Algebraic Characterizaition of Craig Interpolation Properties for Substructural Logics

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Contents

- the strong Craig interpolation property (SCIP)
  - relations among several interpolation properties
- several types of amalgamation property
  - the commutative homomorphisms diagrams (CHD)

This is a joint work with H. Ono.

Craig interpolation property

- A substructural logic $\mathbf{L}$ has the strong Craig interpolation property (SCIP), if for any set of formulas $\Gamma \cup \Sigma \cup \{\phi, \psi\}$,

  if $\Gamma, \Sigma \vdash_{\mathbf{L}} \phi \backslash \psi$ holds then there exists a formula δ such that

  - $\Gamma \vdash_{\mathbf{L}} \phi \backslash \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$,
  - $Var(\delta) \subseteq Var(\Gamma \cup \{\phi\}) \cap Var(\Sigma \cup \{\psi\})$,

  where $Var(\gamma)$ denotes the set of all propositional variables in a formula γ.

- In the above, $\mathbf{L}$ is said to have the Craig interpolation property (CIP) when both Γ and Σ are empty.

Several types of interpolation properties

- Besides the SCIP and CIP, there exist several types of interpolation properties.

  For example,

  - the deductive interpolation property (DIP),
  - the strong deductive interpolation property (SDIP),
  - the Robinson property (RP),
  - the strong Robinson property (SRP),
  - etc...

Interpolation and Robinson properties

- For intermediate logics

$$\begin{array}{ccccc} \text{SCIP} & \Leftrightarrow & \text{SRP} & \Leftrightarrow & \text{CIP} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ \text{SDIP} & \Leftrightarrow & \text{RP} & \Leftrightarrow & \text{DIP} \end{array}$$

Interpolation and Robinson properties

- For logics over $\mathbf{FL_e}$

$$\boxed{\text{SCIP} \Leftrightarrow \text{SRP} \Leftrightarrow \text{CIP}}$$

$$\Downarrow$$

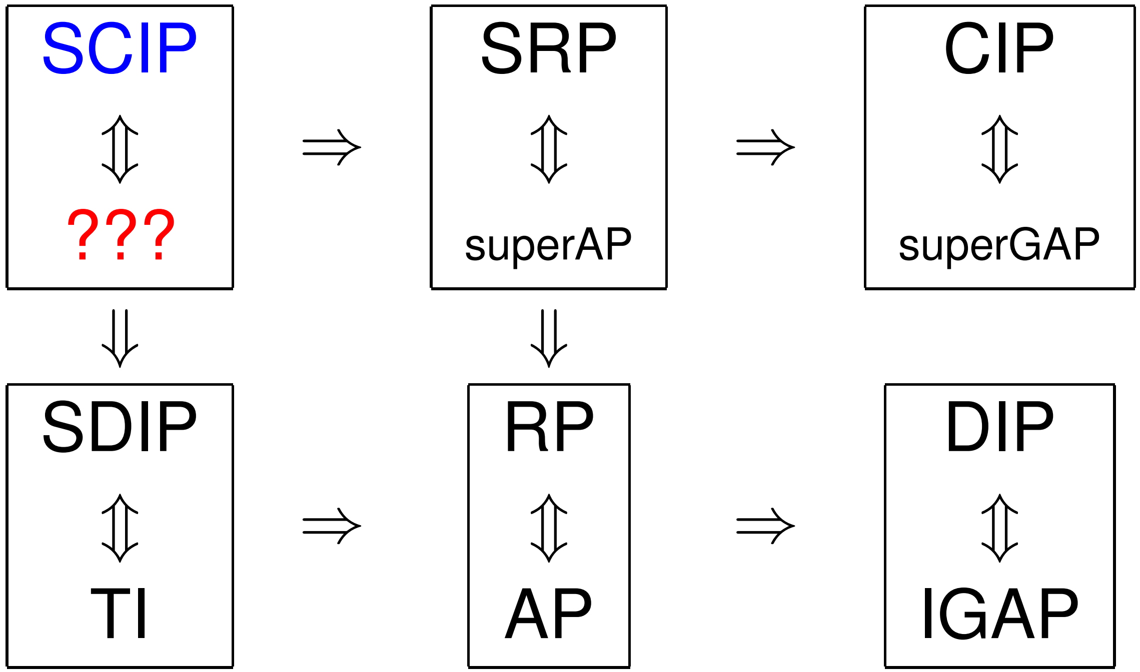
$$\boxed{\text{SDIP} \Leftrightarrow \text{RP} \Leftrightarrow \text{DIP}}$$

Interpolation and Robinson properties

- For logics over **FL**

$$\begin{array}{ccccc} \text{SCIP} & \Rightarrow & \text{SRP} & \Rightarrow & \text{CIP} \\ \Downarrow & & \Downarrow & & \\ \text{SDIP} & \Rightarrow & \text{RP} & \Rightarrow & \text{DIP} \end{array}$$

