

Coalgebraic Semantics of
Non-Classical Logics

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Coalgebras generalise relational structures

Universal Coalgebra ⁽¹⁾ studies these **uniformly**

Coalgebraic Logic: **uniform** proofs of
completeness, decidability,
complexity of satisfiability, ...

(1) [Rotten, 2000]

Overview

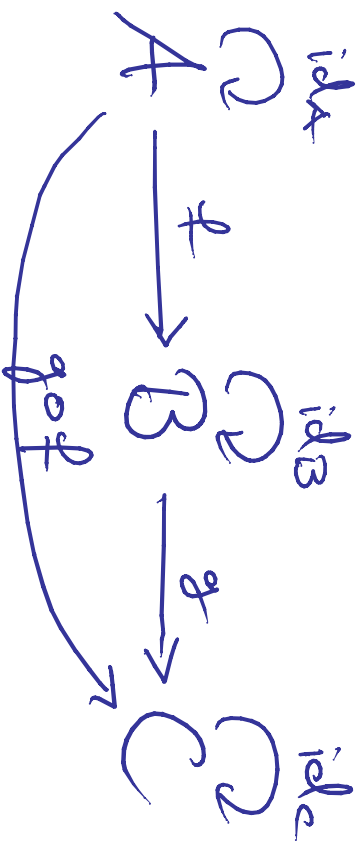
1. Coalgebras
2. Coalgebraic Logic
3. Some results

Coalgebras of type T

For each type T a notion of

- transition system (relational structure)
- bisimilarity (behavioural / observational equiv.)
- final coalgebra
- dual algebra, of type T^{op}

category :



functor T :

$$TA \xrightarrow{TF} TB$$

$$T(id_A) = id_{TA}$$

$$T(g \circ f) = Tg \circ Tf$$

isomorphism :



- the inverse f' is uniquely determined
- T preserves isomorphisms

Coalgebras $\text{Coalg}(T)$

category \mathcal{R} , functor $T: \mathcal{R} \rightarrow \mathcal{R}$

coalgebra: $X \rightarrow TX$

morphism from $X \rightarrow TX$ to $X' \rightarrow TX'$:

$$\begin{array}{ccc} X & \longrightarrow & TX \\ \downarrow f & & \downarrow Tf \\ X' & \longrightarrow & TX' \end{array}$$

Examples

$$\mathcal{X} = \mathcal{S}et$$

$X \rightarrow \mathcal{S}X$ Kripke frames ⁽¹⁾ (\mathcal{S} for powerset)

$X \rightarrow 2^{2^{(X)}}$ neighbourhood frames

$X \rightarrow K \times X$ streams (infinite lists) over K

$X \rightarrow 2 \times X^K$ deterministic automata

$X \rightarrow K_1 \times (\mathcal{S}X)^{K_2}$ Kripke models

$X \rightarrow 1 + \mathcal{D}X$ (\mathcal{D} for probability distributions)

(1) with bounded morphisms (aka ρ -morphisms)

Examples

$\mathcal{R} = \text{Stone}$

\mathcal{P} extends from finite sets to $\overline{\mathcal{P}}$: Stone \rightarrow Stone
 $X \rightarrow \overline{\mathcal{P}X}$ descriptive general frame

$\mathcal{R} = \text{Priestley}$

The convex-subsets functor Conv extends from finite posets to $\overline{\text{Conv}}$: Priestley \rightarrow Priestley
 $X \rightarrow \overline{\text{Conv} X}$ K^+ -spaces of Celani/Jansana

Algebras are Dual to Coalgebras

Remark:

\mathcal{A} is dual to \mathcal{A}^{op}

T is dual to T^{op}

$\text{Alg}(T)$ is dual to $\text{Coalg}(T^{op})$

algebra $T X \rightarrow X$

$\mathcal{A} = \text{Set}$:

algebras for a signature are T -algebras

T -algebras are algebras for a signature

and equations $\text{op}_1(x_1, \dots, x_n) \approx \text{op}_2(y_1, \dots, y_n)$
of "rank 1"

Examples

Set^{op} \cong CABA (complete atomic Bool. alg.)

