Coalgebraic Semantics of Non-Classical Logics

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Coalgebras generalise relational structures

Universal Coalgebra[1] studies these uniformly

Coalgebraic Logic: uniform proofs of
completeness, definability,
complexity of satisfiability, ...

(1) [Rutten, 2000]

Overview

1. Coalgebras

2. Coalgebraic Logic

3. Some results

Coalgebras of type T

For each type T a notion of

- transition system (relational structure)
- bisimilarity (behavioural / observational equiv.)
- final coalgebra
- dual algebra, of type T^{op}

category:

$$\begin{tikzcd} A \arrow[loop above, "id_A"] \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"] & B \arrow[loop above, "id_B"] \arrow[r, "g"] & C \arrow[loop above, "id_C"] \end{tikzcd}$$

isomorphism:

$$\begin{tikzcd} A \arrow[loop left, "id_A"] \arrow[r, bend left, "f"] & B \arrow[loop right, "id_B"] \arrow[l, bend left, "f'"] \end{tikzcd}$$

- the inverse f' is uniquely determined

- T preserves isomorphisms

functor T:

$$TA \xrightarrow{Tf} TB$$

$$T(id_A) = id_{TA}$$

$$T(g \circ f) = Tg \circ Tf$$

Coalgebras $\mathbf{Coalg}(T)$

category $\mathscr{X}$, functor $T : \mathscr{X} \longrightarrow \mathscr{X}$

coalgebra: $X \longrightarrow TX$

morphism from $X \to TX$ to $X' \longrightarrow TX'$:

$$X \xrightarrow{f} X' \qquad \begin{array}{ccc} X & \longrightarrow & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ X' & \longrightarrow & TX' \end{array}$$

Examples $\qquad \mathcal{X} = \mathbf{Set}$

$X \longrightarrow \mathcal{P}X$ Kripke frames[1] ($\mathcal{P}$ for powerset)

$X \longrightarrow 2^{(2^X)}$ neighbourhood frames

$X \longrightarrow K \times X$ streams (infinite lists) over K

$X \longrightarrow 2 \times X^K$ deterministic automata

$X \longrightarrow (K_1 \times (\mathcal{P}X))^{K_2}$ Kripke models

$X \longrightarrow 1 + \mathcal{D}X$ ($\mathcal{D}$ for probability distributions)

(1) with bounded morphisms (aka p-morphisms)

Examples $\mathcal{H} = \mathbf{Stone}$

$\mathcal{P}$ extends from finite sets to $\overline{\mathcal{P}} : \mathrm{Stone} \to \mathrm{Stone}$

$X \longrightarrow \overline{\mathcal{P}}X$ descriptive general frame

$\mathcal{H} = \mathbf{Priestley}$

The convex-subsets functor Conv extends from finite posets to $\overline{\mathrm{Conv}} : \mathrm{Priestley} \to \mathrm{Priestley}$

$X \longrightarrow \overline{\mathrm{Conv}}\,X$ K^+-spaces of Celani/Jansana

Algebras are Dual to Coalgebras

algebra $TX \to X$

Remark:

$\mathcal{X}$ is dual to $\mathcal{X}^{op}$

T is dual to T^{op}

$\mathrm{Alg}(T)$ is dual to $\mathrm{Coalg}(T^{op})$

$\mathcal{X} = \mathsf{Set}$:

algebras for a signature are T-algebras

T-algebras are algebras for a signature

and equations $op_1(x_1, \dots x_m) \approx op_2(y_1, \dots y_n)$

of "rank 1"

Examples

$\mathbf{Set}^{op} \simeq \mathbf{CABA}$ (complete atomic Bool. alg.)

$\mathbf{Stone}^{op} \simeq \mathbf{BA}$ (Boolean algebras)

$\mathbf{Priestley}^{op} \simeq \mathbf{DL}$ (bounded distr. lattices)

$\vdots$

many more are studied eg in Domain Theory, see the work of Abramsky, Jung, Moshier, ...

