On Projective MV-algebras

Antonio Di Nola and Revaz Grigolia

Department of Mathematics and Informatics
University of Salerno
e-mail: adinola@unisa.it
Institute of Computer Sciences
Tbilisi State University
e-mail: grigolia@yahoo.com

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Abstract

We characterize finitely generated projective MV-algebra and give also sufficient conditions to be a finitely generated projective MV-algebra.

1 Introduction and preliminaries

It is known that the variety $\mathbb{MV}$ of all MV-algebras is not locally finite and that, remarkably, it is generated by all simple finite MV-algebras.

Recall that an algebra $A = (A; \oplus, \cdot, \neg, 0, 1)$, is said to be an MV-algebra iff it satisfies the following equations:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
2. $x \oplus y = y \oplus x$;
3. $x \oplus 0 = x$;
4. $x \oplus 1 = 1$;

5. $\neg 0 = 1$;

6. $\neg 1 = 0$;

7. $x \odot y = \neg(\neg x \oplus \neg y)$;

8. $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Every MV-algebra has an underlying ordered structure defined by

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

$(A; \leq, 0, 1)$ is a bounded distributive lattice. Moreover, the following property holds in any MV-algebra:

$$xy \leq x \wedge y \leq x \vee y \leq x \oplus y.$$

The unit interval of real numbers $[0, 1]$ endowed with the following operations: $x \oplus y = \min(1, x + y), x \odot y = \max(0, x + y - 1), \neg x = 1 - x$, becomes an MV-algebra. It is well known that the MV-algebra $S = ([0, 1], \oplus, \odot, \neg, 0, 1)$ generate the variety $\mathbb{MV}$ of all MV-algebras, i. e. $\mathcal{V}(S) = \mathbb{MV}$. Let Q denote the set of rational numbers, for $(0 \neq) n \in \omega$ we set

$$S_n = (S_n; \oplus, \odot, \neg, 0, 1),$$

where

$$S_n = \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}.$$

Let $F_{\mathbb{V}_n}(m)$ be m-generated free MV-algebra in the variety

$$\mathbb{V}_n = \mathcal{V}(\{S_1, ..., S_n\}).$$

Let $g_1^{(n)}, ..., g_m^{(n)} \in F_{\mathbb{V}_n}(m)$ be free generators of $F_{\mathbb{V}_n}(m)$.

On Z^+ we define the function $v_m(x)$ as follows: $v_m(1) = 2^m, v_m(2) = 3^m - 2^m, ..., v_m(n) = (n+1)^m - (v_m n_1 + ... v_m(n_{k-1}))$, where $n_1 (= 1), ..., n_{k-1}$ are all the divisors of n distinct from $n (= n_k)$. Then by [?] (Lemma 22)

$$F_{\mathbb{V}_n}(m) \cong S_1^{v_m(1)} \times ... \times S_n^{v_m(n)}.$$

Let $F_{\mathbb{MV}}(m)$ be m-generated free MV-algebra in the variety $\mathbb{MV}$. Let $g_1, ..., g_m \in F_{\mathbb{MV}}(m)$ be free generators of $F_{\mathbb{MV}}(m)$.

In

A. Di Nola , R. Grigolia, G. Panti, *Finitely generated free MV-algebras and their automorphism groups*, **Studia Logica**, vol.61, N1, 65-78(1998).

A. Di Nola and R. Grigolia, *Projective MV-Algebras and Their Automorphism Groups*, **J. of Mult.-Valued Logic** & **Soft Computing**, Vol. 9(2003), pp. 291-317

a characterization of finitely generated free MV-algebras as subalgebras of an inverse limit of a chain of order type $\omega*$ of free algebras $F_{\mathbb{V}_n}(m)$ is given.

Notice, that R. McNaughton

McNaughton R, *A theorem about infinite-valued sentential logics.* **J.S.L.**, 16(1951), 113.

have described a set of special functions $f : [0,1]^m \to [0,1]$, endowed with MV-operations, that represents the m-generated free MV-algebra. More precisely, McNaughton has proved that a function has an MV polynomial representation $q(x_1, ..., x_m)$ such that $f = q$ iff f satisfies the following conditions:

(i) f is continuous,

(ii) there exists a finite number of affine linear distinct polynomials $\ell_1, ..., \ell_n$, each having the form

$$\ell_j = b_j + n_{j_1}x_1 + ... + n_{j_m}x_m$$

where all b's and n's are integers such that for every $(x_1, ..., x_m) \in [0,1]^m$ there is $j, 1 \leq j \leq n$ such that

$$f(x_1, , x_m) = \ell_j(x_1, , x_m).$$

It is worth to stress that several descriptions of the free MV-algebras are known.

