Effect algebras and AF C*-algebras

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The aim of this lecture is to show that there is a close connection between effect algebras with the Riesz decomposition property and dimension theory of AF C*-algebras.

Definition 1. *(Foulis and Bennett, 1994) An* **effect algebra** *is an algebraic system* $(E; 0, 1, \oplus)$*, where* $\oplus$ *is a partial binary operation and* 0 *and* 1 *are constants, such that the following axioms are satisfied for every* $a, b, c \in E$*:*

(i) if $a \oplus b$ *is defined then* $b \oplus a$ *is defined and* $a \oplus b = b \oplus a$ *(commutativity);*

(ii) if $a \oplus b$ *and* $(a \oplus b) \oplus c$ *is defined then* $a \oplus (b \oplus c)$ *is defined and* $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ *(associativity);*

(iii) for every $a \in E$ *there is a unique* $a' \in E$ *such that* $a \oplus a' = 1$*;*

(iv) $a \oplus 1$ *is defined iff* $a = 0$*.*

In an effect algebra E, we define:

- $a \leq b$ if there is $c \in E$ with $a \oplus c = b$.

- $\leq$ is a partial order, $0 \leq a \leq 1$ for all $a \in E$.

- Cancelation: $a \oplus c_1 = a \oplus c_2$, then $c_1 = c_2$, and we define $c = b \ominus a$ iff $a \oplus c = b$.

- $1 - a = a'$ is called the **orthosupplement** of a.

E is:

- orthoalgebra iff $a \perp a$ implies $a = 0$;

- orthomodular poset iff $a \oplus b = a \vee b$;

- orthomodular lattice if it is a lattice-ordered orthomodular poset;

- MV-effect algebra if it is lattice-ordered and $a \wedge b = 0 \implies a \perp b$;

- boolean algebra if it is an MV-effect algebra and an orthoalgebra in the same time.

Notice that MV-effect algebras are equivalent with MV-algebras introduced by *Chang (1958)* as algebraic bases for many-valued logic.

Let (G, G^+) be a partially ordered abelian group, $v \in G^+$,

$$G^+[0, v] = \{a \in G : 0 \leq a \leq v\}.$$

A partial binary operation on $G^+[0, v]$:

$$a \oplus b = a + b, \text{ defined iff } a + b \leq v.$$

Then $(G^+[0, v]; 0, v, \oplus)$ becomes an effect algebra.

Effect algebras arising this way are called **interval effect algebras**. Two important examples of interval effect algebras are the following.

Example 1 Let H be a Hilbert space and let $G = \mathcal{B}_s(H)$ be the group of self-adjoint operators on H ordered by $A \leq B$ iff $\langle Ax, x\rangle \leq \langle Bx, x\rangle$ for all $x \in H$. The interval $\mathcal{E}(H) := \{A \in \mathcal{B}_s(H) : 0 \leq A \leq I\}$ (0 is the zero, I is the identity operator) is an effect algebra of so called **Hilbert space effects**.

Example 2 Let (G, G^+) be a lattice ordered group, $v \in G^+$. Then the interval $G^+[0, v]$ becomes a **MV-effect algebra**. By *Mundici (1989)*, there is a categorical equivalence between the category of MV-algebras with MV-algebra homomorphisms and ℓ-groups with ℓ-group homomorphisms.

Let (G, G^+) be a partially ordered abelian group.

- G has the **Riesz interpolation property (RIP)** or is an **interpolation group** if given a_i, b_j $(1 \leq i \leq m, 1 \leq j \leq n)$ with $a_i \leq b_j$ for all i, j, there must be a $c \in G$ with $a_i \leq c \leq b_j$ for all i, j.

The Riesz interpolation property is equivalent to the

- Riesz decomposition property (RDP): given a_i, b_j in G^+ $(1 \leq i \leq m, 1 \leq j \leq n)$ with $\sum a_i = \sum b_j$, there exist $c_{ij} \in G^+$ with $a_i = \sum_j c_{ij}$, $b_j = \sum_i c_{ij}$.

