On varieties of distributive lattice effect algebras

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1. Introduction

2. Solution
The Kolmogorov system gave the axiomatical base of probability theory to be a rigorous part of mathematics.

Its basic assumption is that the set of probabilistically interesting events can be modelled by a \( \sigma \)-algebra of subsets of a set.

However it was recognized that in a new physics, quantum mechanics, the Heisenberg uncertainty principle holds asserting that the momentum of an elementary particle and its position can not be measured simultaneously with an arbitrarily prescribed accuracy.
This shows that the Kolmogorov model is not appliable for this process, or that quantum mechanical events do not satisfy the axioms of Boolean algebras.

Hence a new theory, nowadays a theory of quantum structures was found to describe quantum mechanical events, starting by a paper of Birkhoff and von Neumann.

Today a similar phenomena as those in quantum mechanics can be observed in other scientific branches, like social sciences, psychology, economy etc.

We have a system (described as a set) tested by a family of tests (or hypotheses on the tested system) and the result of such a test is an effect, a function dominated by some test. Combining natural equivalences of effects, we obtain an effect algebra.
Effect algebras

Effect algebras, introduced by D. Foulis and M. K. Bennett in 1994, have been recognized to be the appropriate algebraic tool for considerations in quantum mechanics.

An **effect algebra** is a system \( E = (E; +, 0, 1) \) where 0 and 1 are two distinguished elements of \( E \), + is a partial binary operation on \( E \) satisfying the conditions:

1. **(EA1)** \( a + b = b + a \) whenever \( a + b \) exists;
2. **(EA2)** \( a + (b + c) = (a + b) + c \) if one of the sides is defined;
3. **(EA3)** for every \( a \in E \) there exists a unique \( a' \in E \) with \( a + a' = 1 \);
4. **(EA4)** if \( a + 1 \) is defined then \( a = 0 \).
Given an effect algebra $\mathcal{E} = (E; +, 0, 1)$, the relation $\leq$ on $E$ defined by

$$a \leq b \iff b = a + c \text{ for some } c \in E$$

(1)

is a partial order on $E$ with 0 and 1 a least or a greatest element of $E$, respectively. If $(E; \leq)$ is a lattice then $\mathcal{E}$ is said to be a lattice-ordered effect algebra or a lattice effect algebra in brief. Orthomodular lattices and MV-algebras serve as natural examples of lattice effect algebras, namely:
Effect algebras

(1) If \((L; \lor, \land, \bot, 0, 1)\) is an orthomodular lattice then defining

\[
  a + b := a \lor b \text{ iff } a \leq b^\perp,
\]

\((L; +, 0, 1)\) turns out to be a lattice effect algebra

(2) Given an MV-algebra \(A = (A; \oplus, \neg, 0)\), defining

\[
  a + b := a \oplus b \text{ iff } a \leq \neg b,
\]

\((A; +, 0, 1)\) is a lattice effect algebra, where \(a' = \neg a\).

Effect algebras can be equivalently defined as so-called D-posets (Kôpka, Chovanec): one can define on every effect algebra a partial operation ”−” (called the difference) as follows:

\[
  a - b = c \text{ iff } a = b + c.
\]
Two elements \( a, b \) of an effect algebra \( \mathcal{E} \) are said to commute \( (aCb) \), if there are \( c, d \in E \) such that \( c \leq a, b \leq d \) and \( d - a = b - c \). In a lattice ordered effect algebra, \( aCb \) iff \( (a \lor b) - b = a - (a \land b) \). The relation \( C \) has the following basic properties:

\[
\begin{align*}
  aCa, \quad aC0, \quad aC1 \\
  aCb \text{ iff } bCa \\
  aCb \text{ iff } aCb' \\
  \text{if } aCb \text{ and } aCc \text{ then } aC(b + c) \text{ whenever } b + c \text{ exists.}
\end{align*}
\]

By a **block** of \( \mathcal{E} \) is meant a maximal subset \( B \) of pairwise commuting elements of \( \mathcal{E} \). Z. Riečanová has shown that every lattice effect algebra is a union of its blocks, which are in fact MV-algebras.
Although effect algebras are useful in axiomatization of unsharp quantum logic, their disadvantage is that they are **partial algebras**, thus for their investigation one can not use very well developed tools known for total algebras. This essential disadvantage of effect algebras has been successfully overcome as follows:

**A lattice with sectional antitone involutions** is a system \( \mathcal{L} = (L; \lor, \land, (^a)_{a \in L}, 0, 1) \), where \((L; \lor, \land, 0, 1)\) is a bounded lattice such that every principal order-filter \([a, 1]\) (which is called a **section**) possesses an antitone involution \(x \mapsto x^a\).

