CAPTURING ALEXANDROFFNESS
WITH AN INTUITIONISTIC MODALITY

MAMUKA JIBLADZE

Abstract. It is easy to see that a topological space is Alexandroff if and only if the Cantor-Bendixson derivative operator on the Boolean algebra of its subsets has an adjoint. We address the question whether Alexandroffness can be detected by existence of an adjoint for the restriction of the dual operator to the Heyting algebra of open sets. The answer turns out to be negative—the corresponding class of spaces is strictly larger than that of Alexandroff spaces; in fact moreover spaces whose lattice of closed sets is Heyting, and spaces whose lattice of closed sets is isomorphic to the lattice of open sets of (another) space also lie strictly in between.

1. The "classical" case

Recall that an *Alexandroff space* is a topological space homeomorphic to a space obtained from a preorder $(X, \lesssim)$ by equipping X with the *Alexandroff topology*—the topology whose open sets are all *upsets* w. r. t. $\lesssim$, i. e. sets $U \subseteq X$ with

$$\forall_{x \lesssim y}\ x \in U \Rightarrow y \in U.$$

In that case, $\lesssim$ coincides with the *specialization preorder* of the topology:

$$x \lesssim y \iff x \in \mathbf{C}\{y\} \iff \mathbf{C}\{x\} \subseteq \mathbf{C}\{y\}$$

for any $x, y \in X$, where $\mathbf{C}$ is the closure operator of the topology.

One can find many equivalent characterizations of Alexandroff spaces in the literature, the most common being that any intersection of open sets is open, or every point possessing least neighborhood. The one which is probably most interesting from the point of view of **modal logic** is in terms of adjoints to unary operators. Recall that for maps $f : X \to Y$, $g : Y \to X$ between posets $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$, the following conditions are equivalent:

(1) for any $x \in X$, $y \in Y$ one has

$$x \leqslant_X g(y) \iff f(x) \leqslant_Y y;$$

(2) both f and g are monotone and satisfy

$$x \leqslant_X gf(x), \quad fg(y) \leqslant_Y y$$

for any $x \in X$, $y \in Y$.

Under these circumstances one says that (f, g) *form an adjoint pair*, with f *left adjoint to* g and g *right adjoint to* f. Notation is $f \dashv g$.

Recall that a map between complete posets (those admitting all suprema and infima) has a right (resp. left) adjoint iff it preserves all suprema (resp. infima). More precisely, $f : X \to Y$ has a right adjoint iff for any $y \in Y$ the set

$$\{x \in X \mid f(x) \leqslant_Y y\}$$

has a supremum $g(x)$ preserved by f; similarly, $g : Y \to X$ has a left adjoint iff

$$\{y \in Y \mid x \leqslant_X g(y)\}$$

has an infimum $f(x)$ preserved by g, for any $x \in X$.

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One has the following

(1.1) Proposition. *A topology on a set X is Alexandroff if and only if the corresponding interior operator $\mathbf{I}:\mathscr{P}(X)\to\mathscr{P}(X)$ on the powerset of X (with the subset inclusion order) has a left adjoint. An equivalent requirement is that the closure operator $\mathbf{C}:\mathscr{P}(X)\to\mathscr{P}(X)$ has a right adjoint.*

Proof. Although this is a well known (folklore?) exercise, we reproduce the proof here. Suppose the topology is Alexandroff, obtained from the preorder $\lesssim$ on X. Then the corresponding interior operator is given by

$$\mathbf{I}_{\lesssim}(S)=\{x\in X\ \mid\ \forall_{x\lesssim s}\ s\in S\}$$

and the closure operator by

$$\mathbf{C}_{\lesssim}(S)=\{x\in X\ \mid\ \exists_{x\lesssim s}\ s\in S\},$$

and it is straightforward to verify that there are adjunctions

$$\mathbf{C}_{\gtrsim}\dashv\mathbf{I}_{\lesssim},\quad \mathbf{C}_{\lesssim}\dashv\mathbf{I}_{\gtrsim}\,.$$

