Canonical Extensions in the setting of Lattice-based Algebras

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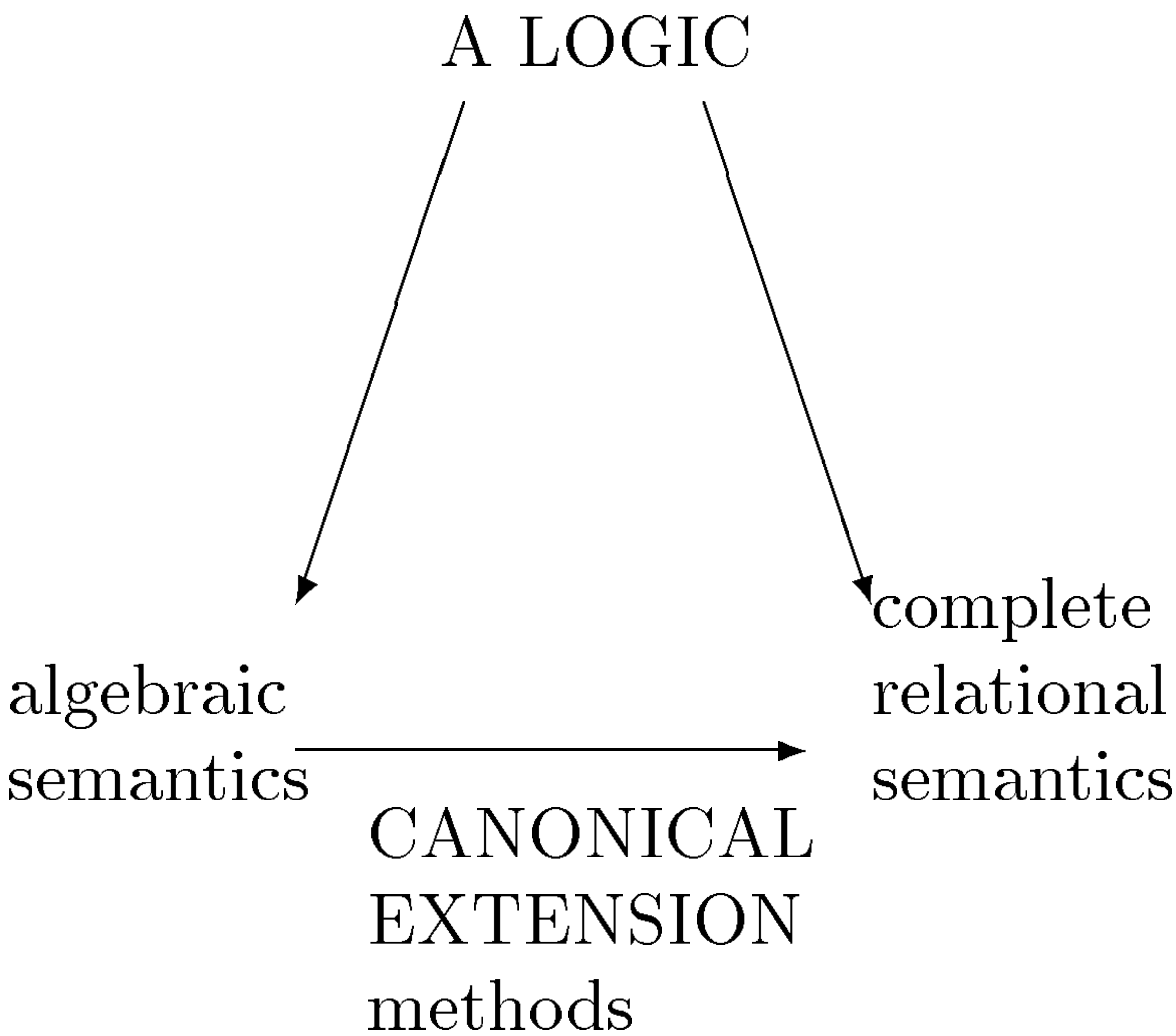
What has it got to do with logic?



