Which commutative chain basic algebras are MV-algebras

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Outline

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Introduction

The concept of **MV-algebra** as an algebraic axiomatization of the Lukasiewicz many-valued propositional logic was introduced by C.C. Chang.

In the Nineties, the Slovak school of quantum structures generalized the concept of MV-algebra with the concept of **D-poset** or equivalently with the concept of **effect algebra**.

Another approach was recently used by the Olomouc school, namely by R. Halaš, I.Chajda and J.Kühr. They introduced the concept of **lattice with section antitone involutions** and some special class of lattices with section antitone involutions called **basic algebras**.

The aim of this lecture is to establish some connections between this structures using a common generalization of them.

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Definition

An MV-algebra is an algebra $A = (A, \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities:

(MV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(MV2) $x \oplus y = y \oplus x$

(MV3) $x \oplus 0 = x$

(MV4) $\neg\neg x = x$

(MV5) $x \oplus \neg 0 = \neg 0$

(MV6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Definition

A *difference on* a bounded poset $(P, \leq, 0, 1)$ is a partial binary operation $\ominus$ on P such that $b \ominus a$ is defined if and only if $a \leq b$ subject to conditions

- (D1) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.
- (D2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A *D-poset* is a bounded poset with a difference.

Basic definitions – lattices with section antitone involutions

Definition

A lattice with section antitone involutions is a system $L = (L, \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order filter $[a, 1]$ (called a section) possesses an antitone involution $x \mapsto x^a$.

The family $({}^a)_{a \in L}$ of section antitone involutions being partial unary operations on L can be equivalently replaced by a single binary operation $\to$ defined by $x \to y := (x \vee y)^y$.

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Proposition

(i) *Let* $L = (A, \vee, \wedge, ({}^a)_{a\in A}, 0, 1)$ *be a lattice with section antitone involutions. Then the assigned algebra* $\mathscr{A}(L) = (L, \oplus, \neg, 0)$*, where* $x \oplus y := (x^0 \vee y)^y$ *and* $\neg x := x^0$ *satisfies the identites*

- (A1) $x \oplus 0 = x$
- (A2) $\neg\neg x = x$
- (A3) $x \oplus 1 = 1 \oplus x = 1$
- (A4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$
- (A5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1.$

(ii) *Conversely, given an algebra* $A = (A, \oplus, \neg, 0)$ *satisfying the identites (A1)-(A5), then for every* $a \in A$*, the mapping* $x \mapsto x^a := \neg x \oplus a$ *is an antitone involution on the section* $[a, 1]$*, and the structure* $\mathscr{L}(A) = (A, \vee, \wedge, ({}^a)_{a\in A}, 0, 1)$ *is a lattice with section antitone involutions.*

(iii) *The corespondence is one-to-one, i.e.* $\mathscr{L}(\mathscr{A}(L)) = L$ *and* $\mathscr{A}(\mathscr{L}(A)) = A$*.*

Definition

Algebras satisfying the identities (A1)-(A5) are called *basic algebras*.

Given a basic algebra A and $x, y \in A$, the elements x, y are said to *commute* if $x \oplus y = y \oplus x$ holds. If every two elements of A commute then A is called a *commutative basic algebra*.

A basic algebra A is called *complete* if the underlying lattice $\mathscr{L}(A)$ is complete. A is said to be a *chain basic alebra* whenever $\mathscr{L}(A)$ is a chain.

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Basic definitions – Commutative basic algebras

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Commutative basic posets

Definition

A *bounded poset with section antitone involutions* (shortly a *basic poset*) is a system $P = (P, \leq, (^a)_{a \in P}, 0, 1)$ where $(P, \leq, 0, 1)$ is a bounded poset such that every principal order filter $[a, 1]$ possesses an antitone involution $x \mapsto x^a$.

We shall sometimes denote by $a^\perp$ the element a^0. A basic poset P is called *commutative* if, for all $a \leq b$, we have $b^a = (a^\perp)^{(b^\perp)}$.

Lemma

Let P be a commutative basic poset and let $a, x, y \in P$, $a \leq x, y$. Then $x \geq y$ if and only if $a^x \geq a^y$.

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Commutative basic posets and lattices

The following proposition shows relations between commutative basic algebras and commutative basic posets which are lattices.

Proposition

Let P be a commutative basic poset. Then the following conditions are equivalent:

1. *P is a commutative basic algebra.*
2. *P is a lattice such that, for all $a, b \in P$, we have $a^{a \wedge b} = (a \vee b)^b$.*

Corollary

Any commutative chain basic poset is a commutative basic algebra.

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The compatibility relation

The preceding proposition indicates the importance of the compatibility relation $a \leftrightarrow b$ iff $a^{a \wedge b} = (a \vee b)^b$ known for example from the theory of effect algebras.

Problem

There arises a natural question whether any lattice that is a commutative basic poset is a set-theoretical union of its blocks (maximal subsets of mutually compatible elements).

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We have the following easy observation.

Proposition

Horizontal sums of commutative basic posets are commutative basic posets.

In contrast to commutative basic algebras we have:

Corollary

There is a commmutative basic poset that is a non-distributive lattice. In fact, such an example is the diamond or the pentagon.

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1. Introduction
2. Commutative basic posets
3. WD-posets
4. Applications and related results

WD-posets

Definition

A *weak difference on* a bounded poset $(P, \leq, 0, 1)$ is a partial binary operation $\ominus$ on P such that $b \ominus a$ is defined if and only if $a \leq b$ subject to conditions

(WD1) If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.

(WD2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(1 \ominus a) \ominus (1 \ominus b) = b \ominus a$.

A *WD-poset* is a bounded poset with a weak difference.

Note that there is an example of a complete chain WD-poset that is not a D-poset.

In what follows, we will sometimes write $b \ominus a$, tacitly supposing that the latter expression is defined.

Lemma

Let $P = (P, \leq, 0, 1, \ominus)$ be a WD-poset and $a, b \in P$. If $a \leq b$, then $b \ominus a \leq 1 \ominus a$ and $(1 \ominus a) \ominus (b \ominus a) = 1 \ominus b$.

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The following propositions show relations between commutative basic posets and WD-posets.

Proposition

Let $P = (P, \leq, (^a)_{a \in P}, 0, 1)$ be a commutative basic poset. Define a partial binary operation $\ominus$ on P by $b \ominus a = (b^a)^\perp$ for $a, b \in P, a \leq b$. Then $\ominus$ is a weak difference on P and P is a WD-poset.

Proposition

Let P be a WD-poset. Define, for all $a, b \in P$, $a \leq b$, $b^a := 1 \ominus (b \ominus a)$. Then the system $P = (P, \leq, (^a)_{a \in P}, 0, 1)$ is a commutative basic poset.

Commutative basic posets and WD-posets

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Let $\mathbf{P} = (P, \leq, (^a)_{a \in P}, 0, 1)$ *be a commutative basic poset. Define a partial binary operation* $\ominus$ *on* P *by* $b \ominus a = (b^a)^\perp$ *for* $a, b \in P, a \leq b$. *Then* $\ominus$ *is a weak difference on* P *and* P *is a WD-poset.*

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Related results

Halaš and Botur have shown that any finite commutative basic algebra is a MV-algebra.

This can be improved as follows:

Theorem

Every atomic Archimedean commutative basic algebra A is associative and therefore an MV-algebra.

Problem

It is possible to omit the assumption to be Archimedean in the above theorem?

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