For the converse, first note that a map f between Boolean algebras with their canonical ordering has a left (resp. right) adjoint iff the dual map $\neg f\neg$ has a right (resp. left) adjoint; this in particular applies to our situation as the interior and closure operators are mutually dual. So suppose that $\mathbf{C}$ has a right adjoint. Then it preserves all suprema; in particular, for any $S\in\mathscr{P}(X)$ one has

$$\mathbf{C}\,S=\mathbf{C}\bigcup\{\{s\}\ \mid\ s\in S\}=\bigcup\{\mathbf{C}\,\{s\}\ \mid\ s\in S\}=\big\{x\in X\ \mid\ \exists_{x\in\mathbf{C}\{s\}}\ s\in S\big\}=\mathbf{C}_{\lesssim}(S),$$

where $\lesssim$ is the specialization preorder corresponding to our topology. □

2. Switching to the derivative

In case one is interested in semantics of **intuitionistic** rather than classical modal logic, then the natural algebras to look at would be not the powersets $\mathscr{P}(X)$ but the Heyting algebras $\mathscr{O}(X)$ of open sets for a topology. Now there is no meaningful way to restrict the above adjunction condition to open sets, as the interior operator restricts to the identity there. Observe however that in the above proposition we might equivalently pick instead of the interior and closure operators the *Cantor-Bendixson derivative* operator $\boldsymbol{\delta}$ and its dual $\neg\,\boldsymbol{\delta}\,\neg$. Recall that for a subset $S\subseteq X$ of a topological space X, $\boldsymbol{\delta}\,S$ is the set of *limit points* of S—those points $x\in X$ with the property that any neighborhood of x meets $S\backslash\{x\}$. Thus the closure operator is definable through $\boldsymbol{\delta}$ via $\mathbf{C}\,S=S\cup\boldsymbol{\delta}\,S$ for any $S\subseteq X$.

The dual operator $\boldsymbol{\tau}=\neg\,\boldsymbol{\delta}\,\neg$, although thoroughly studied by several authors (for just one nice example, [2]), does not seem to have commonly recognized name. This operator assigns to a set S the set $\boldsymbol{\tau}\,S$ of those points x of X which are *surrounded by S*—i. e. there is a neighborhood $U\ni x$ of x with $U\backslash\{x\}\subseteq S$. Dually to the above one has $\mathbf{I}\,S=S\cap\boldsymbol{\tau}\,S$ for any $S\in\mathscr{P}(X)$.

Then,

(2.1) Proposition. *A topology on X is Alexandroff iff the Cantor-Bendixson derivative $\boldsymbol{\delta}$ has a right adjoint.*

Proof. For the Alexandroff topology corresponding to a preorder $\lesssim$, the Cantor-Bendixson derivative is given by

$$\boldsymbol{\delta}_{\lesssim}(S)=\{x\in X\ \mid\ \exists_{x\lesssim s}\ s\neq x\ \&\ s\in S\},$$

and the antiderivative by

$$\boldsymbol{\tau}_{\lesssim}(S)=\{x\in X\ \mid\ \forall_{x\lesssim s}\ s\neq x\Rightarrow s\in S\}.$$

Thus just as above it is easy to check that one has

$$\boldsymbol{\delta}_{\lesssim}\dashv\boldsymbol{\tau}_{\gtrsim}\ \text{ and }\ \boldsymbol{\delta}_{\gtrsim}\dashv\boldsymbol{\tau}_{\lesssim}\,.$$

Conversely, suppose that δ has a right adjoint $\bar{\tau}$. Then one has

$$\begin{aligned} \mathbf{C}\,S \subseteq T &\iff S \subseteq T \;\&\; \delta\,S \subseteq T \\ &\iff S \subseteq T \;\&\; S \subseteq \bar{\tau}\,T, \end{aligned}$$

so $\mathbf{C}$ has a right adjoint given by $T \mapsto T \cap \bar{\tau}\,T$. Hence by 1.1 the topology is Alexandroff. □

Given that, we might ask whether Alexandroffness can be captured by only requiring existence of adjoint for the trace of τ on open sets. Now τ indeed restricts to opens, in the sense that τU is open for any open U (equivalently, $\delta\,C$ is closed for any closed C). This is because by definition $x \in \tau U$ happens iff there is a neighborhood $V \ni x$ with $V \setminus \{x\} \subseteq U$. But openness of U is the same as $U \subseteq \tau U$, so we have $V \setminus \{x\} \subseteq \tau U$ and $\{x\} \subseteq \tau U$, hence $V \subseteq \tau U$.