Bisimilarity, Behavioural / Observational Equivalence

Given $X_1 \xrightarrow{\xi_1} TX_1$, $X_2 \xrightarrow{\xi_2} TX_2$, $x_1 \in X_1$, $x_2 \in X_2$

say that $x_1 \simeq x_2$ (x_1 bisimilar to x_2)

if there are coalgebra morphisms f_1, f_2

$$\begin{tikzcd}
(X_1, \xi_1) \arrow[dr, "f_1"'] & & (X_2, \xi_2) \arrow[dl, "f_2"] \\
& (X, \xi) &
\end{tikzcd}$$

such that $f_1(x_1) = f_2(x_2)$

Final (cofree) Coalgebras

$Z \xrightarrow{\xi} TZ$ is <u>final</u> in $\mathrm{Coalg}(T)$

if for all $X \longrightarrow TX$ there is a unique

morphism

$$\begin{tikzcd} X \arrow[r] \arrow[d, "b"'] & TX \arrow[d, "Tb"] \\ Z \arrow[r] & TZ \end{tikzcd}$$

b identifies precisely the bisimilar elements of X

Terminology

$X \to TX$	a system
$x \in X$	a state
$(X \to TX, x)$	a process

bisimilarity is an equivalence relation on processes

final coalgebra Z contains all processes up to $\simeq$

$X \xrightarrow{b} Z$ maps a process to its behaviour

Final Coalgebras generalise largest fixed points

Lambek's Lemma: The structure map ξ of a final coalgebra $Z \xrightarrow{\xi} TZ$ is an isomorphism

Proof:

$$\begin{tikzcd}
Z \arrow[r, "\xi"] \arrow[d, "\xi"'] & TZ \arrow[d] \\
TZ \arrow[r, "T\xi"] \arrow[d, "b"'] & TTZ \arrow[d] \\
Z \arrow[r, "\xi"] & TZ
\end{tikzcd}$$

Remark: Z solves the 'domain equation'

$$Z \cong TZ$$

Examples of Final Coalgebras

$Z \longrightarrow \mathcal{P}Z$ non-well founded sets [Aczel, 1989]

$Z \longrightarrow \overline{\mathcal{P}}Z$ canonical model [Abramsky, 1989]

$Z \longrightarrow K \times Z$ infinite lists (eg Rutten)

$Z \longrightarrow 2 \times Z^K$ elements of Z are languages

$X \xrightarrow{b} Z$ maps a state $x \in X$ to the accepted language $b(x)$

[Rutten, 1998]

Part 2 Coalgebraic Logic

Aim:
- Develop logics for coalgebras uniformly in T.

$$T ::= \mathrm{Id} \mid K \mid \mathcal{P} \mid 2^{2^{-}} \mid \mathcal{D} \mid \ldots \mid T + T \mid T \times T \mid T \circ T$$

- Results should be compositional (compose logics and proof systems)

The logic of Moss (1997)

given $T : \mathsf{Set} \longrightarrow \mathsf{Set}$ preserving weak pullbacks

language $\mathcal{L}$: $\dfrac{\Phi \in T\mathcal{L}}{\nabla\Phi \in \mathcal{L}}$ $\dfrac{\Phi \subseteq \mathcal{L}}{\bigwedge\Phi \in \mathcal{L}}$

semantics: given $X \xrightarrow{\xi} TX$ define $\Vdash \subseteq X \times \mathcal{L}$

via $x \Vdash \nabla\Phi$ if $(\xi(x), \Phi) \in T(\Vdash)$

Theorem:
- formulas are invariant under bisimilarity
- $\forall x . \exists \varphi_x . \forall y . \; y \Vdash \varphi_x \Rightarrow y \simeq x$

Example: $T = \mathcal{P}$ $\quad \nabla \Phi = \Box \bigvee \Phi \wedge \bigwedge \Diamond \Phi$

Reasons to look for other coalgebraic logic:

∇ not convenient for specifying properties

no proof system / completeness

not compositional

weak pullback preservation excludes $2^{2^{-}}$

how to generalise from Set

Coalgebraic Logic via Duality

A logic for T-coalgebras is given by

$$\begin{tikzcd} L \arrow[loop left] \; BA & Set \arrow[l, bend left, "P"'] \arrow[loop right, "T"] \end{tikzcd}$$

BA, DL, HA, Set, ...

$$LP \xrightarrow{\delta} PT$$

Set, Stone, ...