Algebraic characterizations

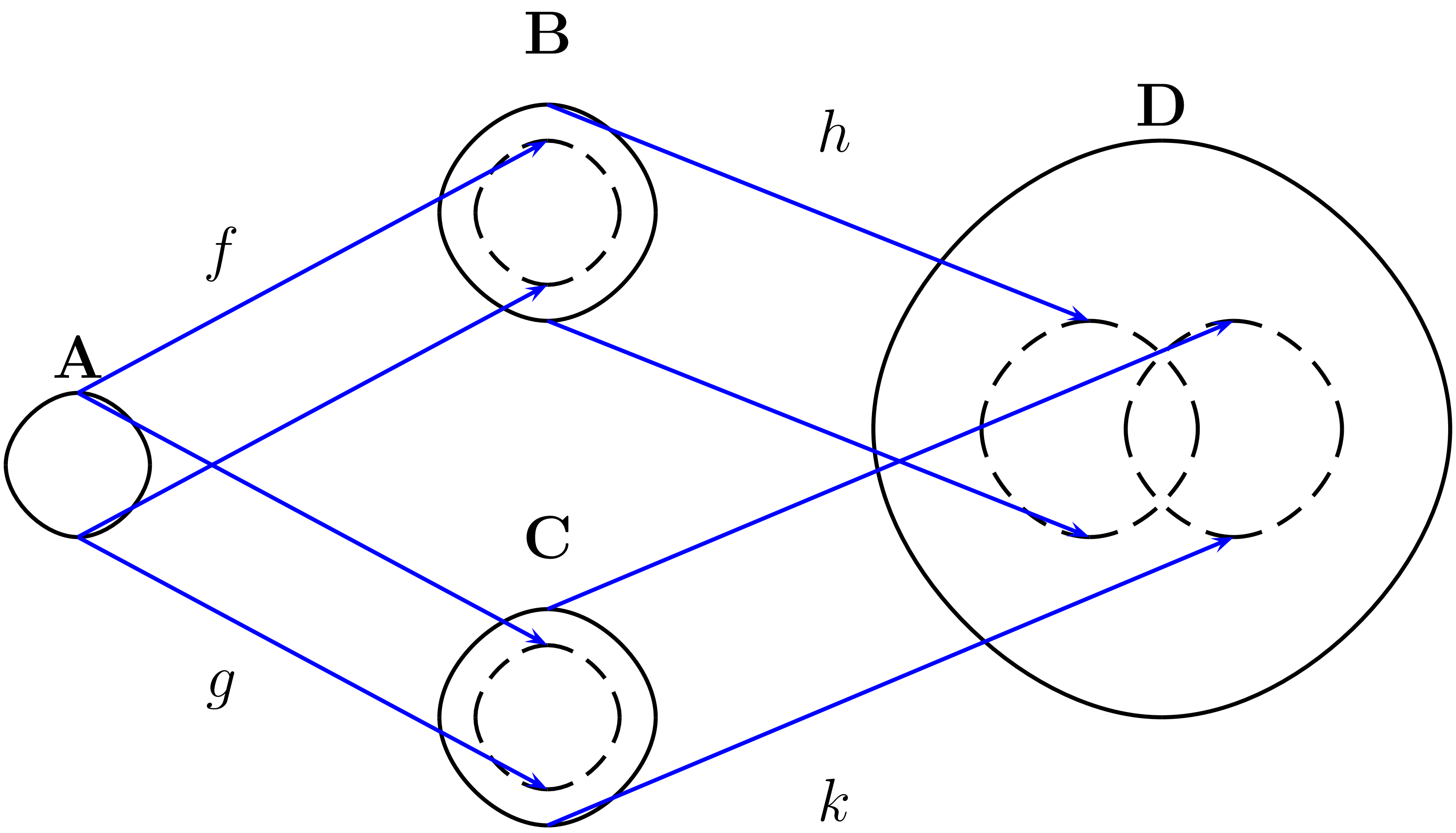


- SDIP ⇔ TI · · · · · · Wroński
- RP ⇔ AP · · · · · · H. Ono, Czelakowski-Pigozzi
- CIP ⇔ SuperGAP
DIP ⇔ IGAP · · · · · · H. Ono and H. Kihara

Amalgamation property

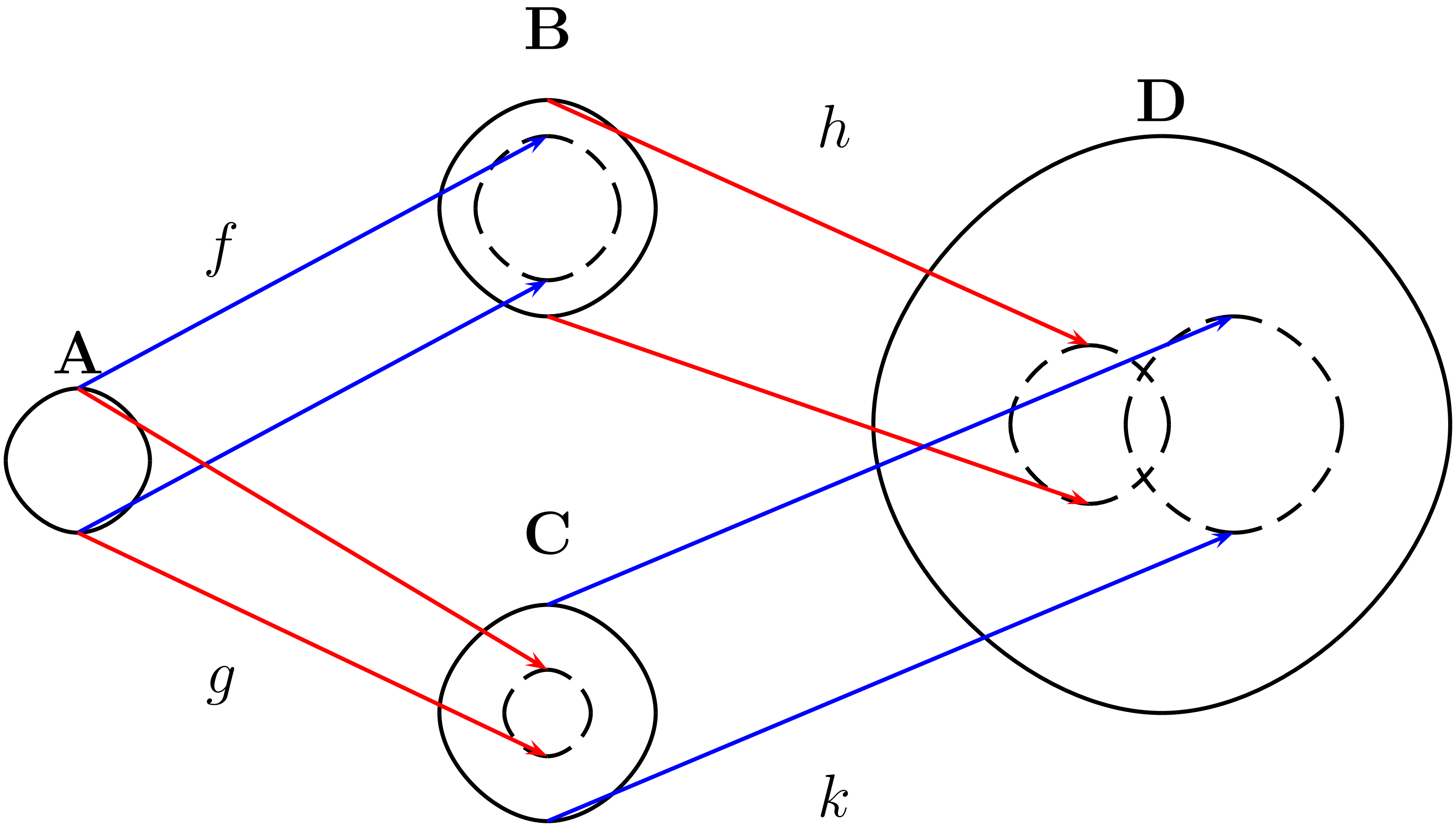
- A variety $\mathbf{V}$ of FL-algebras has the amalgamation property (AP) , if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbf{V}$ and for all embeddings $f : \mathbf{A} \to \mathbf{B}$ and $g : \mathbf{A} \to \mathbf{C}$ there exist an algebra $\mathbf{D}$ in $\mathbf{V}$ and embeddings $h : \mathbf{B} \to \mathbf{D}$ and $k : \mathbf{C} \to \mathbf{D}$ such that
  - $h \circ f = k \circ g$.
- When both g and h are homomorphisms, $\mathbf{V}$ is said to have the transferable injections (TI).