We thus have the restricted operator $\tau_{\mathscr{O}(X)} : \mathscr{O}(X) \to \mathscr{O}(X)$. Several authors (see, for example, [1]) have noticed that in fact this restricted operator does not depend on X, i. e. can be defined purely in terms of the complete Heyting algebra $\mathscr{O}(X)$. Indeed, as shown in [1], τU can be characterized as the largest among those $V \supseteq U$ in $\mathscr{O}(X)$ for which the lattice $[U, V]$ is Boolean. Alternatively, one also has

$$\tau U = \bigwedge \{W \in \mathscr{O}(X) \mid W \to U = U\}$$

in $\mathscr{O}(X)$. It is clear that these constructions make sense in any complete Heyting algebra H, and are known to yield the same result there. Let us reproduce here this result for completeness. To formulate it, recall that an element d of a lattice is called *dense* if $d \wedge x = 0$ implies $x = 0$ for any x in the lattice. In a Heyting algebra, this happens if and only if $\neg d = 0$.

(2.2) Proposition. *For any element a of a complete Heyting algebra H, the set of those $b \in H$ for which the lattice $[a \wedge b, b]$ is Boolean has a largest element. This element is equal to*

$$\tau_H(a) := \bigwedge \{d \geqslant a \mid d \text{ is a dense element of the lattice } [a, 1] \text{—equivalently, } d \to a = a\}.$$

Proof. Evidently

$$\tau_H(a) = \tau_{[a,1]}(0_{[a,1]});$$

moreover for any b one evidently has

$$\{b \geqslant d \geqslant a \wedge b \mid d \text{ is dense in } [a \wedge b, b]\} = \{b \wedge d \mid d \text{ is dense in } [a, 1]\},$$

hence one has

$$\tau_{[a \wedge b, b]}(0_{[a \wedge b, b]}) = b \wedge \tau_H(a).$$

In particular, one has

$$b \leqslant \tau_H(a) \iff \tau_{[a \wedge b, b]}(0_{[a \wedge b, b]}) = b.$$

On the other hand, a Heyting algebra H is Boolean if and only if $\tau_H(0_H) = 1_H$, i. e. 1_H is the only dense element of H. Indeed, all elements of the form $a \vee \neg a$ are dense as $\neg(a \vee \neg a) = \neg a \wedge \neg\neg a = 0$.

Thus we obtain

$$b \leqslant \tau_H(a) \iff [a \wedge b, b] \text{ is Boolean.}$$

□

We then make the following

(2.3) Definition. A topological space X is called *pseudoalexandroff* if the above operator $\tau_{\mathscr{O}(X)}$ has a left adjoint $\Diamond : \mathscr{O}(X) \to \mathscr{O}(X)$.

3. Characterization

Note that because of the above pseudoalexandroffness of a space only depends on the algebra of its open sets; in particular, a space is pseudoalexandroff if and only if its T_0-reflection is. In the following theorem we thus restrict attention to T_0 spaces; later we will easily handle the general case.

(3.1) Theorem. *A* T_0*-space* X *is pseudoalexandroff if and only if in any open set* U *of* X*, the subset of points of* U *which are closed in the induced topology on* U *is discrete.*

Proof. Suppose $\boldsymbol{\tau}_{\mathscr{O}(X)}$ has a left adjoint $\Diamond$. Then necessarily

$$\begin{aligned}\Diamond V &= \bigwedge\{U \mid V \leqslant \boldsymbol{\tau} U\}\\ &= \bigwedge\{U \mid [U \wedge V, V] \text{ is Boolean}\}\\ &= \bigwedge\{U \leqslant V \mid [U, V] \text{ is Boolean}\}\,.\end{aligned}$$

