Games over Formulas in Łukasiewicz Logic

Tomáš Kroupa

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
Prague

Motivation

Game-theoretic insights into Łukasiewicz propositional calculus:

- Ulam game: a 2-player game of questions and (possibly false) answers (Ulam; Mundici)
- Dutch-book theorem: no sure losers and winners in bookmaking over infinite-valued events (Paris; Gerla; Mundici)

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non-cooperative games

vs.

cooperative games ?

Cooperative Game Theory

Coalition games first studied by J. von Neumann in 1928:

- **Players** form coalitions to maximize their profit in a certain social environment
- **Coalition** acts in the common players' interest on specific issues
- **Worth** of each coalition can be obtained by acting in concert towards the common objective

- players may simultaneously belong to many coalitions which can have conflicting interests

Cooperative Game Theory (cont.)

The main problem is to find a set of final payoffs of coalitions.

Core is a set of payoffs satisfying

coalition rationality - every payoff of each coalition is not smaller than the worth of the coalition

social rationality - every payoff of the "grand coalition" equals its worth

- the role of coalitions is predominating in games with a "large" (infinite) number of players whose power is negligible
- e.g. stock market games, voting games

Coalition Game over Formulas

- every coalition substantiates a principle of behavior φ: e.g. "I am a minor shareholder of the company A", "I am a faithful voter of the political party B"
- every player V expresses a level of conformity $V(\varphi)$ with the principle φ
- a worth $\mu(\cdot)$ of each coalition should depend only on the "meaning" of φ

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Definition

Let Φ be a set of formulas with $\overline{1} \in \Phi$, and $\mathcal{F}$ be the set of corresponding equivalence classes. A (coalition) game is a pair (Φ, μ), where $\mu : \mathcal{F} \to \mathbb{R}$ is such that $m(0) = 0$, whenever $0 \in \mathcal{F}$.

Łukasiewicz Logic

- formulas are obtained from propositional variables $\omega_1, \ldots, \omega_k$ by applying negation $\neg$, disjunction $\oplus$, and conjunction $\odot$
- a valuation is a function $V : \mathrm{Form}(\omega_1, \ldots, \omega_k) \to [0,1]$ s.t.

$$\begin{aligned} V(\neg\varphi) &= 1 - V(\varphi) \\ V(\varphi \oplus \psi) &= \min(1, V(\varphi) + V(\psi)) \\ V(\varphi \odot \psi) &= \max(0, V(\varphi) + V(\psi) - 1) \end{aligned}$$

- Lindenbaum algebra $\mathcal{L}_k$ is an MV-algebra

Theorem (McNaughton)

$\mathcal{L}_k$ is the MV-algebra of all k-variable McNaughton functions: continuous piecewise linear functions $[0,1]^k \to [0,1]$, each piece having integer coefficients.

Coalition Game over Formulas (cont.)

Player is a valuation V or a point $x_V \in [0,1]^k$ under the bijection

$$V \mapsto (V(\omega_1, \ldots, \omega_k))$$

Coalition is a k-variable McNaughton function $f \in \mathcal{F}$ corresponding to $\varphi \in \Phi$

Worth of a coalition $f \in \mathcal{F}$ is given by $\mu(f) \in \mathbb{R}$

An acceptable solution is any "distribution" of worth $m : \mathcal{F} \to \mathbb{R}$ such that $m(1) = \mu(1)$ and $m(f) \geq \mu(f)$, for each $f \in \mathcal{F}$.

Measures on MV-algebras

"Distribution" of worth should satisfy the axiom of a measure:

Definition

A measure on $\mathcal{L}_k$ is a mapping $m : \mathcal{L}_k \to \mathbb{R}$ such that

$$\text{if } f \odot g = 0 \text{ for } f, g \in \mathcal{L}_k, \text{ then } m(f \oplus g) = m(f) + m(g).$$

A measure m is called a state if it is nonnegative and $m(1) = 1$.

Properties

- $m(0) = 0$
- m is nonnegative iff it is monotone
- every homomorphism $\mathcal{L}_k \to [0,1]$ is a state

Representation of Measures

Theorem

- *If s is a state on $\mathcal{L}_k$, then there is a Borel probability measure P such that*

$$s(f) = \int_{[0,1]^k} f \, dP, \quad \text{for every } f \in \mathcal{L}_k.$$

- *Each bounded nonnegative measure that is nonzero is a positive multiple of a state.*

Solution of Games

Definition

Let (Φ, μ) be a game, where μ is nonnegative. A core of (Φ, μ) is a set

$$C(\Phi, \mu) = \{m \in \mathscr{M}^+(\mathcal{L}_k) \mid m(1) = \mu(1),\ m(f) \geq \mu(f),\ \text{for each } f \in \mathcal{F}\}$$

Theorem

1. *The core $C(\Phi, \mu)$ is a compact convex subset of $\mathbb{R}^{\mathcal{L}_k}$.*
2. *Each of the following sets is a closed face of $C(\Phi, \mu)$:*

$$\begin{aligned} F_i &= \{m \in C(\Phi, \mu) \mid m(f_i) = \mu(f_i)\}, \quad i = 1, \ldots, n \\ F &= \bigcap_{i \in I} F_i, \quad I \subseteq \{1, \ldots, n\}. \end{aligned}$$

A Game with no Solution

Example

$\Phi = \{\omega, \neg\omega, \bar{1}\}, \quad \mathcal{F} = \{\text{id}, 1 - \text{id}, 1\} \subseteq \mathcal{L}_1$
$\mu(\text{id}) = \mu(1) = 10, \quad \mu(1 - \text{id}) = 5$

$C(\Phi, \mu) = \emptyset$ since

$$\text{id} + (1 - \text{id}) = 1 \quad \text{but} \quad \mu(\text{id}) + \mu(1 - \text{id}) > \mu(1)$$

The coalition corresponding to ω is too demanding...

A Game with a Solution

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$C(\Phi, \mu) \neq \emptyset$ since both these mappings are acceptable distributions of worth:

$$m_1 : f \in \mathcal{L}_1 \mapsto 10 \int_0^1 f(x)dx$$

$$m_2 : f \in \mathcal{L}_1 \mapsto 10f\left(\frac{1}{2}\right)$$

A Game with a Solution

Example

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Both m_1 and m_2 are the "least acceptable" since $\mu = m_1 = m_2$

Checking Nonemptiness of Core

Theorem

Let (Φ, μ) be a game, where $\Phi = \{\varphi_1, \dots, \varphi_n\}$, and μ is nonnegative. The following assertions are equivalent:

1. *There is $m \in C(\Phi, \mu)$ such that $m(f_i) = \mu(f_i)$ for each $i = 1, \dots, n$.*
2. *There is no payoff $\sigma : \mathcal{F} \to \mathbb{R}$ such that*

$$\sum_{i=1}^{n} \sigma(f_i) \max_{V \in \mathcal{V}} V(\varphi_i) \; < \; \sum_{i=1}^{n} \sigma(f_i) V(\varphi_i)$$

for every valuation (player) V.

Incompatible Coalitions

If

$$\varphi_1 \odot \varphi_2 \equiv \overline{0}$$

(coalitions f_1 and f_2 are based on incompatible principles), then

$$V(\varphi_1) \odot V(\varphi_2) = 0$$

for every player V.

An "imaginary player" might try to increase his average payoff by setting his level of conformity to the value

$$\max_{V \in \mathcal{V}} V(\varphi)$$

for each $\varphi \in \Phi$.