The family \((^a)_{a \in L}\) of sectional antitone involutions, which are partial unary operations on \(L\), can be equivalently replaced by a single binary operation defined by

\[
x \rightarrow y := (x \lor y)^y.
\]

This allows one to treat lattices with sectional antitone involutions as total algebras \((L; \lor, \land, \rightarrow, 0, 1)\), or even \((L; \rightarrow, 0, 1)\), that form a variety.
Lemma

Let $\mathcal{L} = (L; \lor, \land, (a)_{a \in L}, 0, 1)$ be a lattice with sectional antitone involutions. Then the assigned algebra $\mathcal{A}(\mathcal{L}) = (L; \oplus, \neg, 0)$, where $x \oplus y := (x^0 \lor y)^y$ and $\neg x := x^0$, satisfies the identities

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$;

(BA3) $x \oplus 1 = 1 \oplus x = 1$;

(BA4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(BA5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

By a basic algebra we mean an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities (BA1)–(BA5) (where $1 := \neg 0$).
Conversely, given a basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$, relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad \neg x \oplus y = 1$$

is a lattice order on $A$ with 0 and 1 a least and a greatest element of $A$, respectively. Observe also that the mapping

$$x \mapsto x^a := \neg x \oplus a$$

is an antitone involution on the section $[a, 1]$.

**Lemma**

*Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra with the induced lattice $\mathcal{L}(\mathcal{A}) = (A; \lor, \land, 0, 1)$, and define for $a \in A$, $x \in [a, 1]$, $x^a := \neg x \oplus a$. Then the structure $\mathcal{L}(\mathcal{A}) = (A; \lor, \land, (a)_{a \in A}, 0, 1)$ is a bounded lattice with sectional antitone involutions.*

Moreover, the correspondence is 1-1: $\mathcal{L}(\mathcal{A}(\mathcal{L})) = \mathcal{L}$ and $\mathcal{A}(\mathcal{L}(\mathcal{A})) = \mathcal{A}$.
Next step is to relate basic algebras to lattice effect algebras:

**Lemma**

A basic algebra $\mathcal{A}$ is a lattice effect algebra iff it satisfies the identity

\[(x \land \neg y) \oplus [(\neg (x \oplus y) \land z) \oplus y] = (x \oplus y) \oplus (\neg (x \oplus y) \land z).\]  

Hence the class $EBA$ of lattice effect algebras forms a subvariety of the variety $BA$ of basic algebras.
The variety $EBA$ of effect basic algebras arithmetical and regular. This yields its congruence distributivity and, by the Baker-Pixley theorem, each subvariety generated by a finite effect algebra is finitely based, i.e. it can be presented by a finite number of identities. Moreover, using Jónsson lemma, all subdirectly irreducible algebras of this variety are contained among its subalgebras and quotient algebras. By a block of a basic algebra $A$ is meant a maximal subset $B \subseteq A$ of pairwise commuting elements of $A$, i.e. $x \oplus y = y \oplus x$ for all $x, y \in B$. Remark that a block of a basic algebra need not be its subalgebra.
Lemma

For a basic algebra \( \mathcal{A} \) the following are equivalent:

(i) \( \mathcal{A} \) is a lattice effect algebra;
(ii) every block of \( \mathcal{A} \) is a subalgebra which is an MV-algebra.

For finite basic algebras the above condition (ii) can be even simplified:

Lemma

Every finite commutative basic algebra is an MV-algebra.

Last two statements now yield:

Lemma

For a finite basic algebra \( \mathcal{A} \) the following are equivalent:

(i) \( \mathcal{A} \) is a lattice effect algebra;
(ii) every block of \( \mathcal{A} \) is its subalgebra.
Denote further \( DEBA \) the variety of lattice effect basic algebras satisfying the distributive lattice identity. In order to describe the bottom of the lattice \( \mathcal{L}(DEBA) \) of all subvarieties of \( DEBA \), we ask for small SI members of \( DEBA \).

