Notes on Complexity of Monoidal T-norm Based Logic and its Extensions

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ML and MTL

- Monoidal Logic (ML) is Full Lambek calculus with exchange and weakening (FL_{ew}). Also known as IMALLW (multiplicative additive fragment of Intuitionistic Linear Logic with weakening).
- Monoidal T-norm Based Logic (MTL) is a schematic extension of ML by the following axiom schema:

$$(\varphi \to \psi) \vee (\psi \to \varphi)$$

- C_nML (resp. C_nMTL) is an extension of ML (resp. MTL) by the following axiom schema:

$$\varphi^{n-1} \to \varphi^n$$

ML-algebras and MTL-algebras

Definition

An ML-algebra is an algebra $\mathbf{A} = (A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ where the following conditions are satisfied:

- $(A, *, \to, \wedge, \vee, \mathbf{1})$ is a commutative integral residuated lattice,
- $\mathbf{0}$ is a bottom element.

Definition

An MTL-algebra is an ML-algebra $\mathbf{A} = (A, *, \to, \wedge, \vee, \mathbf{0}, \mathbf{1})$ such that

- $(x \to y) \vee (y \to x) = \mathbf{1}$ for all $x, y \in A$.

In other words, an MTL-algebra is a representable ML-algebra.

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- IMALL (ML without weakening) is known to be PSPACE-hard, hence PSPACE-complete (Lincoln, Mitchell, Scedrov, Shankar 94)

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- IMALL (ML without weakening) is undecidable (as full ILL)

Motivation

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- Hypersequent calculus for MTL is not suitable for proof search.
- On the other hand, ML has a nice sequent calculus.
- Is it possible somehow to translate provability between MTL and ML?

Main result

Let φ be a formula in the language of MTL and $S = \{\psi_1, \dots, \psi_n\}$ a set of all subformulas of φ.

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$$\psi_1 \leq \psi_2 \leq \cdots \leq \psi_n$$

can be coded as follows:

$$T = \{\psi_1 \to \psi_2, \psi_2 \to \psi_3, \ldots, \psi_{n-1} \to \psi_n\}$$

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Theorem

$\vdash_{\mathrm{MTL}} \varphi$ *iff for all* $T \in \mathcal{O}$ *we have* $T \vdash_{\mathrm{ML}} \varphi$.

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Theorem

$\vdash_{\mathrm{C_nMTL}} \varphi$ *iff for all* $T \in \mathcal{O}$ *we have* $T \vdash_{\mathrm{C_nML}} \varphi$.

Sketch of the proof

- The right-to-left direction is easy since MTL is complete w.r.t. the class of all MTL-chains.
- Suppose that there is $T \in \mathcal{O}$ such that $T \nvdash_{\mathrm{ML}} \varphi$.

Sketch of the proof

- The right-to-left direction is easy since MTL is complete w.r.t. the class of all MTL-chains.
- Suppose that there is $T \in \mathcal{O}$ such that $T \nvdash_{\mathrm{ML}} \varphi$.
- Since ML has FEP, there is a finite ML-algebra **A** such that $T \not\models_{\mathbf{A}} \varphi$.
- There is an **A**-evaluation e such that $e(T) \subseteq \{1\}$ and $e(\varphi) < 1$.
- Thus the set $e(S)$ is totally ordered, i.e.

$$e(S) = \{1 > a_1 > \cdots > a_n > 0\}$$

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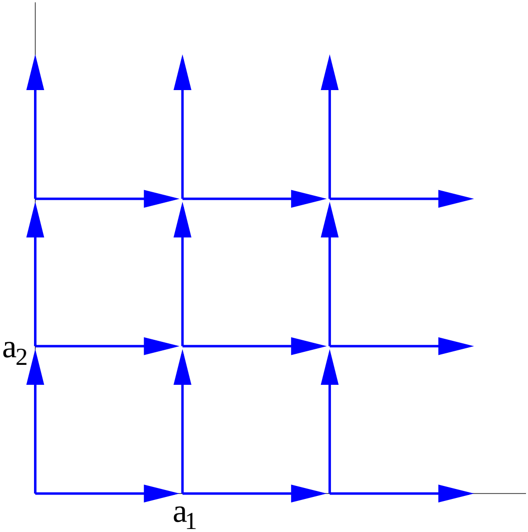
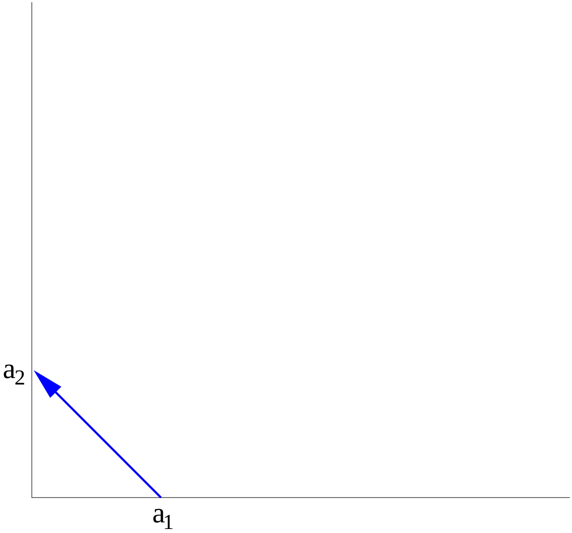
- Note that the quasi-order $\lesssim$ need not be total.

