$\alpha\gamma$ algebras - a new class of residuated lattices

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Abstract

An $\alpha\gamma$ algebra is a residuated lattice satisfying conditions:
$(C_\rightarrow)$: $(x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$ and $(C_\wedge)$: $x \wedge y = [x \odot (x \rightarrow y)] \vee [y \odot (y \rightarrow x)]$,
while an α algebra (γ algebra) is a residuated lattice satisfying condition $(C_\rightarrow)$ ($(C_\wedge)$ respectively).
Recall that a BL algebra is a bounded residuated lattice satisfying conditions:
(prel): $(x \rightarrow y) \vee (y \rightarrow x) = 1$ and (div): $x \wedge y = x \odot (x \rightarrow y)$,
while a MTL algebra (bounded divisible residuated lattice = bounded commutative Rl-monoid) is a bounded residuated lattice satisfying condition (prel) ((div) respectively).
We get: (prel) $\Longleftrightarrow$ $(C_\rightarrow)$ + $(C_\vee)$ $\Longleftrightarrow$ $(C_\wedge)$ + (C_ε) and (div) $\Longleftrightarrow$ $(C_\rightarrow)$+ (C_δ) $\Longleftrightarrow$ $(C_\wedge)$ + (C_π),
where $(C_\vee)$: $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$ and the independent conditions (C_δ), (C_ε), (C_π) must be found (open problem). It follows that: (1) bounded $\alpha\gamma$ algebras are a common generalization of MTL algebras and of bounded divisible residuated lattices; (2) the MTL algebras with condition (DN) (Double Negation): for all x, $(x^-)^- = x$, and the bounded $\alpha\gamma$ algebras with condition (DN) are the IMTL algebras (just like the BL algebras with condition (DN) and the divisible bounded residuated lattices with condition (DN) are the MV algebras); therefore, we have obtained classes of examples of MTL algebras and of bounded $\alpha\gamma$ algebras by starting with IMTL algebras and by using the ordinal product; (3) the ordinal product of two proper bounded $\alpha\gamma$ algebras is again a proper bounded $\alpha\gamma$ algebra; (4) the ordinal product: linearly ordered MTL (BL) algebra $\bigodot$ MTL (BL) algebra is again a MTL (BL) algebra, while the ordinal product: not-linearly ordered MTL (BL) algebra $\bigodot$ MTL (BL) algebra is only a bounded $\alpha\gamma$ algebra (bounded divisible residuated lattice).

We give classes of examples of finite proper IMTL algebras, MTL algebras and bounded $\alpha\gamma$ algebras, satisfying or not satisfying condition (WNM) (Weak Nilpotent Minimum): for all x, y, $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$.

Keywords BCK(P) lattice, residuated lattice, BL algebra, divisible residuated lattice, commutative Rl-monoid, weak-BL algebra, MTL algebra, IMTL algebra

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Content

1. Introduction

Residuated lattices were introduced by Krull, in 1927, as an algebraic counterpart of logics without contraction rule; they have been investigated (cf. Kowalski-Ono) by Krull, Dilworth, Ward and Dilworth, Ward, Balbes and Dwinger, Pavelka, Idziak and others.

Residuated lattices have been known under many names; they have been called *BCK lattices*, *full BCK-algebras*, FL_{ew}-*algebras* and *integral, residuated, commutative l-monoids*; some of those definitions are free of 0.

We consider the following definition:

A *residuated lattice* is an algebra

$$\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 1)$$

of type $(2, 2, 2, 2, 1)$, verifying:

(R1) $(A, \wedge, \vee, 1)$ is a lattice (under $\geq$) with greatest element 1,

(R2) $(A, \odot, 1)$ is an abelian (i.e. commutative) monoid,

(RP) for all $x, y, z \in A$, $y \rightarrow z \geq x \;\Leftrightarrow\; z \geq x \odot y$.

Let **R-L** denote the class of residuated lattices and **R-L**^{b} denote the class of bounded residuated lattices.

Let $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ be a bounded residuated lattice. Define a negation on A by: for all $x \in A$,

$$x^- \stackrel{def}{=} x \rightarrow 0.$$

$$* * *$$

A bounded residuated lattices satisfying condition (DN) (Double Negation):
(DN) for all x, $\quad (x^-)^- = x$,

is called
an **involutive residuated lattice** or
a **Girard monoid**.

Divisible (bounded) residuated lattices,
introduced in dual form by Swamy, in 1965,
are (bounded) residuated lattices satisfying condition (div), where:

(div) for all x, y, $\qquad x \wedge y = x \odot (x \to y)$.