It is well known: given a finite chain \( C_{n+1} = \{0 = a_n < a_{n-1} < \cdots < a_1 < a_0 = 1\} \), one can define on \( C_{n+1} \) in a unique way sectional antitone involutions and hence a structure of a basic algebra as follows:

\[
a_j^{a_k} := a_{k-j}
\]

for \( k \geq j \).

The corresponding basic algebras \( C_{n+1} = (C_{n+1}; \oplus, \neg, 0) \) are known to be simple MV-algebras.
The least SI member of $DEBA$ which is not contained in the variety $MV$ of MV-algebras looks as follows: denote $\mathcal{H} = (H; \oplus, \neg, 0)$, where $H = \{0, 1, a, b\}$, $a \oplus a = b \oplus b = 1$ and $\neg a = a, \neg b = b$. In other words, the underlying lattice of $\mathcal{H}$ is

\[
\begin{array}{c}
1 \\
 \downarrow \\
 a = \neg a \\
 \downarrow \\
 0 \\
 \uparrow \\
 b = \neg b \\
 \uparrow \\
 1
\end{array}
\]

Fig. 1
$H$ is simple and the lattice $L(V(H))$ of subvarieties of the variety $V(H)$ generated by $H$ is visualized by following diagram:

- $V(H)$
- $V(C_2)$
- $V(C_1) =$ Boolean algebras
- $\mathcal{T} =$ Trivial variety

Fig. 2

It is also evident that $MV \cap V(H) = V(C_2)$. Hence a natural question arises:

(P1) Are there any other SI members in the variety DEBA distinct from $H$ and not belonging to $MV$?
Outline

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To solve the above problem, we describe the join $V^* = V(H) \lor MV$ in the lattice $\mathcal{L}(DEBA)$. Since the variety $MV$ is generated by $[0,1]$, the problem (P1) can be also reformulated as

(P2) Is it true that $DEBA = V([0,1] \times H)$?

As usual in MV-algebras, denote for any basic algebra $\mathcal{A}$ by $a \odot b := \neg(\neg a \oplus \neg b)$ and $a \rightarrow b := \neg a \oplus b$. 
Basic properties

Lemma

Every basic algebra $\mathcal{A}$ satisfies the identities

1. $x \leq y \Rightarrow x \circ z \leq y \circ z$;
2. $x \leq y \Rightarrow y \rightarrow z \leq x \rightarrow z$.

Moreover, if $\mathcal{A}$ is a lattice effect algebra, then

3. $((x \rightarrow y) \circ (y \rightarrow x)) \circ ((x \rightarrow y) \circ (y \rightarrow x)) \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$ whenever $z \geq x \geq y$;
4. $x \leq y \Rightarrow x \circ x \leq x \circ y$;
5. $x \circ (x \circ x) = (x \circ x) \circ x$. 
Basic properties

Lemma

The variety $\mathcal{V}^*$ satisfies the identities

1. $(x \to y) \circ (x \to y) \leq (y \to z) \to (x \to z)$;
2. $(x \to y) \circ (x \to y) \leq \neg y \to \neg x$;
3. $x \circ (x \to y) \leq y$;
4. $(x \circ y) \circ (y \lor z) \leq ((x \circ y) \circ y) \lor ((x \circ y) \circ z)$ for $x \leq y$. 
Congruence regularity of $EBA$ implies that we need to describe congruence kernels. First we characterize congruence kernels in $\mathcal{V}^*$:

**Lemma**

Let $\mathcal{V}$ be a subvariety of $EBA$ satisfying the identity (6). Let $\mathcal{A} \in \mathcal{V}$ and $1 \in F \subseteq A$. Then $F$ is a congruence kernel on $\mathcal{A}$ iff

1. $\forall x, y \in F : x \odot y \in F$;

2. $F$ is closed under modus ponens, i.e. $x, x \rightarrow y \in F$ imply $y \in F$.

We call subsets $I \subseteq A$ satisfying these properties filters of $\mathcal{A}$. Denote by $\mathcal{F}(\mathcal{A})$ the set of all filters of $\mathcal{A}$.