Stone^{op} \cong BA (Boolean algebras)

Priestley^{op} \cong DL (bounded distr. lattices)

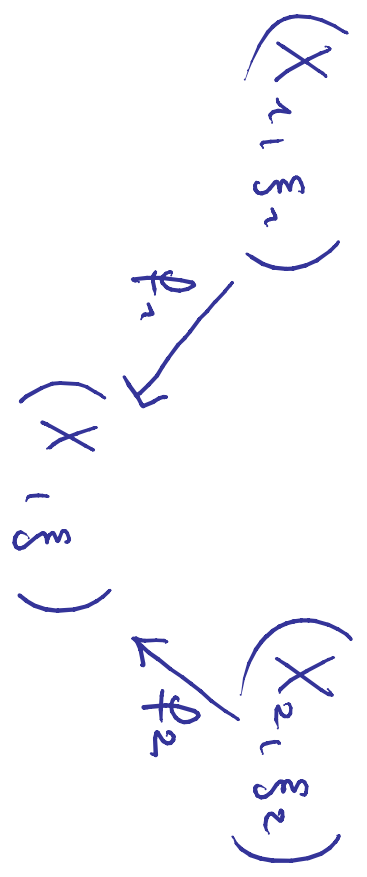
⋮

many more are studied eg in Domain Theory,
see the work of Abramsky, Jung, Hoshier, ...

Bisimilarity, Behavioural / Observational Equivalence

Given $X_1 \xrightarrow{\xi_1} TX_1$, $X_2 \xrightarrow{\xi_2} TX_2$, $x_1 \in X_1$, $x_2 \in X_2$
say that $X_1 \cong X_2$ (X_1 bisimilar to X_2)

if there are coalgebra morphisms f_1, f_2



such that $f_1(x_1) = f_2(x_2)$

Final (cofree) Coalgebras

$Z \xrightarrow{f} T Z$ is final in $\text{Coalg}(T)$

if for all $X \rightarrow T X$ there is a unique

morphism

$$\begin{array}{ccc} X & \longrightarrow & T X \\ \downarrow b & & \downarrow T b \\ Z & \longrightarrow & T Z \end{array}$$

b identifies precisely the bisimilar elements of X

Terminology

$X \rightarrow TX$ a system

$x \in X$ a state

$(X \rightarrow TX, x)$ a process

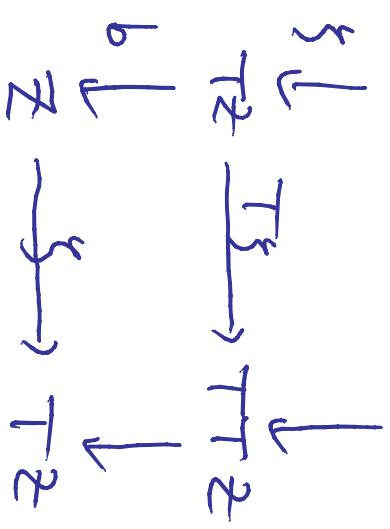
bisimilarity is an equivalence relation on processes
final coalgebra Z contains all processes up to \approx
 $X \xrightarrow{b} Z$ maps a process to its behaviour

Final Coalgebras generalise largest fixed points

Lambek's Lemma: The structure map ζ of a

final coalgebra $Z \xrightarrow{\zeta} TZ$ is an isomorphism

Proof: $Z \xrightarrow{\zeta} TZ$



Remark: Z solves

the 'domain equation'

$$Z \cong TZ$$

Examples of Final Coalgebras

$\mathbb{Z} \rightarrow \mathcal{P}\mathbb{Z}$ non-well founded sets [Aczel, 1987]

$\mathbb{Z} \rightarrow \overline{\mathcal{P}}\mathbb{Z}$ canonical model [Abramsky, 1989]

$\mathbb{Z} \rightarrow K \times \mathbb{Z}$ infinite lists (eg Rutten)

$\mathbb{Z} \rightarrow 2 \times \mathbb{Z}^K$ elements of \mathbb{Z} are languages

$X \xrightarrow{b} \mathbb{Z}$ maps a state $x \in X$
to the accepted language $b(x)$

[Rutten, 1998]

Part 2 Coalgebraic Logic

Aim: • Develop logics for coalgebras uniformly in T .

$$T ::= \text{Id} \mid K \mid \mathcal{B} \mid 2^{_} \mid \mathcal{D} \mid \dots \mid T+T \mid T \times T \mid T \circ T$$

- Results should be compositional (compose logics and proof systems)

