On the structure of linear pseudo-BCK-algebras

Jan Kühr

Department of Algebra and Geometry
Faculty of Science
Palacký University at Olomouc
Olomouc, Czech Republic
kuhr@inf.upol.cz

Joint work with Anatolij Dvurečenskij

1. Every linear hoop/BL-algebra is an ordinal sum of linear Wajsberg hoops [Agliano & Montagna]
2. The $\{\to, 1\}$-subreducts of hoops are BCK-algebras satisfying the identity

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A **porim** (= partially ordered residuated integral monoid) is a structure $(A, \leq, \cdot, \to, \rightsquigarrow, 1)$ where

- $(A, \leq)$ is a poset with greatest element 1,
- $(A, \cdot, 1)$ is a monoid,
- $c \leq a \to b$ iff $c \cdot a \leq b$, and $c \leq a \rightsquigarrow b$ iff $a \cdot c \leq b$.

A **pseudo-hoop** [Georgescu, Leuştean & Preoteasa] is a porim satisfying

$$(x \to y) \cdot x = y \cdot (y \rightsquigarrow x).$$

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Pseudo-MV-algebras = bounded Wajsberg pseudo-hoops

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Porims = algebras $(A, \cdot, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ that satisfy:

$$(x \to y) \rightsquigarrow ((y \to z) \rightsquigarrow (x \to z)) = 1, \tag{1}$$
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$$x \to 1 = 1, \tag{5}$$
$$(x \cdot y) \to z = x \to (y \to z), \tag{6}$$
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Let $\mathbf{A} = (A, \to, \rightsquigarrow, 1)$ be a pseudo-BCK-algebra. The relation $\leq$ given by

$$x \leq y \quad \text{iff} \quad x \to y = 1 \quad (\text{iff} \quad x \rightsquigarrow y = 1)$$

is a partial order on A; 1 is the greatest element of $(A, \leq)$.

If $(A, \leq)$ is a chain, then $\mathbf{A}$ is a **linear** pseudo-BCK-algebra.

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$$(x \to y) \rightsquigarrow y = (y \rightsquigarrow x) \to x,$$

and the following "relative cancellation" property:

$$x \geq y \quad \& \quad x \geq z \quad \& \quad x \to y = x \to z \quad \Rightarrow \quad y = z.$$

- RCP can be replaced by the identity

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Let $(I, \leq)$ be a non-empty chain. The **ordinal sum** of linear pseudo-BCK-algebras $\mathbf{A}_i$ $(i \in I)$ such that $A_i \cap A_j = \{1\}$ for all $i \neq j \in I$ is a pseudo-BCK-algebra $\bigoplus_{i \in I} \mathbf{A}_i = (\bigcup_{i \in I} A_i, \rightarrow, \rightsquigarrow, 1)$ where the operations $\rightarrow, \rightsquigarrow$ are defined as follows:

$$x \rightarrow y = \begin{cases} x \rightarrow_i y & \text{if } x, y \in A_i, \\ 1 & \text{if } x \in A_i \setminus \{1\}, y \in A_j, i < j, \\ y & \text{if } x \in A_i, y \in A_j, i > j, \end{cases}$$

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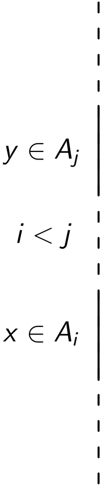

$$x \to y = 1 = x \rightsquigarrow y$$

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Which linear pseudo-BCK-algebras arise as ordinal sums of linear pseudo-ŁBCK-algebras?

A linear pseudo-BCK-algebra is an ordinal sum of linear pseudo-ŁBCK-algebras iff it satisfies the identities

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The identity (H), as well as

$$(x \rightsquigarrow y) \rightsquigarrow (x \rightsquigarrow z) = (y \rightsquigarrow x) \rightsquigarrow (y \rightsquigarrow z), \tag{H'}$$

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- $x < y$ for all $x \in X$ and $y \in A \setminus X$,
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A cut is **trivial** if $X = \emptyset$ or $X = A \setminus \{1\}$.