Moreover so defined $\Diamond$ will indeed be left adjoint to $\boldsymbol{\tau}_{\mathscr{O}(X)}$ if and only if one has

$$V \leqslant \boldsymbol{\tau}\Diamond V$$

for all V, since

$$\Diamond\boldsymbol{\tau} U \leqslant U$$

evidently always holds. It follows that $\boldsymbol{\tau}_{\mathscr{O}(X)}$ has a left adjoint if and only if for all V the interval $[\Diamond V, V]$ is Boolean, with $\Diamond$ defined as above. Equivalently, for any family $U_i \leqslant V$ with $[U_i, V]$ Boolean for all $i \in I$, the interval

$$\left[\bigwedge_{i\in I} U_i, V\right]$$

must be Boolean too. Another equivalent condition is that for any V there is a smallest $U \leqslant V$ with $[U, V]$ boolean.

Now for any open U one has

$$\boldsymbol{\tau} U = U \cup \{x \mid x \text{ is an isolated point of } X\backslash U\}\,.$$

Moreover if $\boldsymbol{\tau}_{\mathscr{O}(X)}$ has a left adjoint $\Diamond$, then one has

$$\begin{aligned}x \notin \Diamond V &\iff \Diamond V \subseteq X\backslash\{x\}\\ &\iff \Diamond V \subseteq \mathbf{I}(X\backslash\{x\}) = X\backslash\mathbf{C}\{x\}\\ &\iff V \subseteq \boldsymbol{\tau}(X\backslash\mathbf{C}\{x\})\\ &\iff [V \cap (X\backslash\mathbf{C}\{x\}), V] \text{ is Boolean}\\ &\iff \mathscr{O}(V \cap \mathbf{C}\{x\}) \text{ is Boolean.}\end{aligned}$$

Here all the equivalences are obvious except probably the last, which is the consequence of the fact that for any $U \subseteq V$ the lattice $\mathscr{O}(V\backslash U)$ is isomorphic to the interval $[U, V]$. Taking here $U = V\backslash\mathbf{C}\{x\}$ gives what we need as $V\backslash\mathbf{C}\{x\} = V \cap (X\backslash\mathbf{C}\{x\})$ and $V\backslash(V\backslash\mathbf{C}\{x\}) = V \cap \mathbf{C}\{x\}$.

Now the lattice of opens of a T_0 space is Boolean if and only if the space is discrete. Moreover, any subspace of a T_0 space is T_0. It follows that for any $V \in \mathscr{O}(X)$ and any $x \in X$, the lattice $\mathscr{O}(V \cap \mathbf{C}\{x\})$ is Boolean if and only if any $x' \in \mathbf{C}\{x\}$ has a neighborhood whose intersection with V is $\{x'\}$. In more detail, this condition reads

$$\forall_{x'\in V}\ \left(\forall_{\mathscr{O}(X)\ni V'\ni x'}\ x \in V'\right) \Rightarrow \left(\exists_{\mathscr{O}(X)\ni V'\ni x'}\ V \cap V' = \{x'\}\right).$$

But $x \in V'$ and $V \cap V' = \{x'\}$ together imply $x' = x$, so one obtains

$$\Diamond V = \{x \in V \mid V \cap \mathbf{C}\{x\} \neq \{x\}\}\,,$$

i. e. $\Diamond V$ is the set of nonclosed points of V.

Now the adjunction condition $V \subseteq \boldsymbol{\tau} \diamondsuit V$, which, since $U \subseteq \boldsymbol{\tau} U$ for all U, is equivalent to $V \setminus \diamondsuit V \subseteq \boldsymbol{\tau} \diamondsuit V$, reads

$$\forall_{x \in V} \left(V \cap \mathbf{C}\{x\} = \{x\}\right) \Rightarrow \left(\exists_{V' \ni x} \forall_{x \neq y \in V'} V \cap \mathbf{C}\{y\} \neq \{y\}\right),$$

i. e. any closed point of an open set has a neighborhood consisting of nonclosed points of this open set. Together with openness of $\diamondsuit V$ this gives

$$\forall_{x \in V} \exists_{V' \ni x} \forall_{x \neq y \in V'} V \cap \mathbf{C}\{y\} \neq \{y\},$$

i. e. any point of an open V has a neighborhood consisting of nonclosed points of V. In other words, the set of closed points of any open V is discrete in the induced topology.