A duplicate name in the literature for
bounded residuated lattices is
bounded commutative R$\boldsymbol{l}$-monoid.

Let **divR-L**^{b} denote the class of divisible bounded residuated lattices.

Divisible bounded residuated lattices satisfying condition (DN) are MV algebras (MV algebras were introduced by C.C. Chang in 1958).

BL algebras,
introduced by P. Hájek in 1996,
are bounded residuated lattices
satisfying the divisibility (div) and prelinearity (prel) conditions,
where :

(prel) for all x, y, $\qquad (x \to y) \vee (y \to x) = 1.$

Let **BL** denote the class of BL algebras.

BL algebras satisfying condition (DN) are MV algebras.

Results

• BL algebras algebras and divisible bounded residuated lattices are closely connected by the property that, by adding the (DN) condition to both of them, we get MV algebras.

Therefore, we have obtained examples of BL algebras and divisible bounded residuated lattices by starting with MV algebras and by using the ordinal product [7].

• They are also connected by [6], Corollary 6.10, saying that:

(1) the ordinal product
"linearly ordered BL algebra $\odot$ BL algebra"
is again a BL algebra, while

(2) the ordinal product
"non-linearly ordered BL algebra $\odot$ BL algebra"
is only a divisible bounded residuated lattice;

(3) the ordinal product
"divisible bounded residuated lattice $\odot$ divisible bounded residuated lattice"
is again a divisible bounded residuated lattice.

MTL (Monoidal t-norm based Logic) algebras
and
weak-BL algebras,
were introduced independently in 2001,
as bounded residuated lattice satisfying condition (prel).

Hence, they are duplicate names for the same structure:

- MTL algebras were introduced by Esteva and Godo starting from the monoidal t-norm based logic, a generalization of Hájek's Basic Logic, while

- weak-BL algebras were introduced by Flondor, Georgescu and Iorgulescu as commutative weak-pseudo-BL algebras (pseudo-BL algebra being a non-commutative generalization of BL algebra).

Let **MTL = weak-BL** denote the class of MTL algebras = weak-BL algebras.

IMTL (Involutive MTL) algebras,
WNM (Weak Nilpotent Minimum) algebras and
NM (Nilpotent Minimum) algebras
were introduced also in 2001 by Esteva and Godo,
as particular cases of MTL algebras:

- IMTL algebras are MTL algebras satisfying condition (DN),

- WNM algebras are MTL algebras satisfying condition (WNM):
for all x, y, $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$,

and

- NM algebras are IMTL algebras satisfying condition (WNM) (or WNM algebras satisfying condition (DN)).

Let **WNM**, **IMTL** and **NM** denote the classes of WNM algberas, IMTL algebras and NM algebras, respectively.

Hence we have:
IMTL = **MTL** + (DN),
WNM = **MTL** + (WNM),
NM = **IMTL** + (WNM) = **WNM** + (DN).

An important class of (bounded) residuated lattices was proved [2], [6] to be the class of **(bounded) $\alpha\gamma$ algebras**,
i.e. the class of (bounded) residuated lattices satisfying conditions $(C_{\rightarrow})$ and $(C_{\wedge})$.

They are closely connected to MTL algebras, as divisible bounded residuated lattices are closely connected to BL algebras:

Results

• $\alpha\beta^b = \alpha\beta\gamma^b =$ MTL algebras and $\alpha\gamma^b$ algebras are closely connected by the property that, by adding the (DN) condition to both of them, we get IMTL algebras.

Therefore, we have obtained examples of MTL algebras and of bounded $\alpha\gamma$ algebras by starting with IMTL algebras and by using the ordinal product [8].

• They are also connected by [6], Corollaries 6.10 and 6.12, saying that:

(1) the ordinal product
"linearly ordered MTL algebra $\odot$ MTL algebra"
is again a MTL algebra, while

(2) the ordinal product
"not-linearly ordered MTL algebra $\odot$ MTL algebra"
is only a bounded $\alpha\gamma$ algebra;

(3) the ordinal product of two bounded $\alpha\gamma$ algebras
is again a bounded $\alpha\gamma$ algebra.

2. New classes of (bounded) residuated lattices

Let $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 1)$ be a residuated lattice.

Let us consider the following conditions [6]:

$$
\begin{aligned}
&(C_{\rightarrow}) \quad (x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x,\\
&(C_{\vee}) \quad x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x],\\
&(C_{\wedge}) \quad x \wedge y = [x \odot (x \rightarrow y)] \vee [y \odot (y \rightarrow x)].
\end{aligned}
$$

We have got the following relations :

(prel) $\Longleftrightarrow (C_{\rightarrow}) + (C_{\vee}) \Longleftrightarrow (C_{\wedge}) + (C_{\varepsilon})$.

