On varieties of distributive lattice effect algebras

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Outline

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Effect algebras

- The Kolmogorov system gave the axiomatical base of probability theory to be a rigorous part of mathematics
- Its basic assumption is that **the set of probabilistically interesting events can be modelled by a σ-algebra of subsets of a set**
- However it was recognized that in a new physics, quantum mechanics, the Heisenberg uncertainty principle holds asserting that the **momentum of an elementary particle and its position can not be measured simultaneously with an arbitrarily prescribed accuracy.**

- This shows that the Kolmogorov model is not appliable for this process, or that **quantum mechanical events do not satisfy the axioms of Boolean algebras**
- Hence a new theory, nowadays a theory of **quantum structures** was found to describe quantum mechanical events, starting by a paper of Birkhoff and von Neumann
- Today a similar phenomena as those in quantum mechanics can be observed in other scientific branches, like social sciences, psychology, economy etc.
- We have a system (described as a set) tested by a family of tests (or hypotheses on the tested system) and the result of such a test is an **effect**, a function dominated by some test. Combining natural equivalences of effects, we obtain an **effect algebra**

Effect algebras

- Effect algebras, introduced by D. Foulis and M. K. Bennett in 1994, have been recognized to be the appropriate algebraic tool for considerations in quantum mechanics.

An **effect algebra** is a system $\mathscr{E} = (E;+,0,1)$ where 0 and 1 are two distinguished elements of E, $+$ is a partial binary operation on E satisfying the conditions:

(EA1) $a+b=b+a$ whenever $a+b$ exists;

(EA2) $a+(b+c)=(a+b)+c$ if one of the sides is defined;

(EA3) for every $a \in E$ there exists a unique $a' \in E$ with $a+a'=1$;

(EA4) if $a+1$ is defined then $a=0$.

Given an effect algebra $\mathscr{E} = (E;+,0,1)$, the relation $\leq$ on E defined by

$$a \leq b \quad \text{iff} \quad b = a + c \text{ for some } c \in E \tag{1}$$

is a partial order on E with 0 and 1 a least or a greatest element of E, respectively. If $(E;\leq)$ is a lattice then $\mathscr{E}$ is said to be a **lattice-ordered effect algebra** or a **lattice effect algebra** in brief.
Orthomodular lattices and MV-algebras serve as natural examples of lattice effect algebras, namely:

(1) If $(L;\vee,\wedge,^{\perp},0,1)$ is an orthomodular lattice then defining

$$a+b := a\vee b \quad \text{iff} \quad a\leq b^{\perp},$$

$(L;+,0,1)$ turns out to be a lattice effect algebra

(2) Given an MV-algebra $\mathscr{A} = (A;\oplus,\neg,0)$, defining

$$a+b := a\oplus b \quad \text{iff} \quad a\leq \neg b,$$

$(A;+,0,1)$ is a lattice effect algebra, where $a' = \neg a$.

Effect algebras can be equivalently defined as so-called D-posets (Kôpka, Chovanec): one can define on every effect algebra a partial operation "$-$" (called the difference) as follows:

$$a-b=c \;\; \text{iff} \;\; a=b+c.$$

Effect algebras

Two elements a, b of an effect algebra $\mathscr{E}$ are said to **commute** (aCb), if there are $c, d \in E$ such that $c \leq a, b \leq d$ and $d - a = b - c$. In a lattice ordered effect algebra, aCb iff $(a \vee b) - b = a - (a \wedge b)$. The relation C has the following basic properties:

aCa, $aC0$, $aC1$
aCb iff bCa
aCb iff aCb'
if aCb and aCc then $aC(b+c)$ whenever $b+c$ exists.

By a **block** of $\mathscr{E}$ is meant a maximal subset B of pairwise commuting elements of $\mathscr{E}$. Z. Riečanová has shown that every lattice effect algebra is a unioun of its blocks, which are in fact MV-algebras.

Although effect algebras are useful in axiomatization of unsharp quantum logic, their disadvantage is that they are **partial algebras**, thus for their investigation one can not use very well developped tools known for total algebras.
This essential disadvantage of effect algebras has been successfully overcomed as follows:
A **lattice with sectional antitone involutions** is a system $\mathscr{L} = (L; \vee, \wedge, (^a)_{a \in L}, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order-filter $[a, 1]$ (which is called a **section**) possesses an antitone involution $x \mapsto x^a$.
The family $(^a)_{a \in L}$ of sectional antitone involutions, which are partial unary operations on L, can be equivalently replaced by a single binary operation defined by

$$x \to y := (x \vee y)^y. \tag{2}$$

This allows one to treat lattices with sectional antitone involutions as total algebras $(L; \vee, \wedge, \to, 0, 1)$, or even $(L; \to, 0, 1)$, that form a variety.