As a corollary we obtain:

**Lemma**

Let $\mathcal{A} \in \mathcal{V}^*$, $F \subseteq A$. Then $F$ is a congruence kernel on $\mathcal{A}$ iff $F$ is a filter of $\mathcal{A}$.
We are ready to describe principal filters of algebras in the variety \( \mathcal{V}^* \):

**Lemma**

Let \( \mathcal{V} \) be a subvariety of EBA satisfying the identities (6) and (8). If \( \mathcal{A} \in \mathcal{V} \) and \( a \in A \), then

\[
F(a) = \{ x \in A; x \geq n \circ a = a \circ \cdots \circ a \text{ for some } n \in \mathbb{N} \}
\]

is a principal filter of \( \mathcal{A} \) generated by \( a \).
The variety $\mathcal{V}^*$ is as a subvariety of DEBA characterized by the identities (6), (8) and (9), i.e.

\[(6) \quad (x \to y) \circ (x \to y) \leq (y \to z) \to (x \to z);\]
\[(8) \quad x \circ (x \to y) \leq y;\]
\[(9) \quad (x \circ y) \circ (y \lor z) \leq ((x \circ y) \circ y) \lor ((x \circ y) \circ z) \text{ for } x \leq y.\]

As a direct corollary we obtain a partial answer to the problem (P2):

**Theorem**

We have $\text{DEBA} = \mathcal{V}([0, 1] \times \mathcal{H})$ iff (6), (8) and (9) hold in DEBA.
Since $\mathcal{V}^*$ is congruence distributive and $\mathcal{V}^* = MV \lor \mathcal{V}(\mathcal{H})$, we conclude

$$(\mathcal{V}^*)_SI = (MV)_SI \cup (\mathcal{V}(\mathcal{H}))[SI] = (MV)_SI \cup \{\mathcal{H}\}.$$ 

Denote further by $\mathcal{V}_*$ the subvariety of $DEBA$ defined by the identities (6),(8) and (9). In order to prove $\mathcal{V}_* = \mathcal{V}^*$, it suffices to show

$$(\mathcal{V}_*)_SI = (MV)_SI \cup \{\mathcal{H}\}.$$
First we show that given $A \in \mathcal{V}_*$ and a block $B$ of $A$ (which is a support of a subalgebra $B$ of $A$), then every filter of $B$ can be extended to a filter of $A$. For a filter $F$ on $B$ denote by $\mathcal{F}(F)$ a filter of $A$ generated by $F$. We state

**Lemma**

Let $A \in \mathcal{V}_*$. Then $\mathcal{F}(F) = \{ x \in A; a \leq x \text{ for some } a \in F \}$. 

**Lemma**

Let $A \in \mathcal{V}_*$, $B$ be a block of $A$ and $F$ a filter of $B$. Then $\mathcal{F}(F) \cap B = F$. 


Lemma

Let $A \in V_*$ be a SI member and denote $\{1\} \neq M$ the monolith of $\mathcal{F}(A)$. Then every block $B$ of $A$ for which $B \cap M \neq \{1\}$ is a chain.

Lemma

Every finite SI $A \in V_*$ is simple.

Lemma

Let $A$ be a finite SI member of $V_*$, and let $x, y \in A$. Then $xCy$ iff $x \parallel y$. 
Finite case can be checked relatively easily:

**Lemma**

*Every finite SI member $\mathcal{A} \in \mathcal{V}_*$ is either a chain (i.e. an MV-algebra) or $\mathcal{A} = \mathcal{H}$.*

The hardest step in proving our main theorem is to show that:

**Lemma**

*Every infinite SI algebra $\mathcal{A} \in \mathcal{V}_*$ is an MV-chain.*
Denote as before by \( \{1\} \neq M \) the monolith of the lattice of all filters of \( \mathcal{A} \). We distinguish two cases:

**[I.]** Assume that \( aCb \) for all \( a, b \in M \).
Then \( M \) is contained in some block \( B \) of \( \mathcal{A} \) and \( B \cap M \neq \{1\} \), \( M \) is a chain.
We discuss two more subcases of I:

- **[Ia.]** Suppose that there is \( x \in A \) such that \( x \) does not commute with \( a \) for some \( a \in M \).
  We have shown that this case can be violated.
  Hence the following holds:
- **[Ib.]** \( xCa \) for all \( a \in M \) and all \( x \in A \).
  In this case it is an infinite MV-algebra, hence SI MV-chain.

**[II.]** There exist \( x, y \in M \) such that \( x \) and \( y \) do not commute (i.e. \( x \parallel y \)).
We have shown that in this case \( \mathcal{A} \) has to be finite (i.e., by our assumptions can be excluded).