Focus of talk is on algebraic and topological methods

The context

Generally shall be considering a variety $\mathcal{V}$:

- members of $\mathcal{V}$ are lattice-based algebras
- $\mathcal{V}$ will be finitely generated

All lattices will be assumed to have $0, 1$ (for simplicity; not an essential restriction)

Which completion?

- **Canonical extension**
- **Profinite completion**
- **Natural extension** — a new kid on the block

The canonical extension and the profinite extension are inherently different:

- For the canonical extension, one forms a completion of the **underlying lattice** and treats additional operations as an overlay.
- to form the profinite completion we work within the given class of algebras.

Building the canonical extension of a lattice

We first form

- the free $\bigvee$-completion $\mathbb{F}_{\vee}(L)$
  ($\cong$ ideal lattice of L, which we order by $\subseteq$)

- the free $\bigwedge$-completion $\mathbb{F}_{\wedge}(L)$
  ($\cong$ filter lattice of L, which we order by $\supseteq$)

This treats join and meet separately

Now we must recombine them

Building the canonical extension of a lattice

From a lattice L to the intermediate structure, Int(L)

$$\begin{tikzcd}
L \arrow[r] \arrow[d, equal] & \mathbb{F}_\vee(L) \arrow[r] \arrow[dr] & \mathbb{F}_\vee(\mathbb{F}_\wedge(L)) \\
L \arrow[r] & \mathbb{F}_\wedge(L) \arrow[r] \arrow[ur] & \mathbb{F}_\wedge(\mathbb{F}_\vee(L))
\end{tikzcd}$$

$$\mathbb{F}_\wedge(\mathbb{F}_\vee(L)) \supseteq \mathbb{F}_\vee(L) \cup \mathbb{F}_\wedge(L) \subseteq \mathbb{F}_\vee(\mathbb{F}_\wedge(L))$$

Furthermore

$$\mathrm{Int}(L) := \mathbb{F}_\vee(L) \cup \mathbb{F}_\wedge(L)$$

acquires a compatible quasi-order.

Building the canonical extension of a lattice

From $\mathsf{Int}(L)$ to the canonical extension L^δ: Form the MacNeille completion of $\mathrm{Int}(L)$.

Completions of lattices

Definitions

Let **L** be a lattice (not necessarily distributive) and **C** a complete lattice, with **L** [isomorphic to] a sublattice of **C**. Then

- **C** is a completion of **L**.
- **C** is a dense completion of **L** if every element of **C** is a join of meets of elements of **L** and a meet of joins of elements of **L**.
- **C** is a compact completion of **L** if for any filter F and ideal J of **L**

$$\bigwedge F \leqslant \bigvee J \implies F \cap J \neq \varnothing.$$

Canonical extensions of lattices, and DLs in particular

Let L be a lattice.

Definition

- C is a canonical extension of L if C is a compact and dense completion of L.

Facts

- L has a dense and compact completion and this is unique up to isomorphism (M. Gehrke & J. Harding (2001)).
- If in addition $L \in \mathfrak{D}$ (distributive lattices) then $L^\sigma := \mathsf{Up}(X_L)$ (= up-set lattice of X_L) is a dense and compact completion of L, where X_L is the Priestley dual space of L (M. Gehrke & B. Jónsson (1994, 2004)), (So $L^\sigma \cong L^\delta$, the canonical extension constructed before.) Furthermore

    - compact $\equiv X_L$ compact;
    - dense $\equiv X_L$ totally order-disconnected.