δ injective means completeness

δ surjective means expressiveness

[Rößiger, Gumm, Jacobs, Cirstea, Pattinson, Bonsangue, Rosicky, Kupke, Palmigiano, Venema, Schröder, Klin, ...]

Semantics

initial algebra (Lindenbaum algebra)

$$\begin{tikzcd}
 & LI \arrow[r] \arrow[dl] & I \arrow[d, "{[\![-]\!]}"] \\
LPX \arrow[r, "\delta_X"] & PTX \arrow[r] & PX
\end{tikzcd}$$

dual of $X \to TX$

Example

$$L_K : BA \longrightarrow BA$$

$L_K(A)$ generated by $\Box a,\ a \in A$

mod out by $\Box \top = \top$

$\Box(a \wedge b) = \Box a \wedge \Box b$

$$L_K PX \xrightarrow{\delta_X} P\mathcal{P}X$$

$$\Box a \longmapsto \{ b \subseteq X \mid b \subseteq a \}$$

Fact: $\mathrm{Alg}(L_{\mathbb{K}}) \cong \mathrm{BAO}$

Fact: $L_{\mathbb{K}} \mathcal{P} X \xrightarrow{\cong} \mathcal{P} \breve{\mathcal{P}} X$ for finite X

Similarly: $L_{\mathbb{K}^+} : \mathrm{DL} \longrightarrow \mathrm{DL}$

Remark: $\mathbb{K}^+$ is the positive fragment of $\mathbb{K}$:

$$\begin{tikzcd} \mathrm{BA} \arrow[d, bend left, "U"] \\ \mathrm{DL} \arrow[u, bend left, "F"] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$

$$F L_{\mathbb{K}^+} \cong L_{\mathbb{K}} F$$

$$L_{\mathbb{K}^+} U \cong U L_{\mathbb{K}}$$

Presenting functors by operations and equations

$L_K : BA \longrightarrow BA$

$L_K(A)$ generated by $\Box a$, $a \in A$

mod out by $\Box \top = \top$

$\Box(a \wedge b) = \Box a \wedge \Box b$

Theorem:[1] Any finitary $L : BA \longrightarrow BA$ has a finitary presentation by equations of rank 1

Remark: Generalises from BA to any variety

(1) [K, Rosicky 06]

Moss's logic revisited

$$\begin{tikzcd} \mathrm{DL} \arrow[loop left, "L"] \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] & \mathrm{Set} \arrow[l, bend left, "P"'] \arrow[loop right, "T"] \\ \mathrm{Set} \arrow[u, bend right, "U"'] \arrow[loop right, "T"] & \end{tikzcd}$$

$L = FTU$

LA generated by

$\nabla \Phi$, $\Phi \in TUA$

$$LPX = (FTU)PX \xrightarrow{\delta_X} PTX$$

is given by a certain natural transformation

$$T(2^X) \longrightarrow 2^{TX}$$

expressiveness, completeness:

$$(FTU) \quad \begin{tikzcd} PX \arrow[r, bend left, "\delta_X"] & PTX \arrow[l, bend left, "\nu_X"] \end{tikzcd}$$