Lemma

Let $\mathscr{L} = (L; \vee, \wedge, ({}^a)_{a \in L}, 0, 1)$ *be a lattice with sectional antitone involutions. Then the assigned algebra* $\mathscr{A}(\mathscr{L}) = (L; \oplus, \neg, 0)$*, where* $x \oplus y := (x^0 \vee y)^y$ *and* $\neg x := x^0$*, satisfies the identities*

(BA1) $x \oplus 0 = x$;

(BA2) $\neg\neg x = x$;

(BA3) $x \oplus 1 = 1 \oplus x = 1$;

(BA4) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;

(BA5) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

By a **basic algebra** we mean an algebra $\mathscr{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the identities (BA1)–(BA5) (where $1 := \neg 0$).

Conversely, given a basic algebra $\mathscr{A} = (A; \oplus, \neg, 0)$, relation $\leq$ defined by

$$x \leq y \quad \text{iff} \quad \neg x \oplus y = 1$$

is a lattice order on A with 0 and 1 a least and a greatest element of A, respectively.
Observe also that the mapping

$$x \mapsto x^a := \neg x \oplus a$$

is an antitone involution on the section $[a, 1]$.

Lemma

Let $\mathscr{A} = (A; \oplus, \neg, 0)$ be a basic algebra with the induced lattice $\ell(\mathscr{A}) = (A; \vee, \wedge, 0, 1)$, and define for $a \in A$, $x \in [a, 1]$, $x^a := \neg x \oplus a$. Then the structure $\mathscr{L}(\mathscr{A}) = (A; \vee, \wedge, (^a)_{a \in A}, 0, 1)$ is a bounded lattice with sectional antitone involutions.

Moreover, the correspondence is 1-1: $\mathscr{L}(\mathscr{A}(\mathscr{L})) = \mathscr{L}$ and $\mathscr{A}(\mathscr{L}(\mathscr{A})) = \mathscr{A}$

Next step is to relate basic algebras to lattice effect algebras:

Lemma

A basic algebra $\mathscr{A}$ is a lattice effect algebra iff it satisfies the identity

$$(x \wedge \neg y) \oplus [(\neg(x \oplus y) \wedge z) \oplus y] = (x \oplus y) \oplus (\neg(x \oplus y) \wedge z). \quad (3)$$

Hence **the class** *EBA* **of lattice effect algebras forms a subvariety of the variety** *BA* **of basic algebras.**

The bottom of $\mathscr{L}$(DEBA)

The variety *EBA* of effect basic algebras **arithmetical and regular**.

This yields its **congruence distributivity** and, by the Baker-Pixley theorem, **each subvariety generated by a finite effect algebra is finitely based**, i.e. it can be presented by a finite number of identities. Moreover, using Jónsson lemma, all subdirectly irreducible algebras of this variety are contained among its subalgebras and quotient algebras.

By a **block** of a basic algebra $\mathscr{A}$ is meant a maximal subset $B \subseteq A$ of pairwise commuting elements of A, i.e. $x \oplus y = y \oplus x$ for all $x, y \in B$. Remark that a **block of a basic algebra need not be its subalgebra**.

Lemma

For a basic algebra $\mathscr{A}$ the following are equivalent:

(i) *$\mathscr{A}$ is a lattice effect algebra;*

(ii) *every block of $\mathscr{A}$ is a subalgebra which is an MV-algebra.*

For finite basic algebras the above condition (ii) can be even simplified:

Lemma

Every finite commutative basic algebra is an MV-algebra.

Last two statements now yield:

Lemma

For a finite basic algebra $\mathscr{A}$ the following are equivalent:

(i) *$\mathscr{A}$ is a lattice effect algebra;*

(ii) *every block of $\mathscr{A}$ is its subalgebra.*

Denote further *DEBA* the variety of lattice effect basic algebras satisfying the distributive lattice identity. In order to describe the bottom of the lattice $\mathscr{L}(DEBA)$ of all subvarieties of *DEBA*, we ask for small SI members of *DEBA*.
It is well known: given a finite chain
$C_{n+1} = \{0 = a_n < a_{n-1} < \cdots < a_1 < a_0 = 1\}$, one can define on C_{n+1} in a unique way sectional antitone involutions and hence a structure of a basic algebra as follows:

$$a_j^{a_k} := a_{k-j}$$

for $k \geq j$.
The corresponding basic algebras $\mathscr{C}_{n+1} = (C_{n+1}; \oplus, \neg, 0)$ are known to be simple MV-algebras.