The Logic of Moss (1997)

given $T: \text{Set} \rightarrow \text{Set}$ preserving weak pullbacks

language \mathcal{L} : $\frac{\Phi \in T\mathcal{L}}{\nabla\Phi \in \mathcal{L}} \quad \frac{\Phi \in \mathcal{L}}{\wedge\Phi \in \mathcal{L}}$

Semantics: given $X \xrightarrow{f} TX$ define $\Vdash \subseteq X \times \mathcal{L}$
via $x \Vdash \nabla\Phi$ if $(\xi^{(x)}, \Phi) \in T(\Vdash)$

Theorem:

- formulas are invariant under bisimilarity
- $\forall x. \exists! \rho_x. \forall y. Y \Vdash \rho_x \Rightarrow Y \simeq x$

Example: $T = \mathcal{S}$ $\nabla \Phi = \square \vee \Phi \wedge \nabla \Phi$

Reasons to look for other categorical logics:

- ▷ not convenient for specifying properties
- no proof system / completeness
- not compositional
- weak pullback preservation excludes 2^2
- how to generalise from Set

Coalgebraic Logic via Duality

Logic for T -coalgebras is given by

$$LGLBA \leftarrow P \text{ Set} \rightarrow T$$

BAL, DL, HA, Set, \dots

$$LP \xrightarrow{\delta} PT$$

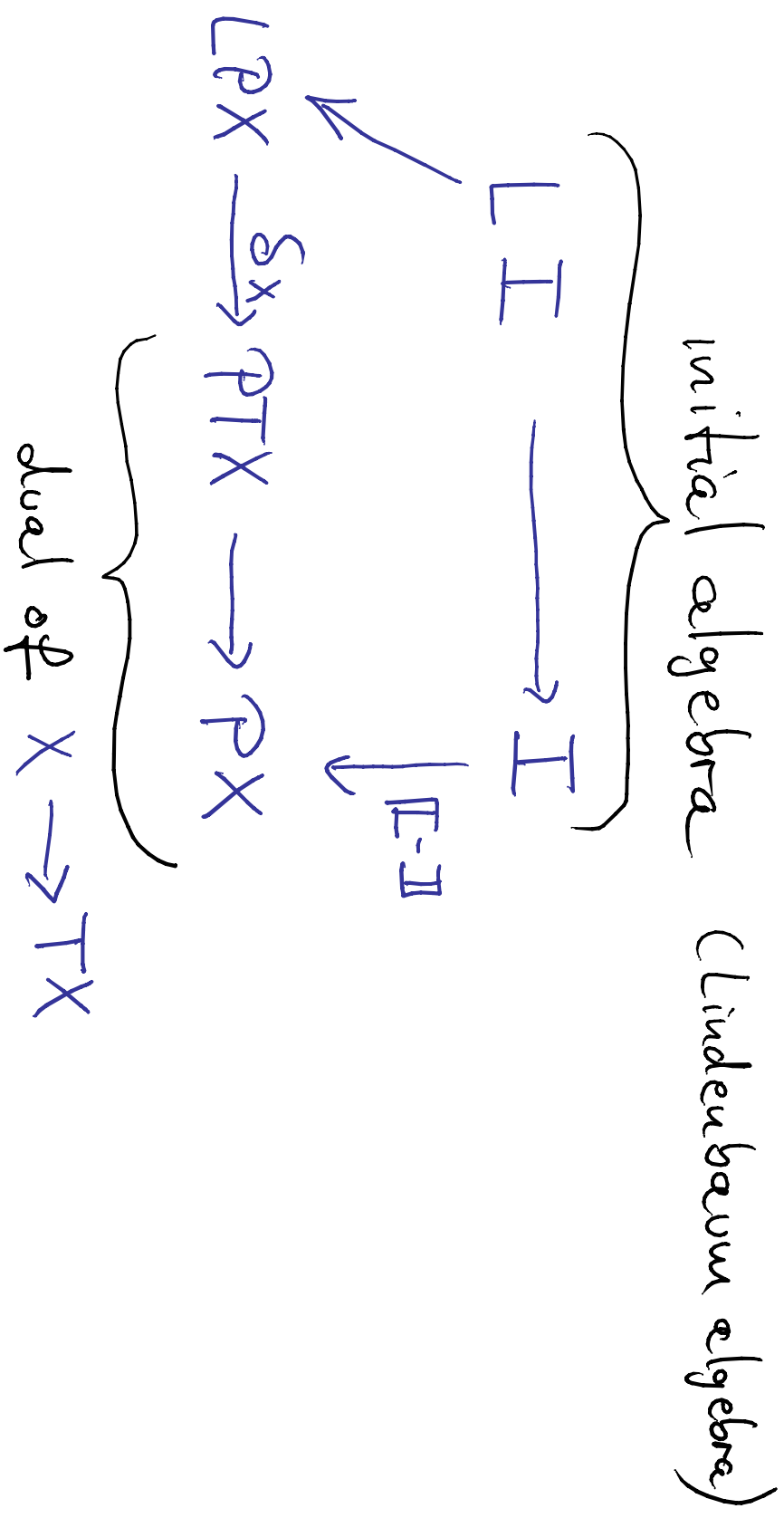
$Set, Stone, \dots$

δ injective means completeness

δ surjective means expressiveness

[Rößiger, Gumm, Jacobs, Cirstea, Pattinson, Bonsangue