Amalgamation property



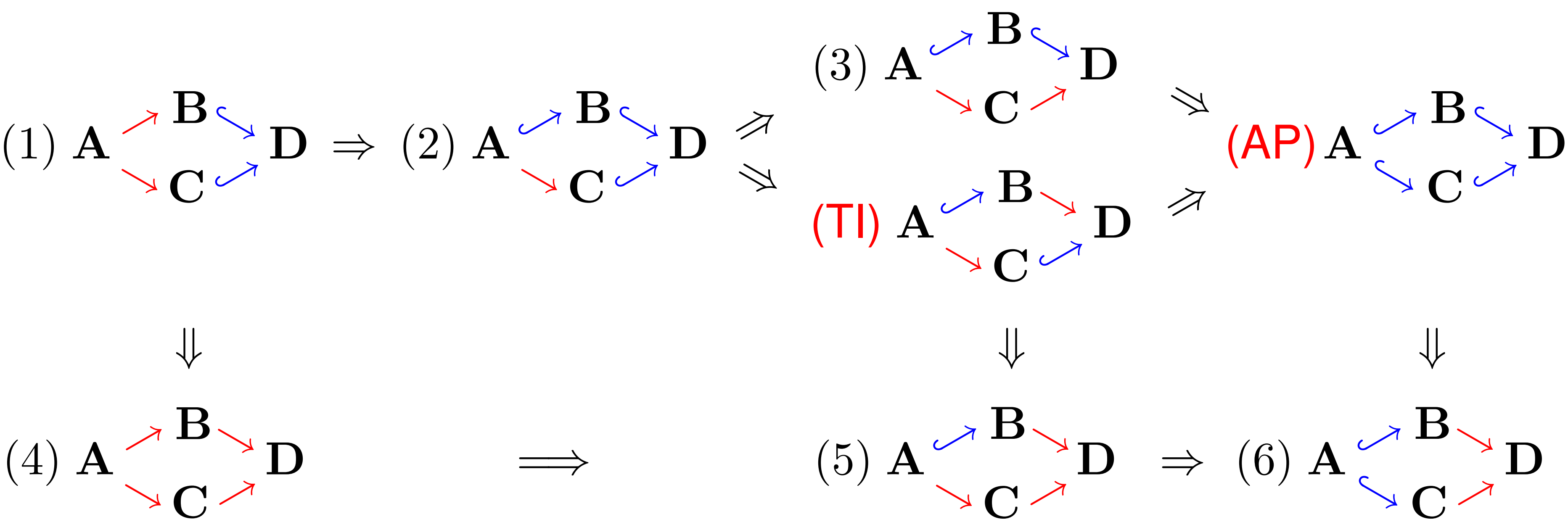
$$h \circ f = k \circ g$$

Transferable injections

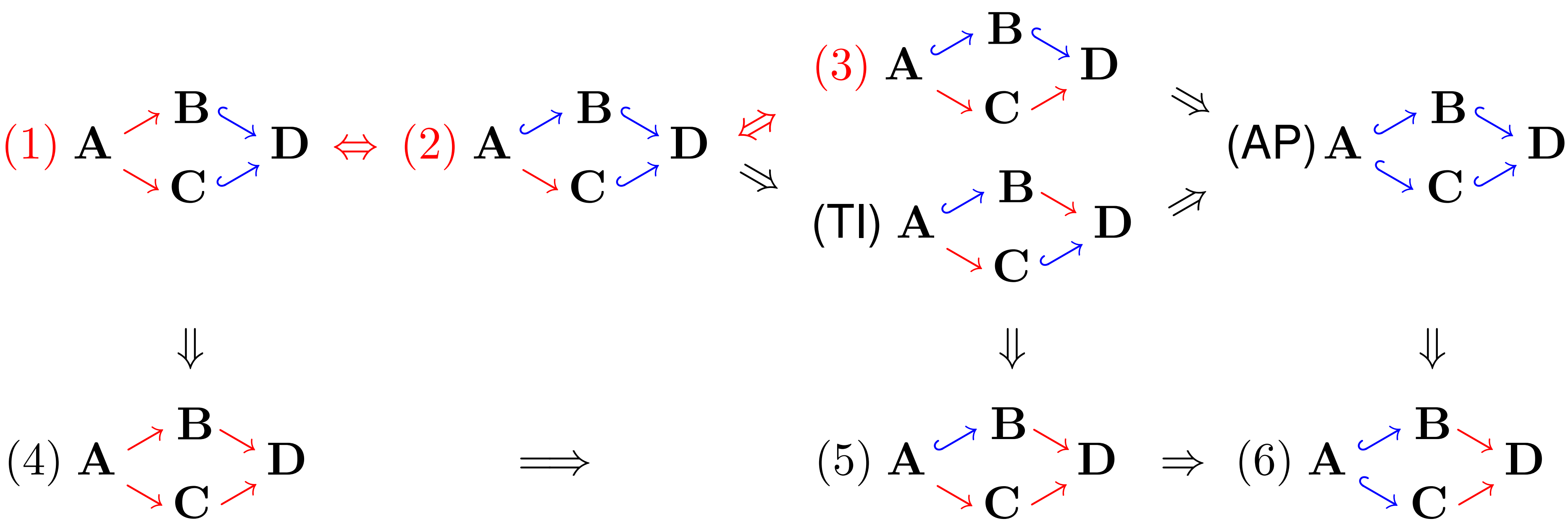


$$h \circ f = k \circ g$$

Algebraic properties

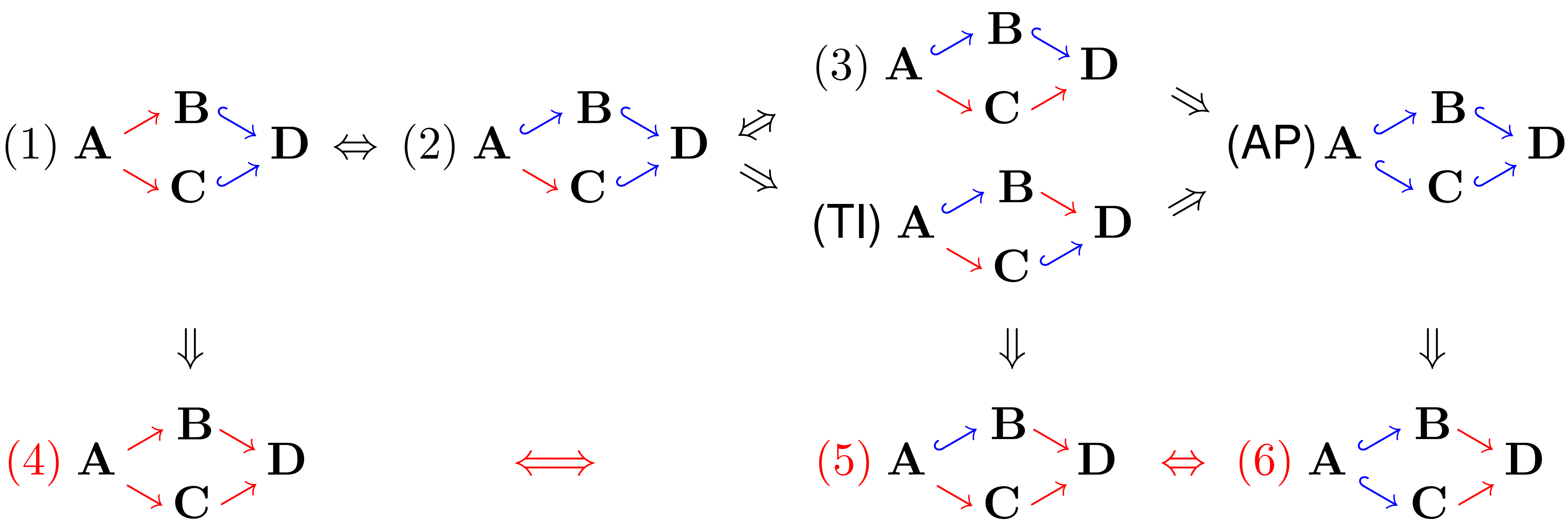


Fact 1



- The above properties (1), (2) and (3) are mutually equivalent. Moreover, the only trivial variety satisfies them.

Fact 2

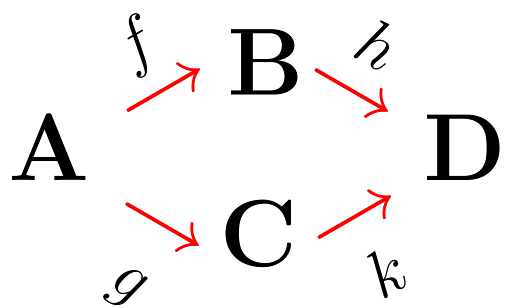


- The above properties (4), (5) and (6) are mutually equivalent. Moreover, every variety satisfies them.

Commutative homomorphisms diagrams

- We say that the commutative homomorphisms diagrams (CHD) is completed in a variety $\mathbf{V}$, if for all $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in $\mathbf{V}$ and homomorphisms $f : \mathbf{A} \rightarrow \mathbf{B}$ and $g : \mathbf{A} \rightarrow \mathbf{C}$, there exist some $\mathbf{D}$ in $\mathbf{V}$ and homomorphisms $h : \mathbf{B} \rightarrow \mathbf{D}$ and $k : \mathbf{C} \rightarrow \mathbf{D}$ such that
  - $h \circ f = k \circ g$.

(CHD)



Theorem 1

(A1): For all $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

1. $h(b) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_1 \in \mathbf{A}$ s.t,
   $b \geq f(a_1)$ and $g(a_1) \geq 1_{\mathbf{C}}$,
2. $k(c) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_2 \in \mathbf{A}$ s.t,
   $f(a_2) \geq 1_{\mathbf{B}}$ and $c \geq g(a_2)$.