The Boolean case, functorially

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{CABA} \arrow[r, shift left, "At"] \arrow[r, phantom, "\text{discrete duality}"] \arrow[d, shift right] & \mathrm{Set} \arrow[l, shift left, "\wp"] \arrow[d, shift left, "\beta"] \\
\mathrm{BA} \arrow[r, shift left, "D"] \arrow[r, phantom, "\text{Stone duality}"] \arrow[u, shift right, "\delta"'] & \mathrm{BS} \arrow[l, shift left, "E"] \arrow[u, shift left, "\flat"]
\end{tikzcd}
$$

complete
atomic BAs

canonical
extension

forgetful
functor

Stone-
Czech
functor

Boolean
algebras

Boolean
spaces

The distributive lattice case, functorially

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{DL}^{+} \arrow[r, shift left, "CJI"] \arrow[d, shift right] & \mathrm{Poset} \arrow[l, shift left, "\mathrm{Upset}"] \arrow[d, shift left, "\beta"] \\
\mathrm{DL} \arrow[u, shift right, "\delta"'] \arrow[r, shift left, "D"] & \mathrm{PS} \arrow[l, shift left, "E"] \arrow[u, shift left, "\flat"]
\end{tikzcd}
$$

discrete duality

Up-set lattices = superalgebraic lattices

canonical extension

Priestley duality

Distributive lattices

Priestley spaces

And likewise with many kinds of additional operations and associated relations added

A well-known theorem on complete lattices ...and its topological manifestations

Theorem

Let $\mathbf{L}$ *be a lattice. The following are equivalent:*

- $\mathbf{L}$ *is isomorphic to* $\mathrm{Up}(P)$, *the lattice of up-sets of some poset* P;
- $\mathbf{L}$ *is isomorphic to a complete sublattice of a power of* $\mathbf{2}$;
- $\mathbf{L}$ *is completely distributive and algebraic;*
- *as a lattice,* $\mathbf{L}$ *is isomorphic to a topologically closed sublattice of a power of* $\mathbf{2}$ *with the discrete topology;*
- *when endowed with the interval topology,* $\mathbf{L}$ *is a topological distributive lattice whose topology is Boolean (*$=$ *compact* 0*-dimensional);*
- $\mathbf{L}$ *is the underlying lattice of a Boolean-topological distributive lattice.*

On canonical extensions, dense means dense

[Recall that a completion of a lattice is (isomorphic to) the canonical extension if it is dense and compact, where these conditions were formulated in ALGEBRAIC terms.]

Restricting to distributive case.

Theorem

Let $\mathbf{L}$ *be a sublattice of* $\mathbf{C}$, *where* $\mathbf{C}$ *is a Boolean-topological distributive lattice.*

- *For* $x \in \mathbf{C}$:

$$x \in \overline{\mathbf{L}} \Leftrightarrow x \text{ is a join of meets of els of } \mathbf{L}$$
$$\Leftrightarrow x \text{ is a meet of joins of els of } \mathbf{L}.$$

- $\mathbf{C}$ *is a dense completion of* $\mathbf{L}$ *if and only if* $\mathbf{L}$ *is topologically dense in* $\mathbf{C}$.

Here a Boolean-topological lattice means a topological lattice in which the topology is compact and totally disconnected

On canonical extensions, compact means compact

Theorem

Let **L** *be a sublattice of a complete lattice* **C**, *where* **C** *is a sublattice of some power* $\mathbf{2}^Z$, *for some set* Z. *Then TFAE*

- *the lattice* **C** *is a compact completion of* **L**;
- *there exists a compact topology on* Z *such that* **L** *is a compact sublattice of the space* $\mathrm{C}(Z, \mathbf{2}_\tau)$ *of continuous maps from* Z *into the 2-point space with the discrete topology* τ.

Some choices for Z:

- Z is the Priestley dual of **L**
- Represent **L** as a sublattice of some $\mathbf{2}^S$, where S is a set, and take $Z = \beta S$, where S is given the discrete topology.

Speaking categorically

We may consider two categories

$$\mathfrak{D}_{\mathrm{CDA}} \quad \left\{ \begin{array}{l} \text{Superalgebraic lattices} = \\ \text{completely distributive algebraic lattices} \\ \text{complete homomorphisms} \end{array} \right.$$

$$\mathfrak{D}_{\mathrm{BTop}} \quad \left\{ \begin{array}{l} \text{Boolean-topological distributive lattices} \\ \text{continuous lattice homomorphisms} \end{array} \right.$$

Theorem

- $\mathfrak{D}_{\mathrm{CDA}}$ and $\mathfrak{D}_{\mathrm{BTop}}$ are ISOMORPHIC categories.

