcal{MGMV}$ is arithmetical.

Definition

An ideal P of a *GMV*-algebra A is called prime if P is a finitely meet-irreducible element in the lattice $\mathcal{I}(A)$.
A prime ideal P is called minimal if P is a minimal element in the set of prime ideals of A ordered by inclusion.

Definition

A *GMV*-algebra *A* is called representable if *A* is isomorphic to a subdirect product of linearly ordered *GMV*-algebras.

Theorem

For a GMV-algebra A the following conditions are equivalent:
(1) A is representable.
(2) There exists a set $\mathcal{S}$ *of normal prime ideals such that* $\bigcap \mathcal{S} = \{0\}$.
(3) Every minimal prime ideal is normal.

Theorem

Let $(A, \exists)$ *be an MGMV-algebra satisfying the identity* $\exists(x \wedge y) = \exists x \wedge \exists y$. *Then* $(A, \exists)$ *is a subdirect product of linearly ordered MGMV-algebras if and only if A is a representable GMV-algebra.*