- For any substructural logic $\mathbf{L}$, the following are equivalent;
  - For any $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ then $\exists \delta$ s.t,
    - $\Gamma \vdash_{\mathbf{L}} \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$,
    - $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \psi)$,
  - $\mathbf{V}(\mathbf{L})$ has the CHD+(A1),

  where $\mathbf{V}(\mathbf{L})$ denotes the variety which corresponds to the logic $\mathbf{L}$.

Theorem 1

(A1): For all $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

1. $h(b) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_1 \in \mathbf{A}$ s.t,
   $b \geq f(a_1)$ and $g(a_1) \geq 1_{\mathbf{C}}$,
2. $k(c) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_2 \in \mathbf{A}$ s.t,
   $f(a_2) \geq 1_{\mathbf{B}}$ and $c \geq g(a_2)$.

- Compare the below logical property with the SCIP and the SDIP.
  - $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{L}} \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$.
  - (SCIP)
    $\Gamma, \Sigma \vdash_{\mathbf{L}} \phi \backslash \psi \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{L}} \phi \backslash \delta$ and $\Sigma \vdash_{\mathbf{L}} \delta \backslash \psi$.
  - (SDIP)
    $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi \quad \Longrightarrow \quad \Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$.

Theorem 2

(Super): For all $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

1. $h(b) \geq k(c) \quad \Rightarrow \quad \exists a_1 \in \mathbf{A}$ s.t,
$h(b) \geq h \circ f(a_1)$ and $k \circ g(a_1) \geq k(c)$,
2. $k(c) \geq h(b) \quad \Rightarrow \quad \exists a_2 \in \mathbf{A}$ s.t,
$k(c) \geq k \circ g(a_2)$ and $h \circ f(a_2) \geq h(b)$.

- For any substructural logic $\mathbf{L}$, the following are equivalent;
  - $\mathbf{L}$ has the SCIP,
  - $\mathbf{V}(\mathbf{L})$ has the CHD+ $\{$(A1) and (Super)$\}$.

Theorem 3

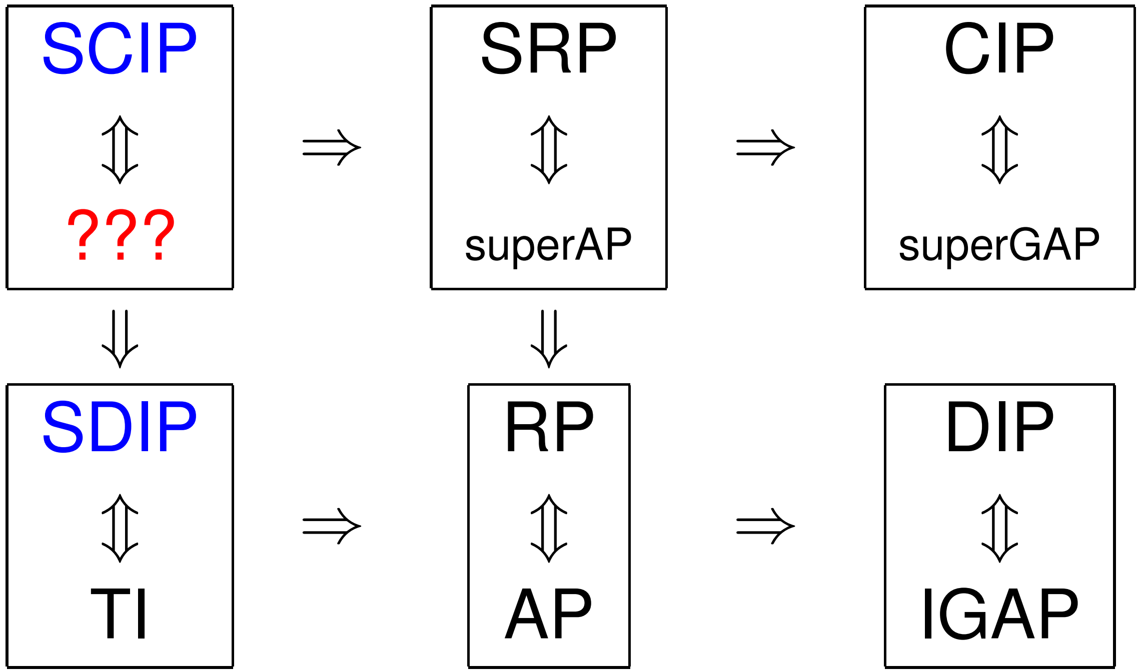
(A2): For all $b \in \mathbf{B}$ and $c \in \mathbf{C}$,

1. $h(b) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_1 \in \mathbf{A}$ s.t,
   $b \in \mathbf{Fil}_{\mathbf{B}}(f(a_1))$ and $g(a_1) \geq 1_{\mathbf{C}}$,
2. $k(c) \geq 1_{\mathbf{D}} \quad \Rightarrow \quad \exists a_2 \in \mathbf{A}$ s.t,
   $f(a_2) \geq 1_{\mathbf{B}}$ and $c \in \mathbf{Fil}_{\mathbf{C}}(g(a_2))$,