Moreover the condition $\diamondsuit \boldsymbol{\tau} U \subseteq U$, or equivalently $\boldsymbol{\tau} U \setminus U \subseteq \boldsymbol{\tau} U \setminus \diamondsuit \boldsymbol{\tau} U$ means in detail

$$\forall x \notin U \left(\exists V \ni x \; V \setminus \{x\} \subseteq U\right) \Rightarrow (x \text{ is a closed point of } \boldsymbol{\tau} U).$$

It is easy to see that this condition is equivalent to requiring that the set of isolated points of any closed subset is T_1. But the latter set is actually always discrete, so the condition is trivially satisfied. □

Now to the general case. We introduce some further notions for that.

(3.2) Definition. A point of a topological space is *cluster-closed* if its closure is antidiscrete.

(3.3) Definition. Call a topological space *0-pseudoalexandroff* if its subspace of cluster-closed points is a topological sum of antidiscrete spaces.

We then have

(3.4) Corollary. *A topological space is pseudoalexandroff if and only if all of its open subspaces are 0-pseudoalexandroff.*

Proof. The point is that a point is cluster-closed iff its image in the T_0-reflection is closed, and a subspace is a topological sum of antidiscrete spaces iff its image in the T_0-reflection is discrete. □

Here is another feature of (T_0) pseudoalexandroff spaces stressing their "near-alexandroff" behavior.

(3.5) Proposition. *In a* T_0 *pseudoalexandroff space one has*

$$\boldsymbol{\tau}(V) = \boldsymbol{\tau}_{\leqslant}(V)$$

for any open set V*, where* $\leqslant$ *is the specialization order.*

Proof. We have seen in the course of proving 3.1 that for any open set V of a T_0 pseudoalexandroff space X the left adjoint $\diamondsuit$ to $\boldsymbol{\tau}_{\mathscr{O}(X)}$ is given by

$$\diamondsuit(V) = \{x \in V \mid V \cap \mathbf{C}\{x\} \neq \{x\}\} = \{x \in V \mid \exists_{y \in V} \, y < x\}.$$

But since all open sets are upsets with respect to the specialization order, the latter set is the same as

$$\{x \in X \mid \exists_{y \in V} \, y < x\} = \boldsymbol{\delta}_{\geqslant}(V).$$

And $\boldsymbol{\tau}_{\mathscr{O}(X)}$, which must be right adjoint to this operator, will then coincide with the restriction of the right adjoint to $\boldsymbol{\delta}_{\geqslant}$ to $\mathscr{O}(X)$, i. e. with $\boldsymbol{\tau}_{\leqslant}$. □

The author does not know whether converse is true.

4. Some intermediate classes

Clearly any Alexandroff space is pseudoalexandroff. Indeed, as we saw in 2.1, in this case $\boldsymbol{\tau}_{\lesssim}$ has a left adjoint $\boldsymbol{\delta}_{\gtrsim}$, and the latter operator restricts to $\gtrsim$-closed sets, which is the same as $\lesssim$-open sets.

However, not only do not the classes of Alexandroff and pseudoalexandroff spaces coincide—there are in fact some distinguishable classes of spaces in between.

(4.1) Definition. A *bispace* is a bitopological space in which open sets of one of its topologies coincide with closed sets of the other topology.

Obviously one has

(4.2) Proposition. *The notion of bispace is equivalent to that of Alexandroff space, in the following sense. Any preorder X produces a bispace by considering Alexandroff topologies of X and X°, and conversely, topologies of any bispace are Alexandroff, corresponding to some preorder and its dual.*

Proof. It suffices to note that for a bispace, any intersection of closed sets is closed, and any union of opens open, for both topologies. □

(4.3) Definition. A topological space X is a *Janus space* if the lattice $\mathscr{O}(X)$ of its open sets is isomorphic to the lattice of closed sets of some topological space.