(div) $\Longleftrightarrow (C_{\rightarrow})+ (C_{\delta}) \Longleftrightarrow (C_{\wedge}) + (C_{\pi})$,

where the independent conditions (C_{δ}), (C_{ε}), (C_{π}) must be found (open problem).

A (bounded) α *algebra*
is a (bounded) residuated lattice verifying condition $(C_{\rightarrow})$.
A (bounded) β *algebra*
is a (bounded) residuated lattice verifying condition $(C_{\vee})$.
A (bounded) γ *algebra*
is a (bounded) residuated lattice verifying condition $(C_{\wedge})$.
A (bounded) δ *algebra*
is a (bounded) residuated lattice verifying condition (C_{δ}).
A (bounded) ε *algebra*
is a (bounded) residuated lattice verifying condition (C_{ε}).
A (bounded) π *algebra*
is a (bounded) residuated lattice verifying condition (C_{π}).

A (bounded) $\alpha\beta$ *algebra*
is a (bounded) residuated lattice verifying $(C_{\rightarrow})$ and $(C_{\vee})$,
a (bounded) $\alpha\gamma$ *algebra*
is a (bounded) residuated lattice verifying $(C_{\rightarrow})$ and $(C_{\wedge})$,
etc.,

a (bounded) $\alpha\beta\gamma\delta\varepsilon\pi$ *algebra*
is a (bounded) residuated lattice verifying all of the six conditions $(C_{\rightarrow})$, $(C_{\vee})$, $(C_{\wedge})$, (C_{δ}), (C_{ε}), (C_{π}).

Their classes, in bounded case, are denoted by
α^b**,** β^b**,** γ^b**,** $\ldots$**,** $\alpha\beta^b$**,** $\ldots$**,** $\alpha\beta\gamma\delta\varepsilon\pi^b$, respectively.

Then we have obtained the following [6]:

$$\alpha\beta^b \;=\; \gamma\varepsilon^b \;\cong\; \mathbf{R\text{-}L}^b + (\mathrm{prel}) = \mathbf{MTL},$$

$$\alpha\delta^b \;=\; \gamma\pi^b \;\cong \mathbf{R\text{-}L}^b + (\mathrm{div}) = \mathbf{divR\text{-}L}^b.$$

$\alpha\beta\gamma\delta\varepsilon\pi^b \;=\; \alpha\beta\delta^b \;=\; \alpha\beta\pi^b \;=\; \alpha\delta\varepsilon^b \;=\; \beta\gamma\pi^b \;=\; \gamma\delta\varepsilon^b \;=\; \gamma\varepsilon\pi^b \cong$ $\mathbf{BL}$.

$$\alpha\beta^b \ + \ (div) \ = \ \alpha\beta^b \ + \ (C_\delta) \ = \ \alpha\beta\delta^b \ \cong \mathbf{BL},$$
$$\alpha\delta^b \ + \ (prel) \ = \alpha\delta^b \ + \ (C_\vee) \ = \ \alpha\delta\beta^b \ \cong \mathbf{BL}.$$

$$\alpha\gamma^b \ + \ (prel) = \alpha\gamma^b \ + \ (C_\vee) = \alpha\gamma\beta^b = \alpha\beta^b \cong \mathbf{MTL},$$
$$\alpha\gamma^b \ + \ (div) = \alpha\gamma^b \ + \ (C_\delta) = \alpha\gamma\delta^b = \alpha\delta^b \cong \mathbf{divR\text{-}L}^b.$$

Since we have obtained in [6] that:

$$(C_\wedge) + \text{(DN)} \Longleftrightarrow (C_\vee) + \text{(DN)} \Longleftrightarrow (C_\wedge) + (C_\vee) + \text{(DN)}.$$

it follows:

$$\beta_{(DN)} = \gamma_{(DN)} = \beta\gamma_{(DN)},$$

$$\alpha\gamma_{(DN)} = \alpha\beta_{(DN)} \cong \mathbf{IMTL},$$

$$\alpha\gamma\delta_{(DN)} = \alpha\beta\delta_{(DN)} \cong \mathbf{MV}.$$

3. Classes of examples of proper bounded $\alpha\gamma$ algebras, with or without condition (WNM)

Recall [6] that these algebras cannot be linearly ordered.

This part can be developed as for divisible bounded residuated lattices [7].