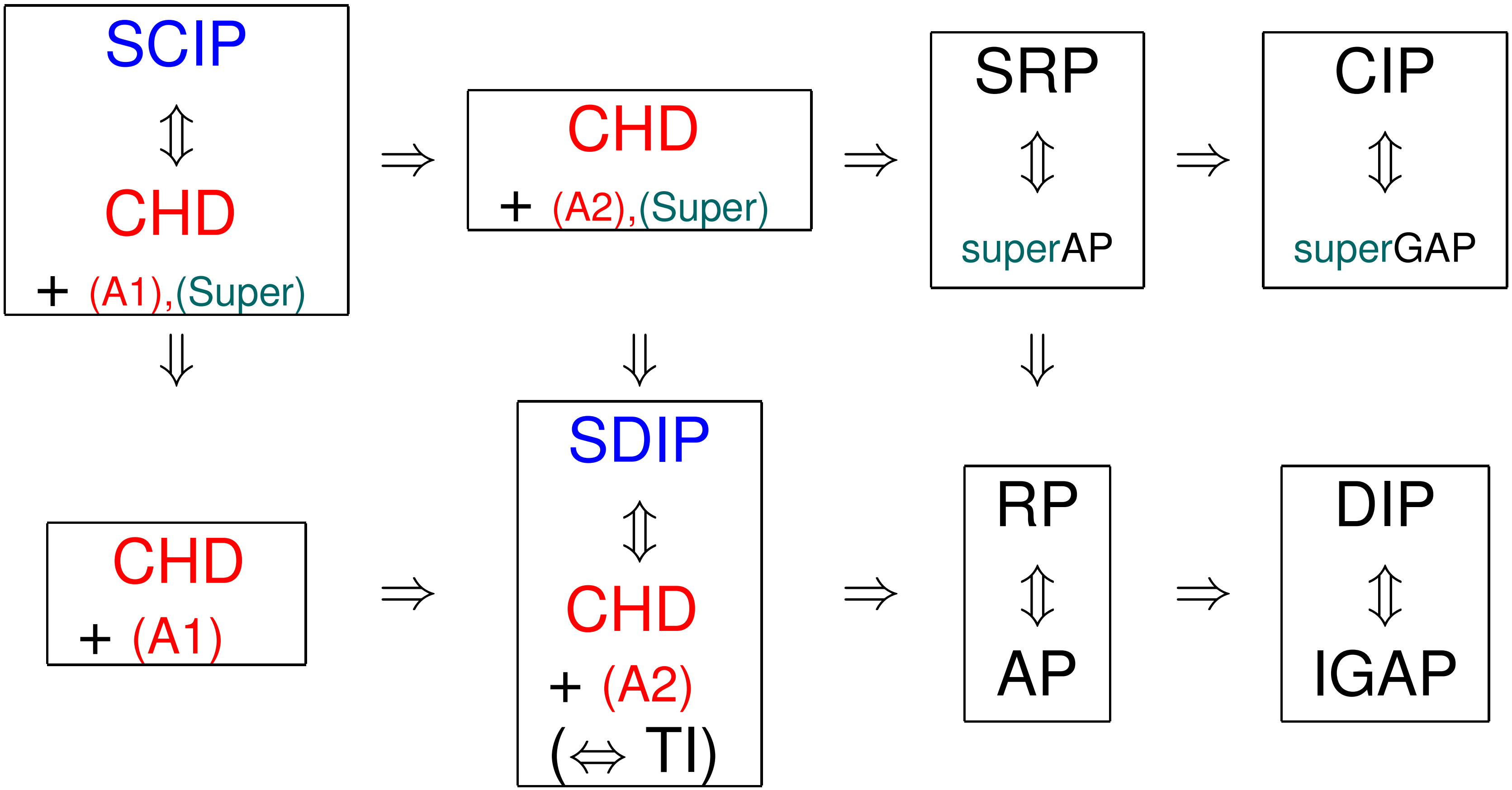
where $\mathbf{Fil}(x)$ denotes the deductive filter generated by x.

- For any substructural logic $\mathbf{L}$, the following are equivalent;
  - $\mathbf{L}$ has the SDIP,
  - $\mathbf{V}(\mathbf{L})$ has the CHD+(A2),
  - $\mathbf{V}(\mathbf{L})$ has the TI.