D. Mundici, *A constructive proof of McNaughton's Theorem in infinite-valued logics*, **J. Symbolic Logic**, 59, (1994), 596-602.

G. Panti, *A geometric proof of the completeness of the Łukasiewcz calculus*, **J. Symbolic Logic**, 60, (1995), 563-578.

G. Jakubik, *Free MV-algebras*, **Czechoslovak Mathematical Journal**, Vol. 53, No. 2, pp. 311-317, 2003

It is well known that MV-algebras are algebraic models of infinitely-valued Łukasiewicz logic L_∞. As well known the structure of non-equivalent formulas of L_∞ forms an ω-generated MV-algebra, which is named Lindenbaum algebra. If we restrict the structure of non-equivalent formulas with m propositional variables, then we will have the m-generated free MV-algebra.

Recall that an algebra $A \in \mathbf{K}$ is said to be a *free algebra* in a variety $\mathbf{K}$, if there exists a set $A_0 \subset A$ such that A_0 generates A and every mapping f from A_0 to any algebra $B \in \mathbf{K}$ is extended to a homomorphism h from A to B. In this case A_0 is said to be *the set of free generators* of A. If the set of free generators is finite then A is said to be a *finitely generated free algebra* .

Recall also that an algebra $A \in \mathbf{K}$ is called *projective*, if for any $B, C \in \mathbf{K}$, any epimorphism (that is an onto homomorphism) $\beta : B \to C$ and any homomorphism $\gamma : A \to C$, there exists a homomorphism $\alpha : A \to B$ such that $\beta\alpha = \gamma$. Notice that in varieties projective algebras are characterized as retracts of free algebras. An algebra A is said to be *a retract* of the algebra B, if there are homomorphisms $\varepsilon : A \to B$ and $h : B \to A$ such that $h\varepsilon = Id_A$.

A subalgebra A of $F_{\mathbb{K}}(m)$ is projective if there exists an endomorphism $h : F_{\mathbb{K}}(m) \to F_{\mathbb{K}}(m)$ such that $h(F_{\mathbb{K}}(m)) = A$ and $h(x) = x$ for every $x \in A$.

2 On Projective MV-algebras

Lemma 1. *If $A(m)$ is an m-generated projective MV-algebra, then it is a retract of the m-generated free MV-algebra $F_{\mathbb{MV}}(m)$.*

Proof. Since $A(m)$ is m-generated, there exists homomorphism onto $h : F_{\mathbb{MV}}(m) \to A(m)$, and we have identity mapping $Id_{A(m)} : A(m) \to A(m)$. So, since $A(m)$ is projective, there exists a homomorphism $\delta : A(m) \to F_{\mathbb{MV}}(m)$ such that $h\delta = Id_{A(m)}$. By this we conclude the proof. □

Lemma 2. *Let $\mathbb{V}$ be a variety of algebras and $A(m)$ an m-generated projective subalgebra of the m-generated free algebra $F_{\mathbb{V}}(m)$ with generators $a_1, ..., a_m \in A(m)$ $(\subset F_{\mathbb{V}}(m))$. Then the one generated subalgebra $A_i(m)$*

of $A(m)$, generated by $a_i \in A(m)$ for $i \in \{1, ..., m\}$, is a projective algebra in a variety $\mathbb{V}$.

Proof. Since $A(m)$ is m-generated subalgebra of $F_{\mathbb{V}}(m)$, we have that there exist homomorphisms h:$F_{\mathbb{MV}}(m) \to A(m)$ and $\varepsilon : A(m) \to F_{\mathbb{V}}(m)$ such that $h(g_i) = a_i$ and $h\varepsilon(a_i) = a_i$ for $i \in \{1, ..., m\}$.