Equivalently, RDP can be expressed as follows: given $a, b_i, i \leq n$ in G^+ with $a \leq \sum_{i \leq n} b_i$, there exist $a_i, i \leq n$ with $a_i \leq b_i, i \leq n$, and $a = \sum_{i \leq n} a_i$.

In order to verify both these properties, it is only necessary to consider the case $m = n = 2$.

An element u of a partially ordered group G is an **order unit** if for all $a \in G$, $a \leq nu$ for some $n \in \mathbb{N}$. An ordered group G is said to be *directed* if $G = G^+ - G^+$. If G has an order unit u then it is directed.

G-partially ordered abelian group.

- G is **unperforated** if given $n \in \mathbb{N}$ and $a \in G$, $na \in G^+$ implies $a \in G^+$.

- G is **archimedean** in $a, b \in G$, $na \leq b \forall n \in \mathbb{N}$ implies $a \leq 0$.

If G is directed and archimedean, or if G is lattice ordered, then G is unperforated (*Goodearl*).

Given partially ordered groups G and H, we say that a group homomorphism $\phi : G \to H$ is **positive** if $\phi(G^+) \subseteq H^+$, and that an isomorphism $\phi : G \to H$ is an *order isomorphism* if $\phi(G^+) = H^+$. If G and H have order units u and v respectively, then a positive homomorphism $\phi : G \to H$ such that $\phi(u) = v$ is called **unital**.

Definition 2. *A partially ordered group G is called a* **Riesz group** *(Fuchs, 1965) if it is directed, unperforated and has the interpolation property.*

An element $u \in G^+$ is called **generative** unit or a **strong unit** if every $g \in G^+$ can be expressed in the form of finite sum of (not necessarily different) elements of the interval $G^+[0, u]$. If G is an interpolation group, then any order unit is generative.

Definition 3. *Let E be an effect algebra. A map $\phi : E \to K$, where K is any abelian group, is called a* K**-valued measure** *if $\phi(a \oplus b) = \phi(a) + \phi(b)$ whenever $a \oplus b$ is defined in E.*

Theorem 4. *(Bennett and Foulis, 1997). Let E be an interval effect algebra. Then there exists a unique (up to isomorphism) partially ordered abelian group G with generating cone G^+ (in the sense that $G = G^+ - G^+$) and an element $u \in G^+$ such that the following conditions are satisfied:*

(i) *E is isomorphic to the interval effect algebra $G^+[0, u]$,*

(ii) *$G^+[0, u]$ generates G^+ (in the sense that every element in G^+ is a finite sum of elements of $G[0, u]$),*

((iii) *every K-valued measure $\phi : E \to K$ can be extended uniquely to a group homomorphism $\phi^* : G \to K$.*

The group G in the preceding theorem is called the **universal group** for E. We will denote it by G_E. The element u is a generative unit in G_E.

We say that an effect algebra E has the **Riesz decomposition property** (RDP) if one of the following equivalent properties is satisfied:

R1 $a \leq b_1 \oplus b_2 \oplus \cdots \oplus b_n$ implies $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ with $a_i \leq b_i$, $i \leq n$;

R2 $\oplus_{i \leq m} a_i = \oplus_{j \leq n} b_j$, $m, n \in \mathbb{Z}$, implies $a_i = \oplus_j c_{ij}$, $i \leq m$, and $b_j = \oplus_i c_{ij}$, $j \leq n$, where $(c_{ij})_{ij}$ are orthogonal elements in E.

Similarly as for partially ordered groups, it suffices to prove the above properties for $m, n = 2$.

A partially ordered set P has the **interpolation property** if $a_1, a_2 \leq b_1, b_2$, there is an element $x \in P$ with $a_1, a_2 \leq x \leq b_1, b_2$.

Every effect algebra with RDP has the interpolation property. But there are lattice ordered effect algebras that do not satisfy the RDP (see e.g. "diamond": $D = \{a, b, 0, 1\}$, $0 \oplus x = x, x \in D$, $a \oplus a = 1 = b \oplus b$).