For example, any Alexandroff space, i. e. any bispace, is a Janus space. Also note that any space whose lattice of opens is self-dual is a Janus space. On the other hand, one has

(4.4) Example. Let X be the interval $(0,1)$ equipped with the intersection of the Alexandroff and Euclidean topologies. Thus the open sets are $(a,1)$ for $0 \leqslant a \leqslant 1$, so that $\mathscr{O}(X)$ is order-isomorphic to $[0,1]$. This is self-dual, so X is a Janus space. However it is obviously not Alexandroff.

(4.5) Definition. A topological space X is a *bi-Heyting space* if $\mathscr{O}(X)$ is a bi-Heyting algebra.

For example, any Janus space is obviously bi-Heyting. We do not know whether converse is true.

Moreover one has

(4.6) Proposition. *For any complete bi-Heyting algebra H, the operator $\boldsymbol{\tau}_H$ has a right adjoint.*

Proof. By 2.2 one has

$$\begin{aligned} a \leqslant \boldsymbol{\tau}_H(b) &\iff [a \wedge b, a] \text{ is Boolean} \\ &\iff [b, a \vee b] \text{ is Boolean} \\ &\iff \boldsymbol{\tau}_{H^\circ}(a) \leqslant b; \end{aligned}$$

thus $\boldsymbol{\tau}_{H^\circ}$ is left adjoint to $\boldsymbol{\tau}_H$. □

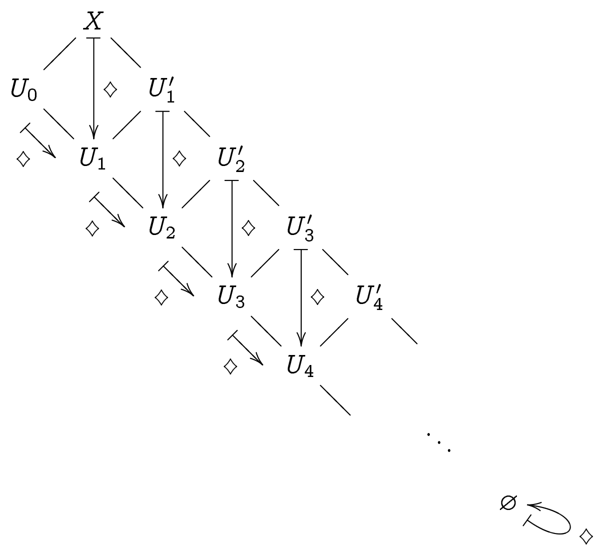
In particular, we have

(4.7) Corollary. *Any bi-Heyting space is pseudoalexandroff.*

□

However, not every pseudoalexandroff space is bi-Heyting:

(4.8) Example. On the one point compactification $X = \mathbb{N} \cup \{*\}$ of the set $\mathbb{N}$ of natural numbers, consider intersection of its topology with the Alexandroff topology of the natural ordering of $\mathbb{N}$, with $*$ incomparable with the rest (rather than being the largest element). Thus the open sets of the resulting space X are $U_n = \{n, n+1, \ldots\}$, $n \geqslant 0$, $U'_n = U_n \cup \{*\}$, with $U'_0 = X$, and the empty set. One has $\boldsymbol{\tau}\varnothing = \varnothing$, $\boldsymbol{\tau}X = X = \boldsymbol{\tau}U_0$, whereas for any $n > 0$ one has $\boldsymbol{\tau}U'_n = U'_{n-1} = \boldsymbol{\tau}U_n$. This operator has a left adjoint $\Diamond$ given by $\Diamond\varnothing = \varnothing$ and $\Diamond U_n = U_{n+1} = \Diamond U'_n$ for all $n \geqslant 0$. Pictorially, $\mathscr{O}(X)$ with $\Diamond$ on it looks like an "upside-down Jacob's ladder":