There exists a homomorphism $h_i : F_{\mathbb{V}}(1) \to A_i(m)$ such that $h_(g) = a_i$. Since $F_{\mathbb{V}}(m)$ is projective, we have that there exists homomorphism $h' : F_{\mathbb{V}}(m) \to F_{\mathbb{V}}(1)$ such that $h_i h' = h$. Let $\delta = h'\varepsilon : A_i(m) \to F_{\mathbb{V}}(1)$. Then $h'\delta = h_i h' \varepsilon = Id_{A_i(m)}$. So, $A_i(m)$ is projective. □

According to Lemma 2 to prove that not every m-generated subalgebra of $F_{\mathbb{MV}}(m)$ is projective, it is enough to show that there exists a one-generated subalgebra of the one-generated free algebra $F_{\mathbb{MV}}(1)$ which is not projective. Indeed, let A be a subalgebra of $F_{\mathbb{MV}}(1)$ generated by $2g$, where g is a free generator of $F_{\mathbb{MV}}(1)$.

Recall that if we have a variety of algebras $\mathbb{V}$ and $\mathbb{V}_1$ is its subvariety, then a homomorphism $\tau : A \to A_1$, where $A \in \mathbb{V}$ and $A_1 \in \mathbb{V}_1$, is said to be $\mathbb{V}_1$-*universal* for A if for any algebra $B \in \mathbb{V}_1$ and any homomorphism $h : A \to B$ there exists a homomorphism $\xi : A_1 \to B$ such that $\xi\tau = h$.

Let us note that

- $F_{\mathbb{V}_1}(n)$ is $\mathbf{V_1}$-morphic image of $F_{\mathbb{V}}(n)$.

Let $\tau : F_{\mathbb{MV}}(1) \to F_{\mathbb{V}_4}(1)$ be $\mathbb{V}_4$-universal for $F_{\mathbb{V}_4}(1)$. Then $\tau(A) = A_4$ is a subalgebra of of $F_{\mathbb{V}_4}(1)$. Notice, that if A is a projective subalgebra of $F_{\mathbb{MV}}(1)$ in $\mathbb{MV}$, then A_4 is a projective subalgebra of $F_{\mathbb{V}_4}(1)$ in $\mathbb{V}_4$.

$F_{\mathbb{V}_4}(1) \cong S_1^2 \times S_2 \times S_3^2 \times S_4^2$, $g^{(4)} = (0, 1, 1/2, 1/3, 2/3, 1/4, 3/4)$ and $2g^{(4)} = (0, 1, 1, 2/3, 1, 1/2, 1)$, $A_4 \cong S_1^2 \times S_3 \times S_4$. From here we see that A_4 is not a retract of $F_{\mathbb{V}_4}(1)$, i. e. there are no homomorphisms, say $h : F_{\mathbb{V}_4}(1) \to A_4$ and $\varepsilon : A_4 \to F_{\mathbb{V}_4}(1)$ such that $h\varepsilon = Id_{A_4}$. So, A is not a retract of $F_{\mathbb{MV}}(1)$. Therefore, according to Lemma 2, A is not a projective MV-algebra.

Let $\mathbb{K}$ be any variety of algebras. In

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is proved the following

Theorem 3. *(Theorem 20). Let $F_{\mathbb{K}}(m)$ be the m-generated free algebra of a variety $\mathbb{K}$ and $g_1, ..., g_m$ be its free generators. Then the m generated subalgebra A of $F_{\mathbb{K}}(m)$ with the generators $a_1, ..., a_m \in A$ is projective if and only if there exist polynomials $P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)$ such that*

$$P_i(g_1, ..., g_m) = a_i$$

and

$$P_i(P_1(x_1, ..., x_m), ..., P_m(x_1, ..., x_m)) = P_i(x_1, ..., x_m),$$

$i = 1, ..., m$.

We say that an MV-algebra polynomial $Q(x_1, ...x_m)$ over an MV-algebra A is *antitone* (*isotone*) if $x_i \leq y_i$, for every $i = 1, ..., m$ implies $Q(x1, ...x_m) \geq Q(y_1, ..., y_m))$ $(Q(x_1, ...x_m) \leq Q(y_1, ..., y_m)))$ for every $(x_1, ...x_m), (y_1, ..., y_m) \in A^m$.