The examples will be of the form:

non-linearly ordered IMTL/MTL algebra $\odot$ IMTL/MTL algebra,

more precisely of one of the following forms:

(1) non-linearly ordered IMTL/NM $\odot$ linearly ordered IMTL/NM,
(2) non-linearly ordered IMTL/NM $\odot$ non-linearly ordered IMTL/NM,
(3) non-linearly ordered IMTL/NM $\odot$ linearly ordered MTL/WNM,
(4) non-linearly ordered IMTL/NM $\odot$ non-linearly ordered MTL/WNM;

(5) non-linearly ordered MTL/WNM $\odot$ linearly ordered IMTL/NM,
(6) non-linearly ordered MTL/WNM $\odot$ non-linearly ordered IMTL/NM,
(7) non-linearly ordered MTL/WNM $\odot$ linearly ordered MTL/WNM,
(8) non-linearly ordered MTL/WNM $\odot$ non-linearly ordered MTL/WNM.

But, there are also other examples.

3.1 Classes of examples of proper ${}_{(WNM)}\alpha\gamma$ algebra

• Example of form (5):

Let us consider the ordinal product of the non-linearly ordered, proper ${}_{(WNM)}\alpha\beta$ algebra (WNM algebra) $\mathcal{A}_1$ and of $\mathcal{F}_2$. Hence, let us consider the set $A = \{0, n, a, b, c, d, 1\}$, organized as a lattice as in Figure 1 and as a bounded residuated lattice with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

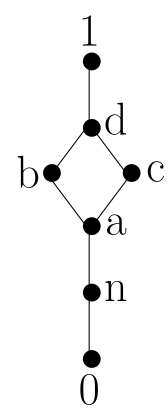

Figure 1: Example of proper ${}_{(WNM)}\alpha\gamma$ algebra

$\rightarrow$	0	n	a	b	c	d	1
0	1	1	1	1	1	1	1
n	c	1	1	1	1	1	1
a	n	n	1	1	1	1	1
b	0	n	c	1	c	1	1
c	n	n	b	b	1	1	1
d	0	n	a	b	c	1	1
1	0	n	a	b	c	d	1

$\odot$	0	n	a	b	c	d	1
0	0	0	0	0	0	0	0
n	0	0	0	n	0	n	n
a	0	0	a	a	a	a	a
b	0	n	a	b	a	b	b
c	0	0	a	a	c	c	c
d	0	n	a	b	c	d	d
1	0	n	a	b	c	d	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a non-linearly ordered bounded residuated lattice, which satisfies conditions $(C_{\rightarrow})$, $(C_{\wedge})$ and (WNM). $\mathcal{A}$ does not satisfy condition $(C_{\vee})$, for b, c.

Consequently, $\mathcal{A}$ is a non-linearly ordered, proper ${}_{(WNM)}\alpha\gamma$ algebra.

3.2 Classes of examples of proper $\alpha\gamma^b$ algebras

We present some classes of examples:

• Examples of the forms (1) and (2): the ordinal product of non-linearly ordered NM or IMTL algebra $\odot$ (linearly ordered or non-linearly ordered) NM or IMTL algebra:

$\mathcal{F}_{2\times 2} \odot \mathcal{F}_4$, $\mathcal{F}_{2\times 2} \odot \mathcal{F}_5$ etc., $\mathcal{F}_{4\times 2} \odot \mathcal{F}_2$, $\mathcal{F}_{2\times 2} \odot \mathcal{F}_{4\times 2}$, $\mathcal{F}_{2\times 2} \odot \mathrm{A}(\mathbf{p}, b_1)$ are examples of proper $\alpha\gamma^b$ algebras.

We develop $\mathcal{F}_{2\times 2} \odot \mathcal{F}_4$. Let us consider the set $A = \{0, a, b, c, 1, 2, 3\} = \{0, a, b, c\} \cup \{c, 1, 2, 3\}$, organized as a bounded lattice as in Figure 2 and as a bounded residuated lattice with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

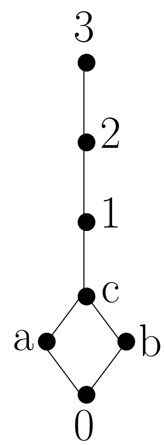

Figure 2: Example of proper $\alpha\gamma^b$ algebra and ${}_{(WNM)}\gamma$ algebra

$\rightarrow$	0	a	b	c	1	2	3
0	3	3	3	3	3	3	3
a	b	3	b	3	3	3	3
b	a	a	3	3	3	3	3
c	0	a	b	3	3	3	3
1	0	a	b	2	3	3	3
2	0	a	b	1	1	3	3
3	0	a	b	c	1	2	3