Theorem 5. *An effect algebra is an MV-effect algebra iff it is lattice ordered and has RDP.*

There are effect algebras that are not interval effect algebras. E.g., there exist orthomodular lattices with no group valued measures (*Navara, 1994*).

Theorem 6. *(Ravindran, 1996) Every effect algebra with the Riesz decomposition property is an interval effect algebra and its universal group is an interpolation group.*

The technique used in the proof goes back to *Baer (1049)* and *Wyler (1966)*. A sketch of the method is as follows.

Let E be an effect algebra with the Riesz decomposition property. A **word** is any sequence

$$W = (a_1, a_2, \dots, a_n)$$

of elements of E. For each $a \in E$, the word (a) is of length 1.

Introduce a binary operation $+$ on the collection $\mathcal{W}(E)$ of all words as follows: for two words
$W_1 = (a_1, a_2, \dots, a_m)$,
$W_2 = (b_1, b_2, \dots, b_n)$, the word
$W_1 + W_2 = (a_1, a_2, \dots, a_m, b_1, b_1, \dots, b)$.

It is easily verified that $\mathcal{W}(E)$ is a semigroup. We will say that two words W_1 and W_2 are **directly similar** (written $W_1 \to W_2$) if $W_1 = (a_1, a_2, \dots, a_n)$, $a_k \perp a_{k+1}$, $W_2 = (a_1, a_2, \dots, a_{k-1}, a_k \oplus a_{k+1}, a_{k+2}, \dots, a_n)$.

Let $\sim$ be the transitive closure of direct similarity. Let $\mathcal{G}^+$ be the collection of the equivalence classes on $\mathcal{W}(E)$ with respect to $\sim$.

Define $[W^1] + [W^2] = [W^1 + W^2]$. This operation is well-defined and so $\mathcal{G}^+$ is the quotient semigroup $\mathcal{W}(E)/\sim$ consisting of the equivalence classes with the operation $+$.

It was proved that $\mathcal{G}^+$ is a positive, cancellative, abelian monoid satisfying RDP, and so there is an interpolation group $\mathcal{G}$ containing $\mathcal{G}^+$ as a positive cone. The mapping $a \mapsto [a]$ is an embedding of E onto $\mathcal{G}^+[[0], [1]]$. By construction, this interval is generative, and $\mathcal{G}$ is the universal group for E.

Theorem 7. *(SP, 1999) Let u be an order unit in an abelian interpolation group G, then G is a unital group with unit u, and G is the universal group for its own unit interval $E = G^+[0, u]$.*

Theorem 8. *(SP, 1999) There is a categorical equivalence between the category of unital interpolation groups with unital group homomorphisms as morphisms, and the category of effect algebras with RDP with effect algebra morphisms as morphisms.*

The functors are $S : (G, u) \to G^+[0, u]$, $T : E \to (G_E, u)$.

The purpose of a dimension theory is to measure the "dimensions" of projections in an algebra.

The dimension of a projection in a matrix algebra is a non-negative integer, while in a finite von Neumann factor one obtains a non-negative real number.

In a C*-algebra $\mathcal{A}$ the dimension function must be given by values in a pre-ordered abelian group (so called K_0 group), rather than in real numbers.

$\mathcal{A}$ - a C*-algebra. Projections $p, q \in \mathcal{A}$ are **equivalent**, written $p \sim g$, if there is a partial isometry u such that $u^*u = p$ and $uu^* = q$.

Let $Proj(\mathcal{A})$ denote the set of all equivalence classes of projections of $\mathcal{A}$.

For two equivalence classes $[p]$ and $[q]$ their "sum" $[p]+[q]$ exists iff there are representatives $p' \in [p]$ and $q' \in [q]$ with $p'q' = 0$, in which case $[p] + [q] = [p' + q']$.

Recall that in the K_0-theory of C*-algebras, the definition of $K_0(\mathcal{A})$ for a C*-algebra $\mathcal{A}$, requires simultaneous consideration of all matrix algebras over $\mathcal{A}$.