Theorem 4. *Let $Q(x_1, ..., x_m)$ be an antitone MV-algebra polynomial. Then the m-generated subalgebras of $F_{\mathbb{MV}}(m)$ generated by*

$$\{g_i \wedge Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{g_i \vee Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{(g_i \wedge \neg(g_i)) \wedge Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{(g_i \vee \neg(g_i)) \vee Q(g_1, ..., g_m)\}_{i=1,...,m},$$

respectively, are projective.

Proof. Set

$$a_i = P_i(g_1, ..., g_m) = g_i \wedge Q(g_1, ..., g_m).$$

Then

$$P_i(P_1(g_1, ..., g_m), ..., P_m(g_1, ..., g_m)) =$$

$$= P_1(g_i \wedge Q(g_1, ..., g_m), ..., g_i \wedge Q(g_1, ..., g_m), ..., g_m \wedge Q(g_1, ..., g_m) =$$

$= g_i \wedge Q(g_1, ..., g_m) \wedge Q(g_1 \wedge Q(g_1, ..., g_m), ..., g_i \wedge Q(g_1, ..., g_m), ..., g_m \wedge Q(g_1, ..., g_m)).$

Since for every $j = 1, ..., m$ $g_j \wedge Q(g_1, ..., g_m) \leq g_j$ and $Q(g_1, ..., g_m)$ is antitone, then we get that

$$Q(g_1, ..., g_m) \leq Q(Q(g_1, ..., g_m), ..., g_i \wedge Q(g_1, ..., g_m), ..., g_m \wedge Q(g_1, ..., g_m))$$

hence

$$P_i(P_1(g_1, ..., g_m), ..., P_m(g_1, ..., g_m)) = g_i \wedge Q(g_1, ..., g_m) = P_i(g_1, ..., g_m).$$

So, by Theorem 3, we get that $A(a_1, ..., a_m)$, the subalgebra of $F_{\mathbb{MV}}(m)$, generated by $\{a_1, ..., a_m\}$, is projective.

The remaining cases can be proved in an analogous way.

□

Theorem 5. *Let $Q(x_1, ..., x_m)$ be an isotone MV-algebra polynomial. Then the m-generated subalgebras of $F_{\mathbb{MV}}(m)$ generated by*

$$\{\neg g_i \wedge Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{\neg g_i \vee Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{(g_i \wedge \neg(g_i)) \wedge Q(g_1, ..., g_m)\}_{i=1,...,m}$$

$$\{(g_i \vee \neg(g_i)) \vee Q(g_1, ..., g_m)\}_{i=1,...,m},$$

respectively, are projective.

Proof. Analogous to the proof of Theorem 4. □

As an example of projective algebras described by Theorems 3 and 4we refer to the 2-generated subalgebra of $F_{\mathbb{MV}}(2)$ which is generated by $\{(g_1 \wedge (\neg g_1)^2 \oplus (\neg g_2)^2), (g_2 \wedge (\neg g_1)^2 \oplus (\neg g_2)^2)\}$.

We recall that to any 1-variable McNaughton function f is associated a partition of the unit interval $[0, 1]$, $\{0 = a_0, a_1, ..., a_n = 1\}$ in such a way that

$a_0 < a_1 < ... < a_n$ and the points $\{(a_0, f(a_0)), (a_1, f(a_1)), ..., (a_n, f(a_n))\}$ are the knots of f and the function f is linear over each interval $[a_{i-1}, a_i]$, with $i = 1, ..., n$. We denote by ℓ_i the linear piece of f defined over the interval $[a_{i-1}, a_i]$. Sometimes we call ℓ_i the *i-th piece of f*. Let $Proj^+$ denote the set of 1-variable McNaughton functions f satisfying the following conditions:

(1) $f \circ f = f$;

(2) $f(0) = 0$.