Two dualities

- (1970) **Priestley's duality**: between

$\mathfrak{D}$ and $\mathfrak{X}$

(DLs) (Priestley spaces)

- (1976) **Banaschewski's duality**: between

$\mathfrak{D}_{\mathrm{BTop}}$ and $\mathfrak{P}$

(Boolean-topological DLs) (posets)

Priestley

$$\begin{tikzcd} \mathfrak{D} \arrow[r, shift left, "H"] & \mathfrak{X} \arrow[l, shift left, "K"] \end{tikzcd}$$

Banaschewski

$$\begin{tikzcd} \mathfrak{P} \arrow[r, shift right, "E"'] & \mathfrak{D}_{\mathrm{BTop}} \arrow[l, shift right, "D"'] \end{tikzcd} \cong \mathfrak{D}_{\mathrm{CDA}}$$

Two dualities, four faces of 2

algebras	relational structures
$\mathbf{2}_{\mathcal{D}} \in \mathcal{D}$ lattice	$\mathbf{2}_{\mathcal{X}} \in \mathcal{X}$ topological poset
$\mathbf{2}_{\mathcal{D}_{\mathrm{BTop}}} \in \mathcal{D}_{\mathrm{BTop}}$ topological lattice	$\mathbf{2}_{\mathcal{P}} \in \mathcal{P}$ poset

$$\mathcal{D} = \mathbb{ISP}(\mathbf{2}_{\mathcal{D}}) \quad \overset{\text{Priestley duality}}{\longleftrightarrow} \quad \mathcal{X} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{2}_{\mathcal{X}})$$

$$\mathcal{D}_{\mathrm{BTop}} = \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{2}_{\mathcal{D}_{\mathrm{BTop}}}) \underset{\text{Banaschewski duality}}{\longleftrightarrow} \mathcal{P} = \mathbb{ISP}(\mathbf{2}_{\mathcal{P}})$$

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Priestley and Banaschewski dualities are related by topology-swapping

Two dualities in tandem

$$\begin{tikzcd} \mathfrak{D} \arrow[r, shift left, "H"] & \mathfrak{X} \arrow[l, shift left, "K"] \arrow[r, "\flat", "\text{forgetful functor}"'] & \mathfrak{P} \arrow[r, shift right, "E"'] & \mathfrak{D}_{\mathrm{BTop}} \cong \mathfrak{D}_{\mathrm{CDA}} \arrow[l, shift right, "D"'] \end{tikzcd}$$

Priestley — Banaschewski

- Composing the right-pointing functors,

$$\mathfrak{D} \xrightarrow{\quad E \circ \flat \circ H \quad} \mathfrak{D}_{\mathrm{BTop}} \cong \mathfrak{D}_{\mathrm{CDA}}$$

  we get a covariant functor $\sigma\colon \mathfrak{D} \to \mathfrak{D}_{\mathrm{CDA}}$.

- Start from $\mathbf{L} \in \mathfrak{D}$. Form its Priestley dual space $\mathbf{X}_L$ *viz.* prime filters of $\mathbf{L}$, suitably topologised. Forget the topology on $\mathbf{X}_L$. Then

$$\sigma\colon \mathbf{L} \mapsto \mathbf{L}^\sigma := \mathrm{Up}(\mathbf{X}_L)$$

  —the concrete canonical extension of $\mathbf{L}$.