Rosicky, Kupke, Palmigiano, Venema, Schröder, Klin, ...]

Semantics



Example

$$\mathbb{L}_K \cong BA \longrightarrow BA$$

$\mathbb{L}_K(A)$ generated by $\square a, a \in A$
mod out by $\square T = T$
 $\square (a \vee b) = \square a \wedge \square b$

$$\mathbb{L}_K P X \xrightarrow{S_X} P \mathcal{B} X$$
$$\square a \longmapsto \{ b \in X \mid b \leq a \}$$

Fact: $\text{Alg}(L_K) \cong \text{BAO}$

Fact: $L_K \text{PX} \xrightarrow{\cong} \text{PSX}$ for finite X

Similarly: $L_{K^+} : \mathcal{DL} \longrightarrow \mathcal{DL}$

Remark: K^+ is the positive fragment of K :

$$F \begin{matrix} \text{BA} \\ \swarrow \quad \searrow \\ (+) \\ \downarrow \\ \text{DL} \end{matrix} U$$

$$F L_{K^+} \cong L_K F$$

$$L_{K^+} U \cong U L_K$$

Presenting functors by operations and equations

$$L_K : BA \longrightarrow BA$$

$L_K(A)$ generated by $\square a, a \in A$

mod out by

$$\begin{aligned} \square T = T \\ \square (a \vee b) = \square a \wedge \square b \end{aligned}$$

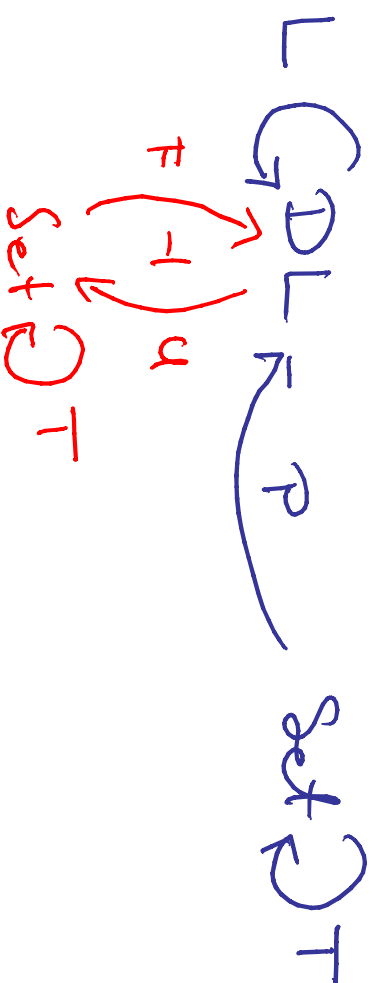
Theorem: ⁽¹⁾

Any finitary $L : BA \longrightarrow BA$ has a finitary presentation by equations of rank 1

Remark: Generalises from BA to any variety

(1) [K1 Resicky 06]