Lemma 6. *Let f be a McNaughton function such that $f \in Proj^+$ then ℓ_2 is decreasing.*

Proof. Assume ℓ_2 increasing, that is for every $x \in]a_1, a_2], x < f(x)$. From the continuity of f there is $k \in]a_1, a_2]$ such that $f(k) = a_2$. Hence $f(f(k)) = f(a_2)$. Since we assumed ℓ_2 increasing we get $a_2 < f(a_2)$. Finally we have:

$$a_2 = f(k) = f(f(k)) = f(a_2) > a_2,$$

which is absurd. Hence ℓ_2 is decreasing. □

Lemma 7. *Let f be a McNaughton function such that $f \in Proj^+$. Assume $i > 2$, if ℓ_i is increasing then for every $x \in [a_{i-1}, a_i]$ $f(x) \neq x$.*

Proof. Assume $2 < i$, $a_i < f(a_i)$ and $f(a_{i-1}) < a_{i-1}$. Then by continuity of ℓ_i there exists $k \in]a_{i-1}, a_i]$ such that $f(k) = k$. For h such that $a_{i-1} < h < k$ and $f(h) = a_{i-1}$, since ℓ_i is increasing we get

$$a_{i-1} = f(h) < h < k$$

and then

$$f(f(h)) < f(h),$$

in contrast with $f \circ f = f$. □

Lemma 8. *Let f be a McNaughton function such that $f \in Proj^+$. Assume $i > 2$, if ℓ_i is decreasing then for every $x \in [a_{i-1}, a_i]$ $f(x) \neq x$.*

Proof. Let $i > 2$ and ℓ_i be decreasing. Then suppose there exists $k \in [a_{i-1}, a_i]$ such that $f(k) = k$. Also we have $a_{i-1} < f(a_{i-1})$, $f(a_i) < a_i$. Hence we can find $h \in]a_{i-1}, k[$ such that $f(h) = a_i$. So we get

$$f(f(h)) = f(a_i) < a_i = f(h)$$

in contrast with the assumption that $f \in Proj^+$. □

Lemma 9. *Let f be a McNaughton function such that $f \in Proj^+$ and for every $x \in [a_0, b[$, $f(x) = 0$. Then $f(x) = 0$ for every $x \in [0,1]$.*

Proof. Suppose there is $k \in [0,1]$ such that $f(k) \neq 0$. Then we can assume that for every $x \in [a_0, a_1]$ $f(x) = 0$. Hence ℓ_2 is increasing on the interval $[a_1, a_2]$. Then there is $h \in]a_1, a_2[$ such that $f(h) = a_1$ and $0 = f(a_1) < f(h)$. Also we have $0 = f(a_1) = f(f(h)) = f(h)$, absurd. □

Proposition 10. *Let f be a McNaughton function such that $f \in Proj^+$. Then, exactly one of the following holds:*

(1) $f(x) = 0$, for every $x \in [0,1]$;

(2) for every $x \in [a_0, a_1]$, $f(x) = x$.

Proof. Trivially the zero function belongs to $Proj^+$. So we assume f be non-zero function in $Proj^+$. By Lemma 9 , for every $x \in]0, a_1]$, $0 < f(x)$. Assume $a_1 < f(a_1)$, then there is $h \in]0, a_1]$ such that $f(h) = a_1$. So we have also $a_1 = f(h) = f(f(h)) = f(a_1)$, which is absurd. Hence the linear piece ℓ_1 of f is non-zero and cannot be up the graph of the identity function. So, ℓ_1 has derivative equal to 1. Hence for every $x \in [0, a_1]$ $f(x) = x$. □

Theorem 11. *Let f be a McNaughton function of one variable the following are equivalent:*

(1) $f \in Proj^+$;

(2) $Max\{f(x), x \in [0,1]\} = f(a_1)$ and for f non-zero function, for every $x \in [0, a_1]$, $f(x) = x$.

Proof. Assume (1) holds. Then if f is the zero function hence (2) trivially holds. If f is non-zero, by Proposition 10, for every $x \in [0, a_1]$, $f(x) = x$. Moreover, by Lemmas 6,7,8 for every $x \in]a_1, 1]$, $f(x) < x$. Hence (2) still holds. Vice-versa, assume (2). If f is the zero function trivially we get (1). Let f be a non-zero function satisfying (2). Then for each $x \in [0,1]$ there is $h \in [0, a_1]$ such that $f(x) = h$. So, $f(f(x)) = f(h) = h$. Hence $f \circ f = f$, and (1) holds. □

It is easy to check that the following corollary holds:

Corollary 12. *Let $f, g \in Proj^+$ then:*

(1) $f \vee g \in Proj^+$

(2) $f \wedge g \in Proj^+$

(3) $id(x) \in Proj^+$ *and* $id(x) = max(Proj^+)$

Theorem 13. *Let* A *be a one-generated subalgebra of* $F_{\mathbb{MV}}(1)$. *Then the following are equivalent:*

(1) A *is projective;*

(2) A *is generated by some* $f \in Proj^+$*;*

Proof. The theorem is trivial when $A = \{f_0, f_1\}$. So assume A be a projective subalgebra of $F_{\mathbb{MV}}(1)$ generated by a non-zero function f and $A \neq \{f_0, f_1\}$. If f is such that $f(0) = 1$ then we can consider the generator f^* of A. So we have that $f^*(0) = 0$. Since f^* is a McNaughton function, then there is an MV-polynomial $P(x) : [0,1] \to [0,1]$ such that for every $x \in [0,1]$ $P(x) = f^*(x)$. By Theorem 3 $P(P(x)) = P(x)$ and then for every $x \in [0,1]$ $f^*(f^*(x)) = f^*(x)$. Hence $f^* \in Proj^+$.

Vice versa, let A be a subalgebra of $F_{\mathbb{MV}}(1)$ generated by the function f and $f \in Proj^+$. Since f is an element of $Free(1)$ then there is an MV-polynomial of one variable $Q(x) : [0,1] \to [0,1]$ such that for every $x \in [0,1]$ $f(x) = P(x)$. Since $f \in Proj^+$, by Theorem 6 $P(P(x)) = P(x)$, for every $x \in [0,1]$. Then, by Theorem 3, A is projective. □

Now we generalize 1-generated case on m-generated. We recall that $F_{\mathbb{MV}}(m)$ is generated by the McNaughton functions

$$g_i : [0,1]^m \to [0,1], \text{ where } g_i(x_1, ..., x_m) = x_i, \quad i = 1, ..., m.$$

Let $A(f_1, ..., f_m)$ be the subalgebra of $F_{\mathbb{MV}}(m)$ generated by $f_1, ..., f_m$, where $f_i(0, ..., 0) = 0$ for every $i = 1, ..., m$. Assume $A(f_1, ..., f_m)$ be projective. Then by Theorem 3 there are m MV-polynomials $P_i : [0,1]^m \to [0,1]$, $i = 1, ..., m$ such that:

$$P_i(g_1(x_1, ..., x_m), ..., g_m(x_1, ..., x_m)) = f_i(x_1, ..., x_m), \quad i = 1, ..., m \qquad (I)$$

and

$$P_i(P_1(x_1,...,x_m),...,P_m(x_1,...,x_m)) = P_i(x_1,...,x_m), \quad i = 1,...,m \qquad (II)$$

Hence, from (I) we get

$$P_i(x_1,...,x_m) = f_i(x_1,...,x_m), \quad i = 1,...,m.$$

Thus we can identify P_i with f_i, i=1,...,m. Then (II) can be settled in the following form:

$$f_i(f_1(x_1,...,x_m),...,f_m(x_1,...,x_m)) = f_i(x_1,...,x_m), \quad i = 1,...,m. \qquad (III)$$

Let us consider the case when $f_i(0,...,0) = 0$, $i = 1,...,m$. Define the subset $I_i(f_i)$ of $[0,1]^m$, $i = 1,...,m$, as follows:

$$I_i(f_i) = \{(x_1,...,x_m) \in [0,1]^m \mid f_i(x_1,...,x_m) = x_i, \quad i = 1,...,m.\}$$

Then the set $I_i(f_i)$ $(i = 1,...,m)$ is nonempty, indeed $(0,...,0) \in I_i(f_i)$. We observe that the points $Q \in I_i(f_i)$ are points in which the value of f_i coincides with $g_i(Q)$, i.e. over such points Q the graph of f_i intersects the graph of g_i.