2 generators



2 generators



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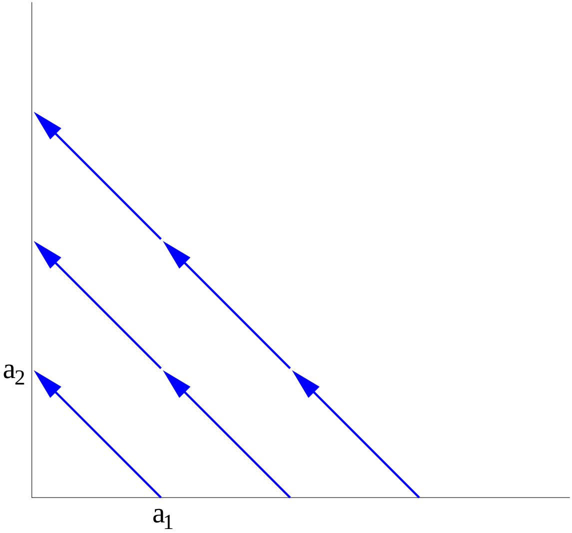

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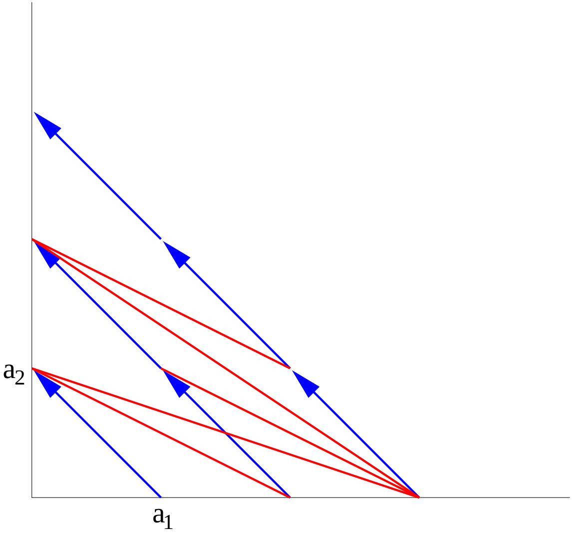
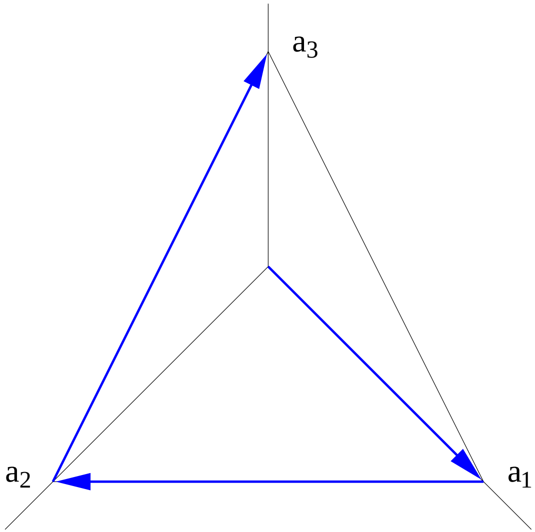


3 generators



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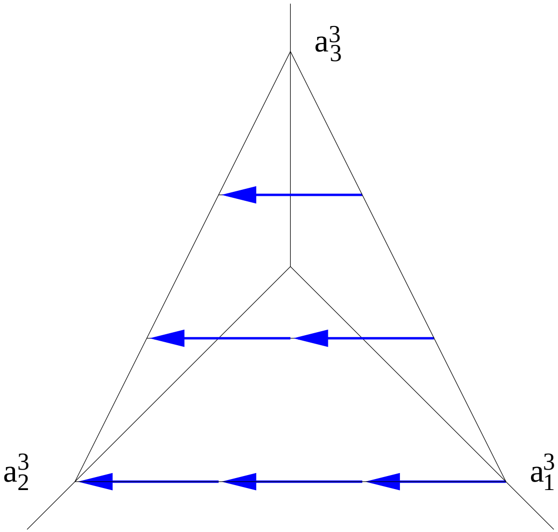