ν_X maps singletons via $TX \xrightarrow{T\{\cdot\}} T(2^X)$

$\delta_X(\nu_X(\{t\})) = \{t\}$:

$$\begin{tikzcd} & \tilde{T}\{\cdot\} \arrow[dl] \arrow[dr] & & \tilde{T}\ni \arrow[dl] \arrow[dr] & \\ TX \arrow[rr, "\nu"'] \arrow[rrrr, bend right, "id"'] & & T(2^X) \arrow[rr, "\delta"'] & & TX \end{tikzcd}$$

gives a method
to prove completeness
(joint work with Kupke, Venema)

Part 3 Some Results

$$\begin{tikzcd} L \arrow[loop left] & \mathcal{A} \arrow[r, bend left, "S"] & \mathcal{X} \arrow[l, bend left, "P"] & T \arrow[l, loop] \end{tikzcd}$$

If $\mathcal{A} \cong PS\mathcal{A}$
$X \cong SPX$
$LP \cong PT$

then $\mathrm{Alg}(L) \simeq \mathrm{Coalg}(T)^{op}$

Thm: Completeness, expressiveness

Example (Goldblatt '76): $\begin{tikzcd} L_K \arrow[loop left] & BA \arrow[r, bend left] & \mathrm{Stone} \arrow[l, bend left] & \overline{\mathcal{P}} \arrow[l, loop] \end{tikzcd}$

Domain Theory in Logical Form (Abramsky '87)

Example: π-calculus in logical form (1)

$$L \circlearrowright \mathcal{A}^{\mathbb{I}^{op}} \rightleftarrows \mathcal{X}^{\mathbb{I}} \circlearrowleft T = S(\ldots)$$

describing L by operations and equations gives a complete and expressive logic for π-calculus

(1) [Bonsangue, K, LICS 07]

Adjunction instead of equivalence

$$\begin{tikzcd} \mathrm{LCBA} \arrow[r, bend left, "S"] & \mathrm{Set} \arrow[l, bend left, "P"] \arrow[loop right, "T"] \end{tikzcd} \qquad \begin{aligned} A &\longrightarrow PSA \\ X &\longrightarrow SPX \end{aligned}$$

Given T obtain $LP \longrightarrow PL$

Completeness ? Definability ?

Suppose T preserves finite sets (+ a technical condition)

Jonsson - Tarski - Thm for coalgebras:

Every L-algebra can be embedded into the complex algebra of a T-coalgebra.

($\rightsquigarrow$ strong completeness of L wrt T)

Goldblatt - Thomason - Thm for coalgebras:

A class of coalgebras closed under ultrafilter extensions is modally definable iff it is closed under $S, H, \coprod$ and reflects ultrafilter extensions

Proof:

$$\begin{tikzcd} BA \arrow[r, bend left] & Set \arrow[l, bend left] \\ BA_\omega \arrow[u, hook] \arrow[r, bend left] & Set_\omega \arrow[u, hook] \arrow[l, bend left] \end{tikzcd}$$

adjunction

equivalence

generalises to:

$\mathcal{V}$ locally finite variety

$\mathcal{V}_\omega$ class of finite algebras

$Pro\text{-}\mathcal{V}_\omega$ pro-completion

$$\begin{tikzcd} \mathcal{V} \arrow[rr, bend left, "S"] \arrow[rr, phantom, "\perp"] & & Pro\text{-}\mathcal{V}_\omega \arrow[ll, bend left, "P"] \\ & \mathcal{V}_\omega \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

[K, Rosicky, 06, 07]

Final / Initial Sequence

$1 \leftarrow T1 \leftarrow \cdots \quad T^{\omega}1$ approximates the final coalgebra

$0 \rightarrow L0 \rightarrow \cdots \quad L^{\omega}0$ $L^{\omega}0$ the Lindenbaum algebra for a finitary language

- completeness for arbitrary T
- methods of Ghilardi 92, 95 work for logics of rank 1
- de Rijke's Lindström theorem generalises

Conclusion

The leading question was :

What results in Modal Logic on completeness, definability, complexity, ... generalise from Kripke frames to Coalg(T)

Aims for the future:

- More applications to modal logic
- Systematic study of the "coalgebraic dimension"