In the special case of $A(g_1,...,g_m)$ we get:

$$I_1(g_1) = ... = I_m(g_m) = [0,1]^m.$$

Coming back to the case of $A(f_1,...,f_m)$ set

$$\Sigma_{A(f_1,...,f_m)} = \{((f_1(Q),...,f_m(Q))\}_{Q\in[0,1]^m}.$$

Then with the above notations we have:

Proposition 14. *Let $A(f_1,...,f_m)$ be a subalgebra of $F_{\mathbb{MV}}(m)$ generated by $f_1,...,f_m$. Assume that $f_i(0,...,0) = 0$ for $i = 1,...,m$. Then the following are equivalent:*

(j) $A(f_1,...,f_m)$ *is projective;*

(jj) $\Sigma_{A(f_1,...,f_m)} \subseteq \bigcap_{i=1}^m I_i(f_i)$.

Proof. Assume (j) holds, then by (III), for every $Q \in [0,1]^m$,

$$f_i(f_i(Q), ..., f_m(Q)) = f_i(Q) \quad i = 1, ..., m.$$

Then the point $K = (f_1(Q), ..., f_m(Q)) \in [0,1]^m$ is such that:

$$K \in \bigcap_{i=1}^{m} I_i(f_i).$$

Hence,

$$\Sigma_{A(f_1,...,f_m)} \subseteq \bigcap_{i=1}^{m} I_i(f_i).$$

Viceversa, assume that

$$\Sigma_{A(f_1,...,f_m)} \subseteq \bigcap_{i=1}^{m} I_i(f_i).$$

then

$$f_i(f_1(Q), ..., f_m(Q)) = f_i(Q) \quad i = 1, ..., m,$$

by (III) and Theorem 3, $A(f_1, ..., f_m)$ turns out to be projective. □

Remark

We observe that for any pair of functions h_1, h_2 such that $h_1(0,0) = 0$, $h_2(0,0) = 0$ we get $I_1(h_1) \cap I_2(h_2) \neq \emptyset$. Hence we can define a mapping

$$\sigma : [0,1]^2 \to [0,1]^2$$

defined by

$$\sigma(Q) = (h_1(Q), h_2(Q))$$

then we get the projectivity of the subalgebra $A(h_1, h_2)$ of $F_{\mathbb{MV}}(2)$, generated by h_1 and h_2, when

$$\sigma([0,1]^2) \subseteq I_1(h_1) \cap I_2(h_2).$$

that is when

$$\sigma \circ \sigma = \sigma$$

.

Now we give a theorem which is valid in every variety of algebras.

Theorem 15. *Let A be a subalgebra of m-generated free algebra $F_{\mathbb{V}}(m)$ in a variety $\mathbb{V}$ generated by $a_1, ..., a_m \in A \subset F_{\mathbb{V}}(m)$ such that the subalgebra $A_i = [a_i]_A \subset A$, generated by a_i, $i = 1, ..., m$, is projective. Then A is a projective algebra in $\mathbb{V}$.*

Proof. Let us consider the case when $m = 2$, since the one can be generalized for the case $m > 2$. So, we have (identity) embeddings $\delta_i : A_i \to A$, $\tau : A \to F_{\mathbb{V}}(2)$, i. e. $\delta_i(a) = a$ $(i = 1, 2)$, $\tau(b) = b$ for every $a \in A_i \subset A \subset F_{\mathbb{V}}(2)$ and $b \in A \subset F_{\mathbb{V}}(2)$.

So, according to Theorem 3 and since $F_{\mathbb{V}}(1)$ is isomorphic to the subalgebra generated by the free generator $g_i \in F_{\mathbb{V}}$ $(i = 1, 2)$, there exist polynomials $P_1'(x_1)$ and $P_2'(x_2)$ such that $P_1'(g_1) = a_1$ and $P_2'(g_2) = a_2$ such that $P_1'(P_1'(g_1)) = P_1'(g_1)$ and $P_2'(P_2'(g_2)) = P_2'(g_2)$. Let $P_1(x_1, x_2) = P_1'(x_1)$ and $P_2(x_1, x_2) = P_2'(x_2)$. Then $P_1(P_1(g_1, g_2), P_2(g_1, g_2)) = P_1(g_1, g_2)$ $(= P_1'(g_1))$ and $P_2(P_1(g_1, g_2), P_2(g_1, g_2)) = P_2(g_1, g_2)$ $(= P_2'(g_2))$. So, according to Theorem 3, the subalgebra A of $F_{\mathbb{V}}(2)$ generated by a_1 and a_2 is projective.

□