Moss's logic revisited



$$L = FTV$$

LA generated by

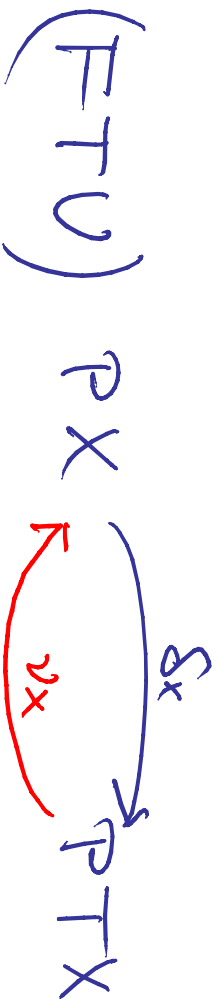
$$\forall \Phi, \Psi \in TVA$$

$$L PX = (FTV)PX \xrightarrow{\delta^x} P TX$$

is given by a certain natural transformation

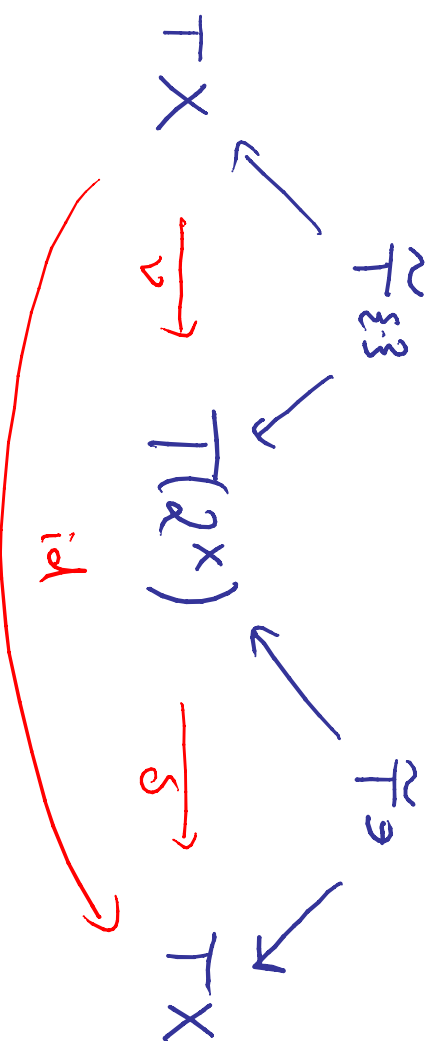
$$T(2^x) \longrightarrow 2^{TX}$$

expressiveness, completeness:



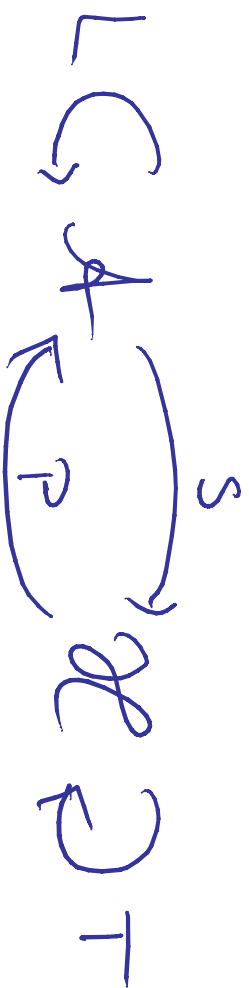
$2x$ maps singletons via $TX \xrightarrow{T\{t\}} T(2^x)$

$$\underline{S_x(2x(\{t\})) = \{t\} :}$$



gives a method
to prove completeness
(don't work with V up to, Venema)

Part 3 Some Results



$$\text{If } A \approx PSA$$

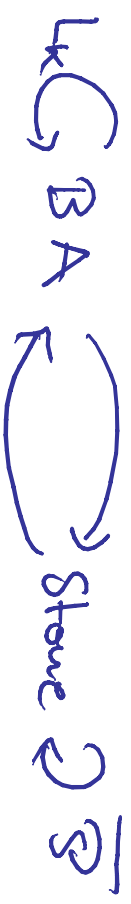
$$X \approx SPX$$

$$LP \approx PT$$

$$\text{then } Alg(L) \approx Coalg(CT)^{op}$$

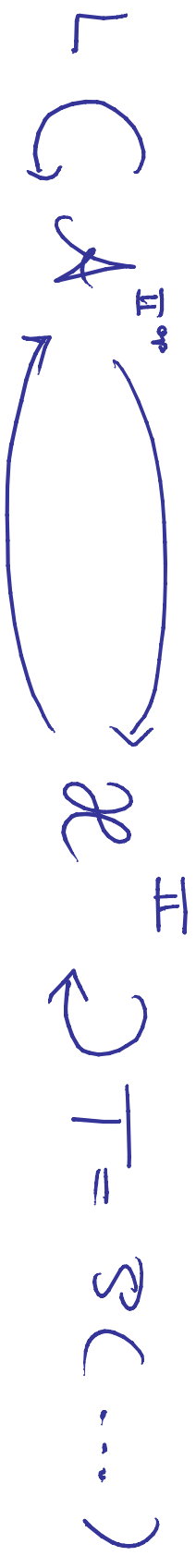
Thm: Completeness | expressiveness

Example (Goldblatt '76):