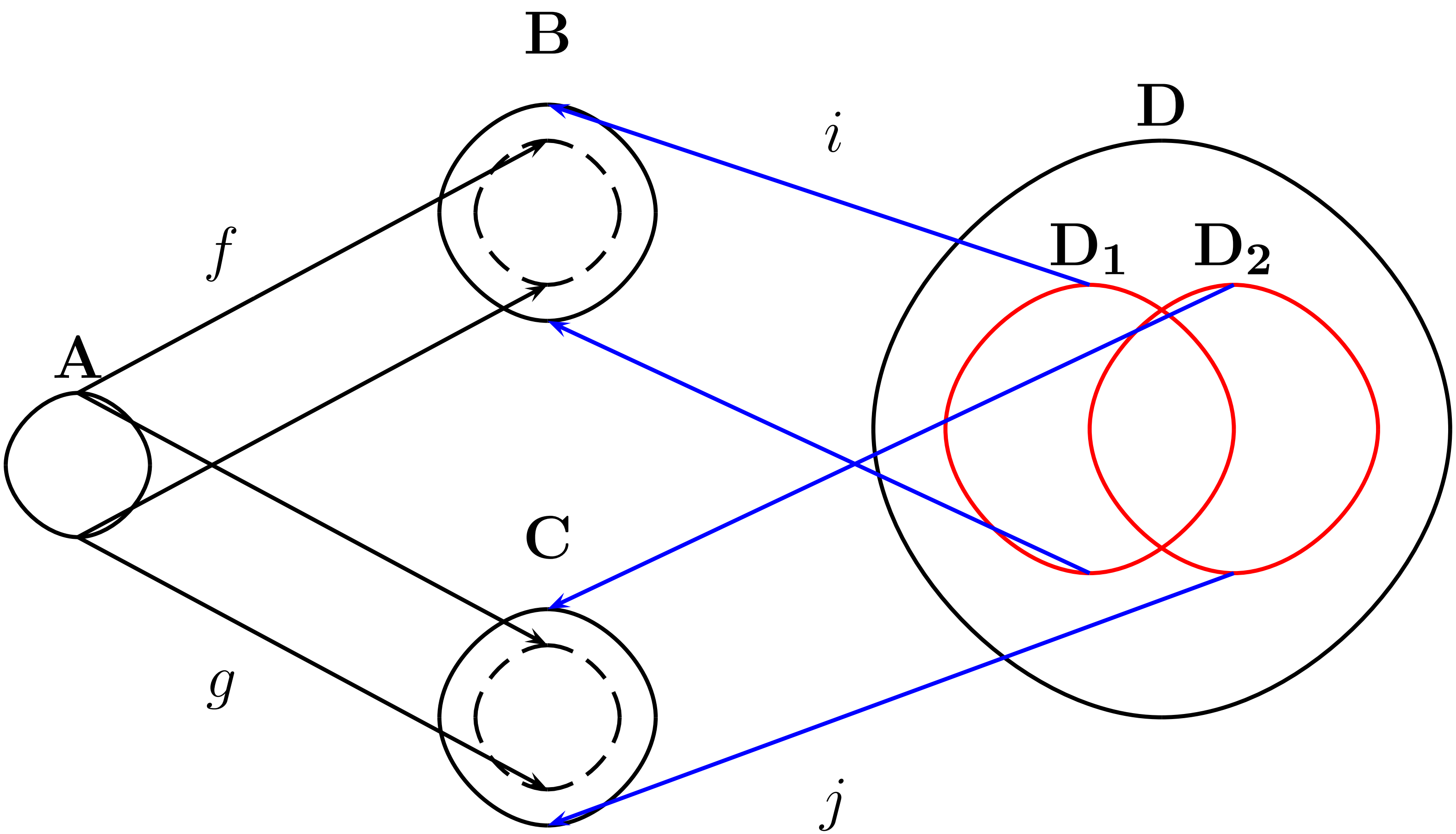
Conclusion



Conclusion

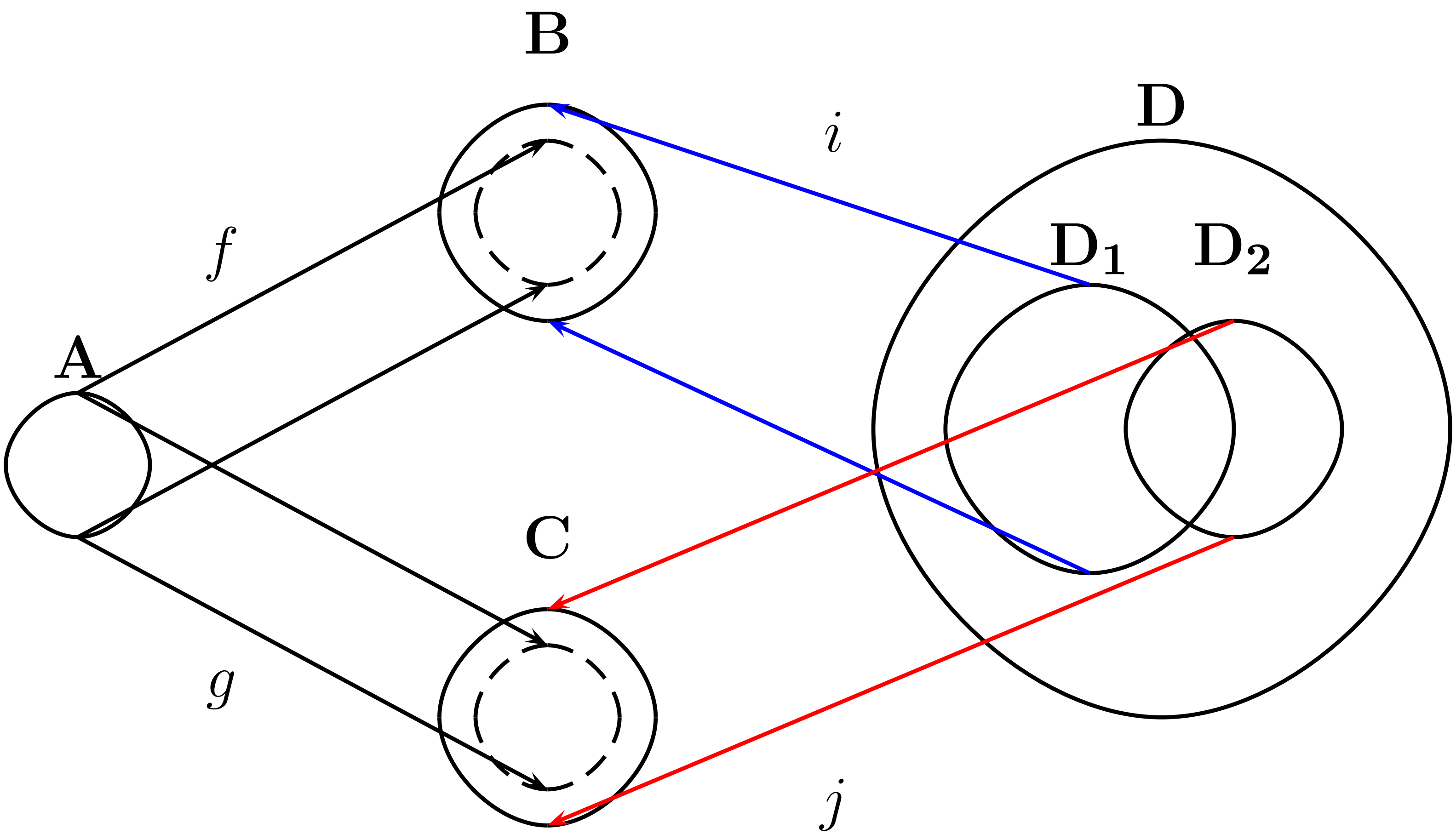

SCIP
⇕
CHD
+ (A1),(Super)
⇒
CHD
+ (A2),(Super)
⇒
SRP
⇕
superAP
⇒
CIP
⇕
superGAP
⇓
⇓
⇓
CHD
+ (A1)
⇒
SDIP
⇕
CHD
+ (A2)
(⇔ TI)
⇒
RP
⇕
AP
⇒
DIP
⇕
IGAP