The least SI member of *DEBA* which is not contained in the variety *MV* of MV-algebras looks as follows:
denote $\mathscr{H} = (H; \oplus, \neg, 0)$, where
$H = \{0, 1, a, b\}, a \oplus a = b \oplus b = 1$ and $\neg a = a, \neg b = b$. In other words, the underlying lattice of $\mathscr{H}$ is

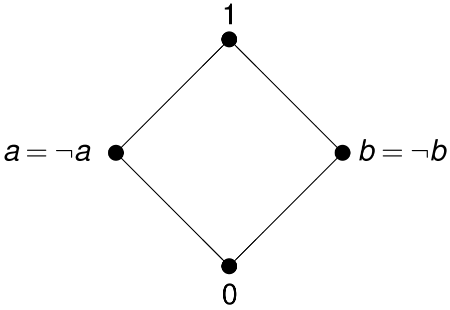

Fig. 1

The bottom of $\mathscr{L}$(DEBA)

$\mathscr{H}$ is simple and the lattice $\mathscr{L}(\mathscr{V}(\mathscr{H}))$ of subvarieties of the variety $\mathscr{V}(\mathscr{H})$ generated by $\mathscr{H}$ is visualized by following diagram:

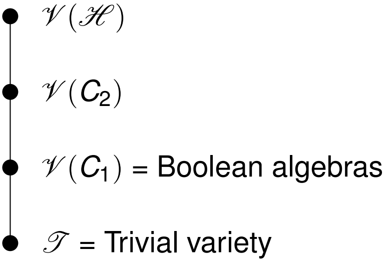

Fig. 2

It is also evident that $MV \cap \mathscr{V}(\mathscr{H}) = \mathscr{V}(C_2)$. Hence a natural question arises:

(P1) Are there any other SI members in the variety *DEBA* distinct from $\mathscr{H}$ and not belonging to *MV*?

Outline

To solve the above problem, we describe the join $\mathscr{V}^* = \mathscr{V}(\mathscr{H}) \vee MV$ in the lattice $\mathscr{L}(DEBA)$. Since the variety MV is generated by $[0,1]$, the problem (P1) can be also reformulated as

(P2) Is it true that $DEBA = \mathscr{V}([0,1] \times \mathscr{H})$?

As usual in MV-algebras, denote for any basic algebra $\mathscr{A}$ by

$a \odot b := \neg(\neg a \oplus \neg b)$ and $a \to b := \neg a \oplus b$.

Basic properties

Lemma

Every basic algebra $\mathscr{A}$ satisfies the identities

(1) $x \leq y \Rightarrow x \odot z \leq y \odot z$;

(2) $x \leq y \Rightarrow y \to z \leq x \to z$.
Moreover, if $\mathscr{A}$ is a lattice effect algebra, then

(3) $((x \to y) \odot (y \to x)) \odot ((x \to y) \odot (y \to x)) \leq (z \to x) \to (z \to y)$ *whenever* $z \geq x \geq y$;

(4) $x \leq y \Rightarrow x \odot x \leq x \odot y$;

(5) $x \odot (x \odot x) = (x \odot x) \odot x$.

Lemma

The variety $\mathscr{V}^*$ *satisfies the identities*

(6) $(x \to y) \odot (x \to y) \leq (y \to z) \to (x \to z)$;

(7) $(x \to y) \odot (x \to y) \leq \neg y \to \neg x$;

(8) $x \odot (x \to y) \leq y$;

(9) $(x \odot y) \odot (y \vee z) \leq ((x \odot y) \odot y) \vee ((x \odot y) \odot z)$ *for* $x \leq y$.

Congruence regularity of *EBA* implies that we need to describe congruence kernels.
First we characterize congruence kernels in $\mathscr{V}^*$:

Lemma

Let $\mathscr{V}$ be a subvariety of EBA satisfying the identity (6). Let $\mathscr{A} \in \mathscr{V}$ and $1 \in F \subseteq A$. Then F is a congruence kernel on $\mathscr{A}$ iff

- *(i)* $\forall x, y \in F : x \odot y \in F$;
- *(ii)* *F is closed under modus ponens, i.e. $x, x \to y \in F$ imply $y \in F$.*

We call subsets $I \subseteq A$ satisfying these properties **filters** of $\mathscr{A}$.
Denote by $\mathscr{F}(\mathscr{A})$ the set of all filters of $\mathscr{A}$.
As a corollary we obtain:

Lemma

Let $\mathscr{A} \in \mathscr{V}^*, F \subseteq A$. Then F is a congruence kernel on $\mathscr{A}$ iff F is a filter of $\mathscr{A}$.