Domain Theory in Logical Form (Abramsky '87)

Example: π -calculus in logical form (1)



describing λ by operations and equations

gives a complete and expressive logic for π -calculus

(1) [Bonsangue, K, LICS 07]

Adjunction instead of equivalence



Given T obtain $LP \longrightarrow PL$

Completeness ? Definability ?

Suppose T preserves finite sets (+ a technical condition)

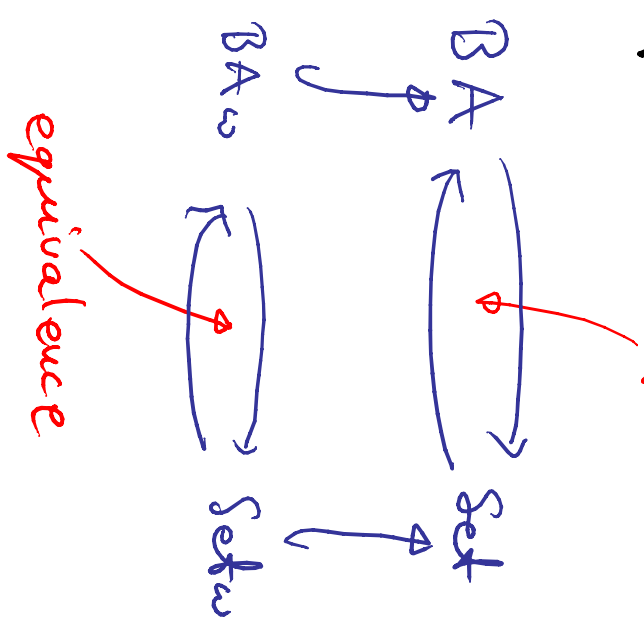
Jonsen - Tarski - Tim for coalgebras:

Every L -algebra can be embedded into the complex algebra of a T -coalgebra.
(\leadsto strong completeness of L w.r.t T)

Goldblatt - Thomason - Tim for coalgebras:

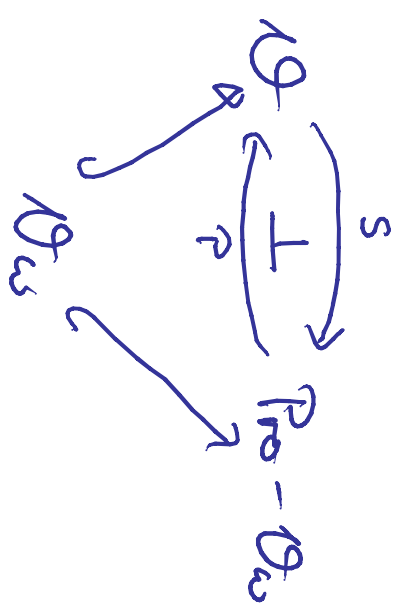
A class of coalgebras closed under ultrafilter extensions is modally definable iff it is closed under S, H, U and reflects ultrafilter extensions

Proof: adjunction



generalises to:

- \mathcal{V} locally finite variety
- \mathcal{A}_ω class of finite algebras
- $Pro\text{-}\mathcal{A}_\omega$ pro-completion



[K, Rosicky, 06, 07]

Final / Initial Sequence

$1 \leftarrow T_1 \leftarrow \dots \leftarrow T_{\omega} 1$
approximates the final coalgebra

$0 \rightarrow L_0 \rightarrow \dots \rightarrow L_{\omega} 0$ the Lindenbaum algebra for a finitary language

- completeness for arbitrary T
- methods of Ghilardi 92, 95 work for logics of rank 1
- de Rijke's Lindström theorem generalises

Conclusion

The leading question was :

What results in Modal Logic on
completeness, decidability, complexity, ...
generalise from Kripke Frames to Coalg (CT)

Aims for the future:

- More applications to modal logic
- Systematic study of the 'coalgebraic dimension'