$\odot$	0	a	b	c	1	2	3
0	0	0	0	0	0	0	0
a	0	a	0	a	a	a	a
b	0	0	b	b	b	b	b
c	0	a	b	c	c	c	c
1	0	a	b	c	c	c	1
2	0	a	b	c	c	2	2
3	0	a	b	c	1	2	3

Then, $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1) = \mathcal{F}_{2\times 2} \odot \mathcal{F}_4$ is a bounded residuated lattice, which satisfies conditions $(C_{\rightarrow})$ and $(C_{\wedge})$, i.e. is an $\alpha\gamma$ algebra.
Note that it is a proper $\alpha\gamma^b$ algebra, since it does not satisfy condition $(C_{\vee})$ for a, b, (WNM) for 1.

- **Other examples**:

We present two other examples.

Example 1. Let us consider the set $A = \{0, a, c, d, m, 1\}$ organized as a lattice as in Figure 3 and as a bounded residuated lattice with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

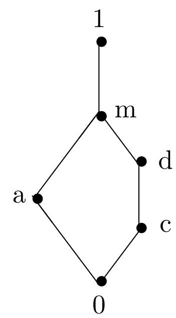

Figure 3: Example 1 of proper $\alpha\gamma^b$ algebra

$\rightarrow$	0	a	c	d	m	1
0	1	1	1	1	1	1
a	d	1	d	d	1	1
c	a	a	1	1	1	1
d	a	a	m	1	1	1
m	0	a	d	d	1	1
1	0	a	c	d	m	1

$\odot$	0	a	c	d	m	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
c	0	0	c	c	c	c
d	0	0	c	c	c	d
m	0	a	c	c	m	m
1	0	a	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded residuated lattice, which satisfies conditions $(C_{\rightarrow})$ and $(C_{\wedge})$, i.e. is is an $\alpha\gamma$ algebra, without condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Note that it is a proper $\alpha\gamma^b$ algebra, since: it does not satisfy condition $(C_{\vee})$ for a, d, (WNM) for d, m.

Example 2. Let us consider the set $A = \{0, a, b, c, d, m, 1\}$ organized as a lattice as in Figure 4 and as a bounded residuated lattice with the operation $\rightarrow$ and $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$ as in the following tables:

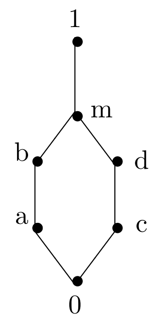

Figure 4: Example 2 of proper $\alpha\gamma^b$ algebra

$\rightarrow$	0	a	b	c	d	m	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	m	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	m	1	1	1
m	0	b	b	d	d	1	1
1	0	a	b	c	d	m	1

$\odot$	0	a	b	c	d	m	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
m	0	a	a	c	c	m	m
1	0	a	b	c	d	m	1

Then $\mathcal{A} = (A, \wedge, \vee, \rightarrow, 0, 1)$ is a bounded residuated lattice, which satisfies conditions $(C_{\rightarrow})$ and $(C_{\wedge})$, i.e. is an $\alpha\gamma$ algebra, without condition (DN) (you have the values of $x^- = x \rightarrow 0$ in the table of $\rightarrow$, column of 0).

Note that it is a proper $\alpha\gamma^b$ algebra, since: it does not satisfy condition $(C_{\vee})$ for b, d, (WNM) for b.

References

[1] A. Iorgulescu, Classes of BCK algebras - Part I, Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 1/2004, 1-33.

[2] A. Iorgulescu, Classes of BCK algebras - Part II, Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 2/2004, 1-27.

[3] A. Iorgulescu, Classes of BCK algebras - Part III, Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 3/2004, 1-37.

[4] A. Iorgulescu, Classes of BCK algebras - Part IV, Preprint Series of the Institute of Mathematics of the Romanian Academy, preprint nr. 4/2004, 1-25.

[5] A. Iorgulescu, On BCK algebras - Part I: An attempt of unifying treatment of algebras of logic, submitted.

[6] A. Iorgulescu, On BCK algebras - Part II: New algebras. The ordinal sum (product) of two bounded BCK algebras, Soft Computing, accepted.

[7] A. Iorgulescu, On BCK algebras - Part III: Classes of examples of finite proper MV algebras, BL algebras and divisible bounded residuated lattices, with or without condition (WNM), manuscript.

[8] A. Iorgulescu, On BCK algebras - Part IV: Classes of examples of IMTL algebras, MTL and bounded $\alpha\gamma$ algebras, with or without condition (WNM), manuscript.