Thus X is pseudoalexandroff. It is however not bi-Heyting—for example, $\mathscr{O}(X)$ does not possess $X \dot{-} U_0$. Indeed one has $U_0 \cup U'_n = X$ for all n but

$$U_0 \cup \bigwedge \left\{U'_n \;\middle|\; n \geqslant 0\right\} = U_0 \neq X,$$

since

$$\bigwedge \left\{U'_n \;\middle|\; n \geqslant 0\right\} = \mathbf{I}\bigcap \left\{U'_n \;\middle|\; n \geqslant 0\right\} = \mathbf{I}\{*\} = \varnothing.$$

Thus the set

$$\{U \in \mathscr{O}(X) \mid U \cup U_0 = X\}$$

does not possess smallest element, i. e. $X \dot{-} U_0$ does not exist.

We finish with

(4.9) Proposition. *A space is a bi-Heyting* $\mathrm{T_D}$ *space iff it is a* $\mathrm{T_0}$ *Alexandroff space.*

Proof. Obviously any poset is bi-Heyting and $\mathrm{T_D}$ in its Alexandroff topology. Conversely, a space X is $\mathrm{T_D}$ iff for any $x \in X$ the set

$$C_x := \mathbf{C}\{x\} \setminus \{x\}$$

is closed. If X is moreover bi-Heyting, then there is an open set

$$U_x := (X \setminus C_x) \dot{-} (X \setminus \mathbf{C}\{x\})$$

which is smallest among those opens U for which

$$U \cup (X \setminus \mathbf{C}\{x\}) \supseteq X \setminus C_x$$

—or, equivalently,

$$(X \setminus C_x) \setminus (X \setminus \mathbf{C}\{x\}) \subseteq U.$$

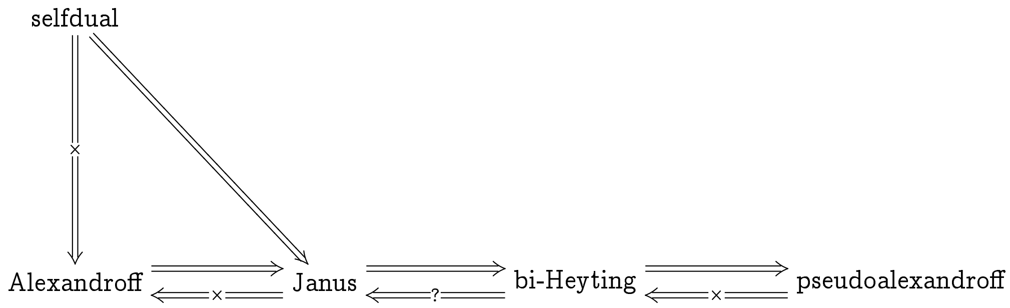
But

$$(X \setminus C_x) \setminus (X \setminus \mathbf{C}\{x\}) = \mathbf{C}\{x\} \setminus C_x = \mathbf{C}\{x\} \setminus (\mathbf{C}\{x\} \setminus \{x\}) = \{x\},$$

so that U_x is the smallest neighborhood of x. And it is well known that a space is Alexandroff iff each of its points has smallest neighborhood. □

Note that the space from 4.8 above is in fact also $\mathrm{T_D}$, so it is not possible to weaken "bi-Heyting" to "pseudoalexandroff" in the last proposition.

To summarize, we have arrived at the following picture:



$$\mathrm{T_0}\ \&\text{Alexandroff} \Longleftrightarrow \mathrm{T_D}\ \&\ \text{Janus} \Longleftrightarrow \mathrm{T_D}\ \&\ \text{bi-Heyting} \underset{\Leftarrow\times=}{\Longrightarrow} \mathrm{T_D}\ \&\ \text{pseudoalexandroff}.$$

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References

[1] Harold Simmons, *An algebraic version of Cantor-Bendixson analysis*, Categorical Aspects of Topology and Analysis, Lecture Notes in Math., vol. 915, Springer-Verlag, Berlin and New York, pp. 310–323.
[2] ______, *The extended Cantor-Bendixson analysis of trees*, available at http://www.cs.man.ac.uk/~hsimmons/DOCUMENTS/PAPERSandNOTES/CB-analysis.pdf.