Generalized amalgamation property



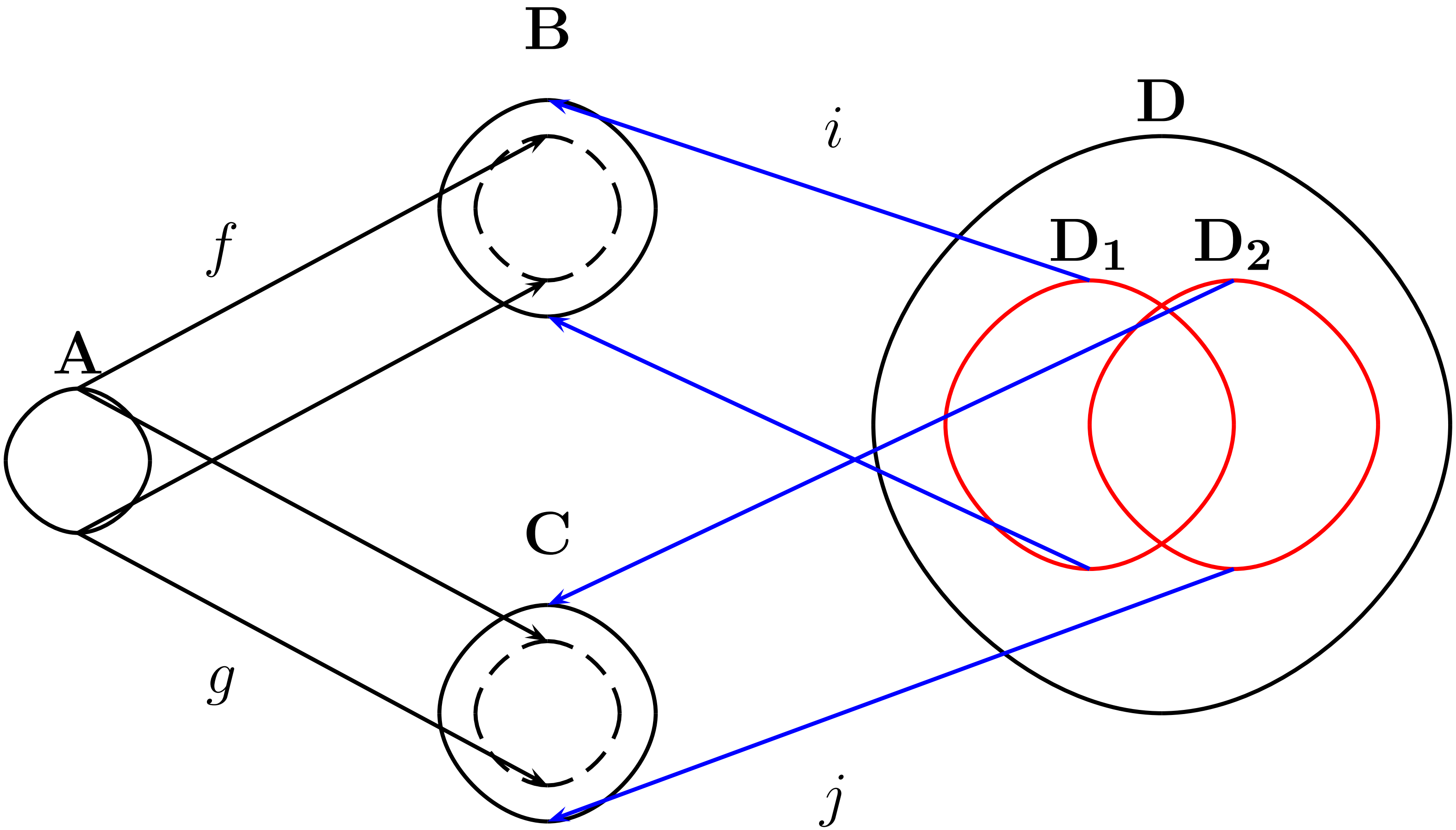
$$\forall a \in A, \exists d \in D_1 \cap D_2 \quad (f(a) = i(d) \text{ and } g(a) = j(d))$$

Injective GAP



$$\forall a \in A, \exists d \in D_1 \cap D_2 \quad (f(a) = i(d) \text{ and } g(a) = j(d))$$

SuperGAP



$$\forall d_1 \in D_1, \forall d_2 \in D_2$$
$$(d_1 \leq d_2 \Rightarrow \exists a \in A(i(d_1) \leq f(a) \text{ and } g(a) \leq j(d_2)))$$

Deductive interpolation property

- A substructural logic $\mathbf{L}$ has the deductive interpolation property (DIP), if for any set of formulas $\Gamma \cup \{\psi\}$, if $\Gamma \vdash_{\mathbf{L}} \psi$ holds then there exists a formula δ such that
  - $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta \vdash_{\mathbf{L}} \psi$,
  - $Var(\delta) \subseteq Var(\Gamma) \cap Var(\psi)$.
- $\mathbf{L}$ is said to have the strong deductive interpolation property (SDIP), if for any set of formulas $\Gamma \cup \Sigma \cup \{\psi\}$, if $\Gamma, \Sigma \vdash_{\mathbf{L}} \psi$ holds then there exists some δ such that
  - $\Gamma \vdash_{\mathbf{L}} \delta$ and $\delta, \Sigma \vdash_{\mathbf{L}} \psi$,
  - $Var(\delta) \subseteq Var(\Gamma) \cap Var(\Sigma \cup \{\psi\})$.

Robinson property

- A substructural logic $\mathbf{L}$ has the Robinson property (RP), if it has the SDIP under the assumption of $\Gamma \cup \Sigma \cup \{\phi, \psi\}$ below;
  - for every formula α with $Var(\alpha) \subseteq Var(\Gamma \cup \{\phi\}) \cap Var(\Sigma \cup \{\psi\})$,

$$\Gamma \vdash_{\mathbf{L}} \alpha \iff \Sigma \vdash_{\mathbf{L}} \alpha.$$

- $\mathbf{L}$ has the strong Robinson property (SRP), if it has the SCIP under the above assumpition.