K_0 is a covariant functor from the category of C*-algebras to the category of abelian semigroups, which preserves direct products and inductive limits.

To every abelian group G there is a C*-algebra $\mathcal{A}$ with $K_0(\mathcal{A}) = G$. But this correspondence is not one-to-one; the C*-algebra $\mathcal{A}$ is not uniquely defined by its $K_0(\mathcal{A})$. The situation is better in the class of approximately finite C*-algebras.

Definition 9. *A C*-algebra* $\mathcal{A}$ *is called* **approximately finite-dimensional (AF)** *if* $\mathcal{A}$ *is the direct limit of an increasing sequence of finite-dimensional C*-algebras.*

We will be concerned with unital AF C*-algebras that arise as direct limits of sequences of unital finite dimensional C*-algebras with the same unit.

Let $M_n(\mathbb{C})$ denote the C*-algebra consisting $n \times n$-matrices with complex entries. Two projections in $M_n(\mathbb{C})$ are equivalent iff they have the same dimension. Hence the range of dimension can be described by a finite chain of integers $(0, 1, \ldots, n)$. This chain can be endowed with a structure of an MV-effect algebra, its universal group is $\mathbb{Z}$, with order unit n.

Every finite dimensional C*-algebra $\mathcal{A}$ is isomorphic to the direct product $M_{n(1)}(\mathbb{C}) \times M_{n(2)}(\mathbb{C}) \times \cdots \times M_{n(k)}(\mathbb{C})$. It is characterized by the k-tuple $(n(1), n(2), \ldots, n(k))$ of positive integers. The range of dimension is then the direct product of finite MV-chains

$$(0, 1, \ldots, n(1)) \times \cdots \times (0, 1, \ldots, n(k)).$$

The universal group for it is $\mathbb{Z}^k$ with order unit $(n(1), \ldots, n(k))$, which is known to be the $K_0(\mathcal{A})$.

We recall that an abelian group G is called **simplicial** if it is isomorphic with $\mathbb{Z}^k$, $k \in \mathbb{N}$. An element $\mathbf{u} = (u_1, \ldots, u_k)$ is an order unit iff $u_i > 0, i \leq k$.

Lemma 10. (i) *Every finite effect algebra with RDP is an MV-algebra isomorphic to a direct product of finite chains.* (ii) *A unital interpolation group* (G, u) *is simplicial iff its unit interval is a finite effect algebra with RDP.*

Let $\mathcal{A}$ be a unital AF C*-algebra which is a direct limit of a directed system

$$A_1 \to A_2 \to \cdots$$

of finite dimensional C*-algebras with the same unit. Then there is a (unique up to isomorphism) directed system

$$G_1 \to G_2 \to \cdots$$

of simplicial groups, the direct limit G of which is the $K_0(\mathcal{A})$, and a directed system of finite effect algebras with RDP

$$D_1 \to D_2 \to \cdots$$

with direct limit D, which is the unit interval for G.

Theorem 11. *(Elliott, 1976) Let $\mathcal{A}$ be the inductive limit of unital finite dimensional algebras C*-algebras. The range of the dimension on $\mathcal{A}$ is isomorphic to a generating interval of a countable partially ordered abelian group which is the inductive limit of a sequence of simplicial groups with order units.*

According to *Elliott*, a group G which is a direct limit (of a sequence) of simplicial groups, is called a **dimension group**.

Theorem 12. *(Efros, Handelman and Shen, 1980) Any countable Riesz group with order unit is isomorphic to a direct limit of a countable sequence of simplicial groups with order unit (in the category of partially ordered abelian groups with order unit).*

A **dimension effect algebra** can be analogously defined as an effect algebra which is the direct limit of a sequence of finite effect algebras with RDP.

A dimension effect algebra is countable, has RDP, and its universal group is a dimension group.

Question. Is there an intrinsic characterization of the countable effect algebras with RDP which are dimension effect algebras?