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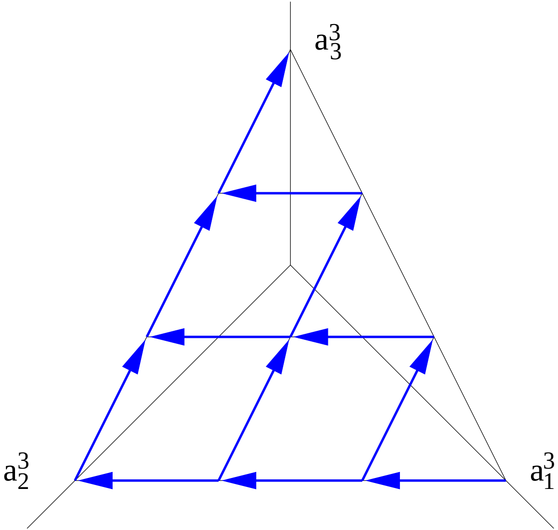

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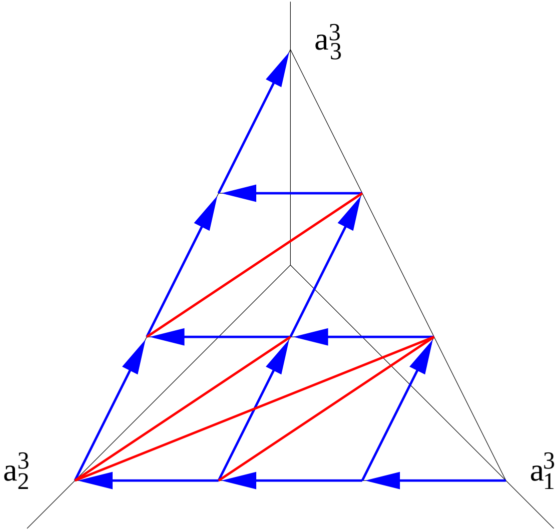

Let $\leq_\ell$ be the component-wise partial order on $\mathbb{N}^n$ and $\leq_{\text{lex}}$ the lexicographic total order on $\mathbb{N}^n$.

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Lemma

Let $x, y \in \mathbb{N}^n$ and

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then the relation $\leq_{\mathbf{R}}$ defined by

$$x \leq_{\mathbf{R}} y \text{ iff } \mathbf{R} \cdot x \geq_\ell \mathbf{R} \cdot y$$

is a partial order monotone w.r.t. $+$.

Moreover, if $x \leq_{\mathbf{R}} y$ then $x \lesssim y$, i.e. $\leq_{\mathbf{R}}$ is a sub-quasi-order of $\lesssim$.

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Let $\sim'$ be the equivalence corresponding to $\lesssim'$. Then $\mathbb{N}^n/\sim'$ is an MTL-algebra into which the partial subalgebra $e(S)$ of $\mathbf{A}$ can be embedded.

Questions for audience

- Is there any bound on counter-models in ML?
- Is it known whether ML is PSPACE complete?

MTL and ML

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- iff $\vdash_{\mathbf{GILLW}} !\alpha_1, \dots, !\alpha_k \Rightarrow \varphi$ by a cut-free proof

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- iff $\vdash_{\mathbf{GILLW}} !\alpha_1, \dots, !\alpha_k \Rightarrow \varphi$ by a cut-free proof
- a standard argument used e.g. in (Lincoln, Mitchell, Scedrov, Shankar 94)
- What is complexity of ILLW?
- We need less than full **GILLW**:

GML!

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Contraction rule for $!\varphi$ admissible

$$\frac{\Gamma, !\varphi, !\varphi \Rightarrow \delta}{\Gamma, !\varphi \Rightarrow \delta} \text{ !-contr}$$

Cut rule can be eliminated

How to create proof search in **GML**! and what is its complexity?

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- iff $\vdash_{\mathbf{GC_nML}} \alpha_1^{n-1}, \dots, \alpha_k^{n-1} \Rightarrow \varphi$ by a cut free proof
- Complexity of proof search in $\mathbf{GC_nML}$?