Principal filters

We are ready to describe principal filters of algebras in the variety $\mathscr{V}^*$:

Lemma

Let $\mathscr{V}$ be a subvariety of EBA satisfying the identities (6) and (8). If $\mathscr{A} \in \mathscr{V}$ and $a \in A$, then

$$F(a) = \{x \in A; x \geq n \odot a = \underbrace{a \odot \cdots \odot a}_{n\text{-times}} \textit{ for some } n \in \mathbb{N}\}$$

is a principal filter of $\mathscr{A}$ generated by a.

Theorem

The variety $\mathscr{V}^*$ is as a subvariety of DEBA characterized by the identities (6), (8) and (9), i.e.

(6) $(x \to y) \odot (x \to y) \leq (y \to z) \to (x \to z)$;

(8) $x \odot (x \to y) \leq y$;

(9) $(x \odot y) \odot (y \vee z) \leq ((x \odot y) \odot y) \vee ((x \odot y) \odot z)$ *for* $x \leq y$.

As a direct corollary we obtain a partial answer to the problem (P2):

Theorem

We have $DEBA = \mathscr{V}([0,1] \times \mathscr{H})$ *iff* (6), (8) *and* (9) *hold in DEBA.*

Since $\mathscr{V}^*$ is congruence distributive and $\mathscr{V}^* = MV \vee \mathscr{V}(\mathscr{H})$, we conclude
$(\mathscr{V}^*)_{SI} = (MV)_{SI} \cup (\mathscr{V}(\mathscr{H}))_{SI} = (MV)_{SI} \cup \{\mathscr{H}\}$.
Denote further by $\mathscr{V}_*$ the subvariety of *DEBA* defined by the identities (6),(8) and (9). In order to prove $\mathscr{V}_* = \mathscr{V}^*$, it suffices to show
$(\mathscr{V}_*)_{SI} = (MV)_{SI} \cup \{\mathscr{H}\}$.

First we show that given $\mathscr{A} \in \mathscr{V}_*$ and a block B of $\mathscr{A}$ (which is a support of a subalgebra $\mathscr{B}$ of $\mathscr{A}$), then every filter of $\mathscr{B}$ can be extended to a filter of $\mathscr{A}$. For a filter F on $\mathscr{B}$ denote by $\mathscr{F}(F)$ a filter of $\mathscr{A}$ generated by F. We state

Lemma

Let $\mathscr{A} \in \mathscr{V}_*$. Then $\mathscr{F}(F) = \{x \in A; a \leq x \text{ for some } a \in F\}$.

Lemma

Let $\mathscr{A} \in \mathscr{V}_*$, B be a block of $\mathscr{A}$ and F a filter of $\mathscr{B}$. Then $\mathscr{F}(F) \cap B = F$.

Lemma

Let $\mathscr{A} \in \mathscr{V}_*$ be a SI member and denote $\{1\} \neq M$ the monolith of $\mathscr{F}(\mathscr{A})$. Then every block B of $\mathscr{A}$ for which $B \cap M \neq \{1\}$ is a chain.

Lemma

Every finite SI $\mathscr{A} \in \mathscr{V}_*$ is simple.

Lemma

Let $\mathscr{A}$ be a finite SI member of $\mathscr{V}_*$, and let $x, y \in A$. Then xCy iff $x \nparallel y$.

SI members

Finite case can be checked relatively easily:

Lemma

Every finite SI member $\mathscr{A} \in \mathscr{V}_*$ *is either a chain* (*i.e. an MV-algebra*) *or* $\mathscr{A} = \mathscr{H}$.

The hardest step in proving our main theorem is to show that:

Lemma

Every infinite SI algebra $\mathscr{A} \in \mathscr{V}_*$ *is an MV-chain.*

Denote as before by $\{1\} \neq M$ the monolith of the lattice of all filters of $\mathscr{A}$. We distinguish two cases:

[I.] Assume that aCb for all $a, b \in M$.
Then M is contained in some block B of $\mathscr{A}$ and $B \cap M \neq \{1\}$, M is a chain.
We discuss two more subcases of I:
[Ia.] Suppose that there is $x \in A$ such that x does not commute with a for some $a \in M$.
We shave shown that this case can be violated.
Hence the following holds:
[Ib.] xCa for all $a \in M$ and all $x \in A$.
In this case it is an infinite MV-algebra, hence SI MV-chain.

[II.] There exist $x, y \in M$ such that x and y do not commute (i.e. $x \parallel y$).
We have shown that in this case $\mathscr{A}$ has to be finite (i.e., by our assumptions can be excluded).