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2. Let **A** be a linear pseudo-BCK-algebra and X be a cut of **A**. Then $\mathbf{A}_1 = (X \cup \{1\}, \to, \rightsquigarrow, 1)$ and $\mathbf{A}_2 = (A \setminus X, \to, \rightsquigarrow, 1)$ are subalgebras of **A**, and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$. If the cut X is non-trivial, then $\mathbf{A}_1$, $\mathbf{A}_2$ are non-trivial pseudo-BCK-algebras.

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2. Let $\mathbf{A}$ be a linear pseudo-BCK-algebra and X be a cut of $\mathbf{A}$. Then $\mathbf{A}_1 = (X \cup \{1\}, \rightarrow, \rightsquigarrow, 1)$ and $\mathbf{A}_2 = (A \setminus X, \rightarrow, \rightsquigarrow, 1)$ are subalgebras of $\mathbf{A}$, and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$. If the cut X is non-trivial, then $\mathbf{A}_1$, $\mathbf{A}_2$ are non-trivial pseudo-BCK-algebras.

Let **A** be a linear pseudo-BCK-algebra. For $a \in A \setminus \{1\}$ we put

$$X_a = \{x \in A \setminus \{1\} \mid a \to x = x\}.$$

We have

$$X_a = \{x \in A \setminus \{1\} \mid a \rightsquigarrow x = x\}.$$

If **A** satisfies the identities (H) and (J), then for every $a \in A \setminus \{1\}$, X_a is a cut of **A**. The cut is non-trivial provided that $X_a \neq \emptyset$.

Let **A** be a linear pseudo-BCK-algebra satisfying (H) and (J). The following statements are equivalent:

1. **A** is sum irreducible.
2. For all $a, b \in A$, if $a \to b = b$ (or $a \rightsquigarrow b = b$), then $a = 1$ or $b = 1$.
3. **A** is a pseudo-ŁBCK-algebra.

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Every non-trivial linear pseudo-BCK-algebra satisfying the equations

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z) \tag{H}$$
$$(((x \to y) \rightsquigarrow y) \to x) \rightsquigarrow x = (((y \rightsquigarrow x) \to x) \rightsquigarrow y) \rightsquigarrow y \tag{J}$$

can uniquely be represented as an ordinal sum of non-trivial linear pseudo-ŁBCK-algebras.

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1. **A** satisfies the identities (H) and (J).
2. **A** is an ordinal sum of linear pseudo-ŁBCK-algebras.
3. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a linear pseudo-hoop.
4. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a linear pseudo-BL-algebra.

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A pseudo-hoop/pseudo-BL-algebra/pseudo-BCK-algebra which is a subdirect product of ones with an underlying linear order is said to be **representable**.

Representable pseudo-BCK-algebras/pseudo-hoops/pseudo-BL-algebras are axiomatized by the identity

$$(x \to y) \to u \leq ((((((y \to x) \to z) \to z) \rightsquigarrow w) \rightsquigarrow w) \to u) \to u. \tag{R}$$

For every pseudo-BCK-algebra **A**, the following are equivalent:

1. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a representable pseudo-BL-algebra;
2. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a representable pseudo-hoop;
3. **A** satisfies the equations (R), (H) and (J).

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For every pseudo-BCK-algebra **A**, the following are equivalent:

1. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a representable pseudo-BL-algebra;
2. **A** is a $\{\to, \rightsquigarrow, 1\}$-subreduct of a representable pseudo-hoop;
3. **A** satisfies the equations (R), (H) and (J).

The class of all $\{\to, \rightsquigarrow, 1\}$-subreducts of representable pseudo-BL-algebras/pseudo-hoops is the variety of pseudo-BCK-algebras satisfying

$$(x \to y) \to u \leq ((((((y \to x) \to z) \to z) \rightsquigarrow w) \rightsquigarrow w) \to u) \to u, \tag{R}$$

$$(x \to y) \to (x \to z) = (y \to x) \to (y \to z), \tag{H}$$

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The class of all $\{\rightarrow, \rightsquigarrow, 1\}$-subreducts of representable pseudo-BL-algebras/pseudo-hoops is the variety of pseudo-BCK-algebras satisfying

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THANK YOU