Profinite completions

To form the profinite completion of a algebra $\mathbf{A}$

- Let $\mathcal{S}_{\mathbf{A}}$ be the set of congruences θ on $\mathbf{A}$ s.t. $\mathbf{A}/\theta$ is finite.
- For $\theta_1, \theta_2 \in \mathcal{A}_{\mathbf{A}}$ with $\theta_1 \subseteq \theta_2$ let $\varphi_{\theta_1,\theta_2} : \mathbf{A}/\theta_1 \to \mathbf{A}/\theta_2$ be the natural map.
- With $\mathcal{S}_{\mathbf{A}}$, ordered by reverse inclusion, and the maps $\varphi_{\theta_1,\theta_2}$ as bonding maps the family $\mathbf{A}_\theta$ for $\theta \in \mathcal{S}_{\mathbf{A}}$ forms an **inverse system**.
- The **profinite completion** of $\mathbf{A}$ is

$$\mathrm{pro}(\mathbf{A}) := \varprojlim \{\, \mathbf{A}/\theta \mid \theta \in \mathcal{S}_{\mathbf{A}} \,\}.$$

Profinite completions

The profinite completion of an algebra $\mathbf{A}$

- is an algebra in the variety generated by $\mathbf{A}$;
- is a compact totally disconnected topological algebra.

A 'natural' consequence of the Priestley and Banaschewski dualities is

Theorem The canonical extension of $L \in \mathfrak{D}$ coincides with $\mathrm{pro}(L)$.

Now go back to dualities-more on profinite completions shortly

Can we play our duality game more generally?

Recap

what works for $\mathfrak{D}$

Start from $\mathfrak{D} = \mathbb{ISP}(\mathbf{2}_{\mathfrak{D}})$

Form topological dual category

$\mathfrak{X} = \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathfrak{X}})$

Form discrete dual category (by forgetting topology)

$\mathfrak{P} = \mathbb{ISP}(\mathbf{2}_{\mathfrak{P}})$

Take $\mathbf{L} \in \mathfrak{D}$

Form dual $\mathbf{X}_L = \mathfrak{D}(\mathbf{L}, \mathbf{2}_{\mathfrak{D}}) \in \mathfrak{X}$

Recapture original algebra

$\mathbf{L} \cong \mathfrak{X}(\mathbf{X}_L, \mathbf{2}_{\mathfrak{X}})$

Lift to

canonical extension

$\mathbf{L}^\sigma := \mathfrak{P}(\mathbf{X}_L^\flat, \mathbf{2}_{\mathfrak{P}})$

Can we play our duality game more generally?

ASSUME $\mathbf{M}$ is a finite lattice-based algebra. Give underlying set M the discrete topology. Assume (just for simplicity) that $\mathcal{V} := \mathbb{ISP}(\mathbf{M})$ is a variety.

What should our dual category $\mathcal{Y} := \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{Y}})$ be? ('defn' looks circular)—natural duality theory tells us how to choose relational structure for $\mathbf{M}_{\mathcal{Y}}$ so that each $A \in \mathcal{V}$ can be recaptured from its dual.

Questions

- Can we get the canonical extension of $\mathbf{A} \in \mathcal{V}$ by forgetting the topology on the dual side?
- Can we play the topology-swapping game?
- What about the profinite completion?

Yes!

Imitate the DL case:

	what works for $\mathcal{D}$	what we'd like for $\mathcal{V}$
Start from	$\mathcal{D} = \mathbb{ISP}(\mathbf{2}_{\mathcal{D}})$	$\mathcal{V} = \mathbb{ISP}(\mathbf{M})$
Form topological dual categories	$\mathcal{X} = \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathcal{X}})$	$\mathcal{Y} = \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{Y}})$
Form discrete dual categories (by forgetting topology)	$\mathcal{P} = \mathbb{ISP}(\mathbf{2}_{\mathcal{P}})$	$\mathcal{Q} = \mathbb{ISP}(\mathbf{M}_{\mathcal{Q}})$
Take	$\mathbf{L} \in \mathcal{D}$	$\mathbf{A} \in \mathcal{V}$
Form duals	$\mathbf{X}_L = \mathcal{D}(\mathbf{L}, \mathbf{2}_{\mathcal{D}}) \in \mathcal{X}$	$\mathbf{Y}_A = \mathcal{V}(\mathbf{A}, \mathbf{M}_{\mathcal{V}}) \in \mathcal{Y}$
Recapture original algebra	$\mathbf{L} \cong \mathcal{X}(\mathbf{X}_L, \mathbf{2}_{\mathcal{X}})$	$\mathbf{A} \cong \mathcal{Y}(\mathbf{Y}_A, \mathbf{M}_{\mathcal{Y}})$
Lift to	canonical extension $\mathbf{L}^\sigma := \mathcal{P}(\mathbf{X}_L^\flat, \mathbf{2}_{\mathcal{P}})$	**natural extension** $n(\mathbf{A}) := \mathcal{Q}(\mathbf{Y}_A^\flat, \mathbf{M}_{\mathcal{Q}})$

