Nabla Algebras and Chu Spaces

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(joint work with Yde Venema)

TANCL'07, Oxford, 8 August 2007

Algebraic and Coalgebraic study on Nabla

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- Vietoris construction reformulated as P-lifting.

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- Vietoris construction reformulated as P-lifting.
- Key for generalizing Vietoris construction to arbitrary set functors.

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- As a *logical connective*: Barwise & Moss on circularity, Janin & Walukiewicz on automata-theory, modal μ-calculus.

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Intuition: The tuple (σ, ∇) is a morphism of suitable Chu spaces.

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- Intrinsic, non roundabout axiomatization for ∇;
- $\Box$ and $\Diamond$ defined as in (1) are normal modal operators.

∇-algebras, negation free

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$\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \nabla \rangle$ is a *positive modal ∇-algebra* if its lattice reduct is a BDL and $\nabla : \mathrm{P}_\omega(A) \to A$ satisfies:

∇1. If $\alpha \overline{\mathrm{P}}(\leq)\beta$, then $\nabla\alpha \leq \nabla\beta$,

∇2. If $\bot \in \alpha$, then $\nabla\alpha = \bot$,

∇3. $\nabla\alpha \wedge \nabla\beta \leq \bigvee\{\nabla\{a \wedge b \mid (a, b) \in Z\} \mid Z \in \alpha \bowtie \beta\}$,

∇4. If $\top \in \alpha \cap \beta$, then
$\nabla\{a \vee b \mid a \in \alpha, b \in \beta\} \leq \nabla\alpha \vee \nabla\beta$,

∇5. $\nabla\varnothing \vee \nabla\{\top\} = \top$,

∇6. $\nabla\alpha \cup \{a \vee b\} \leq$
$\nabla(\alpha \cup \{a\}) \vee \nabla(\alpha \cup \{b\}) \vee \nabla(\alpha \cup \{a, b\})$.

A more compact equivalent axiomatization

$A \in \mathrm{P}\mathrm{P}_\omega(Fm), B \in \mathrm{P}_\omega \mathrm{P}(Fm),$

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∇2′. $\bigwedge\{\nabla\alpha \mid \alpha \in A\} \ \leq \ \bigvee\{\nabla\{\bigwedge\beta \mid \beta \in B\} \mid \bigcup B \subseteq \bigcup A,$ and for every $\alpha \in A,\ \alpha\overline{\mathrm{P}}(\in)B)\}$,

∇3′. $\nabla\{\bigvee\alpha \mid \alpha \in B\} \ \leq \ \bigvee\{\nabla\gamma \mid \gamma\overline{\mathrm{P}}(\in)B\}$.

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- its ∇-free reduct is a BA
- it satisfies ∇1 – ∇6 (or equivalently ∇1, ∇2′, ∇3′),
- and in addition,

$$\nabla 7.\ \ \neg \nabla \alpha = \nabla\{\bigwedge \alpha, \top\} \vee \nabla \varnothing \vee \bigvee \{\nabla\{a\} \mid a \in \alpha\}.$$

Axiomatic equivalences

Using the stipulations

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one shows:

Theorem

- The categories PMA and PMA_∇ are isomorphic.
- The categories MA and MA_∇ are isomorphic.

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$\mathbb{S} = \langle X, S, A\rangle$ and $\mathbb{T} = \langle Y, T, B\rangle$, a Chu transform $\mathbb{S} \to \mathbb{T}$ is a pair $(f : X \to Y, g : B \to A)$ s.t. the (generalized) *adjointness condition* holds

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∇ and Chu, coalgebraically

$$s \Vdash \nabla\Phi \iff \sigma(s)\overline{\mathrm{P}}(\Vdash)\Phi.$$

(σ, ∇) is a *Chu transform* $(S, \Vdash, Fm) \to (\mathrm{P}S, \overline{\mathrm{P}}(\Vdash), \mathrm{P}_\omega(Fm))$.

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Fact: If F preserves weak pullbacks, then $\widetilde{\mathrm{F}}S := \langle \mathrm{F}(X), \overline{\mathrm{F}}(S), \mathrm{F}(A) \rangle$ is an endofunctor on Chu.

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Let $S = \langle X, \in, A \rangle$ be a Stone space:

- The Vietoris space $V(S) = \langle K(X), \in, V(A) \rangle$ can be realized as an instance of a P-lifting construction (via a normalization step).
- $V(A)$ is isomorphic to

$$\mathrm{BA}\langle \ \{\nabla\alpha \mid \alpha \in \mathrm{P}_\omega A\} : \nabla 1 - \nabla 7 \ \rangle$$

Conclusions

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- Case study: Vietoris construction on Stone spaces reformulated as a P-lifting construction for the (finite) power set functor.

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- On Chu spaces: we gave F-lifting constructions on Chu, which are functorial in case F preserves weak pullbacks.
- Case study: Vietoris construction on Stone spaces reformulated as a P-lifting construction for the (finite) power set functor.
- We linked this reformulation to the axiomatization of the modal ∇-algebras.