We want:

Theorem $\mathbf{L}_{n(\mathbf{A})} \cong (\mathbf{L}_A)^\sigma$

Profinite completions and natural extensions

For a variety $\mathcal{V}$ generated by a **finite** algebra $\mathbf{M}$ and $\mathbf{A} \in \mathcal{V}$:

- $n(\mathbf{A})$ is independent of the particular choice of generating algebra for $\mathcal{V}$.
- $\mathbf{A} \mapsto n(\mathbf{A})$ is functorial (for the appropriate categories).

Theorem

- $n(\mathbf{A}) \cong \mathrm{pro}(\mathbf{A})$.
- If $\mathbf{M}$ is lattice-based, then (the underlying lattice of) $n(\mathbf{A})$ is a canonical extension of (the underlying lattice of) $\mathbf{A}$.

(J. Harding (2006) reconciled canonical extension and profinite completion in this setting.)

The finitely generated case is NICE

For $\mathbf{A} \in \mathcal{V}$, the variety generated by a finite lattice-based algebra $\mathbf{M}$.

- Hardwired into the construction: the non-lattice operations of $n(\mathbf{A})$ are obtained 'naturally', by pointwise lifting, and $\mathbf{A} \mapsto n(\mathbf{A})$ is clearly functorial.
  For canonical extension enthusiasts: because $\mathcal{V}$ is finitely generated, each basic operation f is smooth, that is, has an extension $f^\delta := f^\sigma = f^\pi$ (no decision on which extension to take!) Smoothness happens because the extensions are continuous in the interval topology. —ensures best possible behaviour of extended basic operations under weakest possible assumptions.
- In the distributive case (at least) the canonical extension of $\mathbf{A}$ is isomorphic (as an algebra; not just as a lattice) to $n(\mathbf{A})$.
- $\mathcal{V}$ is **canonical** (proved earlier by M. Gehrke & B. Jónsson).

Topology-swapping in general

Let $\mathbf{M}$ be a finite lattice-based algebra and $\mathcal{V} = \mathbb{ISP}(\mathbf{M})$. A natural duality can be set up so we have a dual equivalence between $\mathcal{V}$ and a category $\mathcal{Y} = \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{Y}})$ of topologised relational structures.

Question Do we have a 'partner' duality obtained by topology-swapping, in the same way that the Banaschewski duality partners the Priestley duality in the case $\mathcal{V} = \mathcal{D}$? (If so, we get the canonical extension 'naturally'.)

To do this an extension of natural duality theory from algebras to structures is needed (B. Davey (2006)).

- A rather general topology-swapping theorem can be proved if the duality for $\mathcal{V}$ is strong and ... [too technical to state in full]. This theorem applies, eg, to f.g. varieties of orthomodular lattices.

Not the end of the story

More for another day ...

- Beyond the lattice case to canonical extensions of poset-based algebras (M. Dunn, M. Gehrke & A. Palmigiano (2005)) via natural dualities (where such exist).
- Relational and topologico-relational semantics via natural dualities (will be based on structures which are defined as functions rather than as sets). Developing Sahlqvist theory in this setting.
- Beyond the finitely generated case: natural duality theory has a contribution to make for certain non finitely-generated varieties of lattice- and semilattice-based algebras.