# Behavioral algebraization

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# Outline

# Motivation

- Theory of algebraization of logics

  [Lindenbaum & Tarski, Blok & Pigozzi, Czelakowski, Nemeti et al]

  - Representation of logics in algebraic setting (theory restricted to propositional based logics)

- Many-sorted and non-truth-functional logics

  - In practice, propositional based logics are not enough for reasoning about complex systems

- Generalization of the notion of algebraizable logic

  - Many-sorted behavioral logic

# Algebraization

$\mathcal{L}$ — Strong representation $\Longleftrightarrow$ — **Equational logic**

unisorted — unisorted

# Logic

## Definition

A *structural propositional logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$, where $\Sigma$ is a propositional signature and $\vdash \subseteq \mathcal{P}(L_\Sigma(X)) \times L_\Sigma(X)$ is a *consequence relation* satisfying the following conditions, for every $T_1 \cup T_2 \cup \{\varphi\} \subseteq L_\Sigma(X)$:

**Reflexivity:** if $\varphi \in T_1$ then $T_1 \vdash \varphi$

**Cut:** if $T_1 \vdash \varphi$ for all $\varphi \in T_2$, and $T_2 \vdash \psi$ then $T_1 \vdash \psi$

**Weakening:** if $T_1 \vdash \varphi$ and $T_1 \subseteq T_2$ then $T_2 \vdash \varphi$

**Structurality:** if $T_1 \vdash \varphi$ then $\sigma[T_1] \vdash \sigma(\varphi)$

# Algebraizable logic

$$\boldsymbol{\mathcal{L}} \underset{\Delta(x_1,x_2)}{\overset{\Theta(x)}{\rightleftarrows}} Eqn^{\Sigma}_{K}$$

$T \vdash \varphi$ iff $\Theta[T] \vDash_{Eqn^{\Sigma}_{K}} \Theta(\varphi)$

$\{\Delta(\delta_i, \epsilon_i) : i \in I\} \vdash \Delta(\varphi_1, \varphi_2)$ iff $\{\delta_i \approx \epsilon_i : i \in I\} \vDash_{Eqn^{\Sigma}_{K}} \varphi_1 \approx \varphi_2$

$\varphi \dashv\vdash \Delta[\Theta(\varphi)]$

$\varphi_1 \approx \varphi_2 \;=\!\!|\!\!\vDash_{Eqn^{\Sigma}_{K}} \Theta[\Delta(\varphi_1, \varphi_2)]$

# Leibniz operator

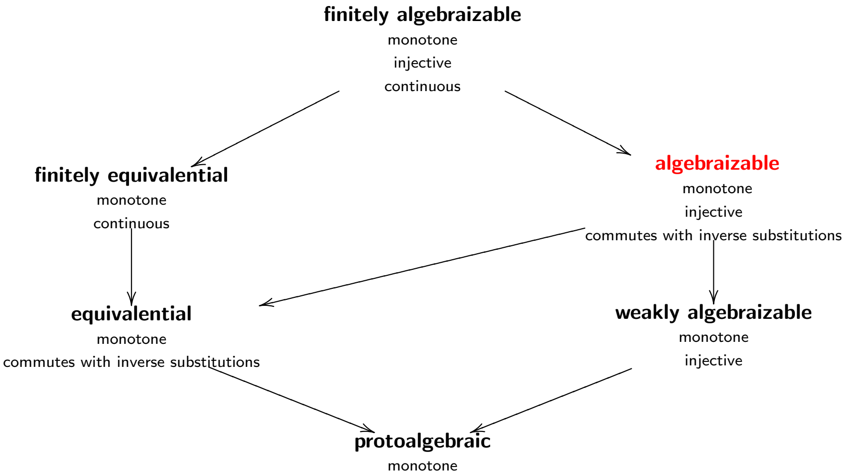
**Definition (Leibniz operator)**

Let $\mathcal{L} = \langle \Sigma, \vdash \rangle$ be a structural propositional logic. Then the *Leibniz operator* on the formula algebra can be given by:

$$\Omega : Th_{\mathcal{L}} \to \mathsf{Congr}_{\mathbf{L}_{\mathbf{\Sigma}}}$$

$$T \mapsto \text{ largest congruence compatible with } T.$$

# Characterization theorems



# Limitations

- First-order logic
    - Its algebraization gives rise to cylindric algebras (Henkin, Monk, Tarski 1971)
    - Strange, since cylindric algebras are unisorted and first-order logic clearly is not
- Limitation applicable to other many-sorted logics
- Even at the propositional level there are some "relatively" well-behaved logics that are not algebraizable

# Paraconsistent logic $\mathcal{C}_1$ of da Costa

- $\mathcal{C}_1$ (da Costa 1963) given by a structural Hilbert-style axiomatization
- Mortensen, Lewin-Mikenberg-Schwarze: $\mathcal{C}_1$ is not algebraizable
- da Costa, Carnielli-Alcantara have proposed an "algebraic counterpart" called da Costa algebras (paraconsistent algebras)
- Caleiro et al: alternative approach using two-sorted algebras

# Many-sorted signatures

**Definition (Many-sorted signature)**

A *many-sorted signature* is a pair $\Sigma = \langle S, F \rangle$ where $S$ is a set (of *sorts*) and $F = \{F_{ws}\}_{w \in S^*, s \in S}$ is an indexed family of sets (of operations).

**Example (FOL)**

$\Sigma_{FOL} = \langle S, F \rangle$ such that

- $S = \{\phi, t\}$
- $F_{\phi\phi} = \{\forall_x : x \in X\} \cup \{\neg\}$
- $F_{\phi\phi\phi} = \{\Rightarrow, \wedge, \vee\}$
- $F_{t^n\phi} = \{P : \ P \ \ n\text{-ary predicate symbol}\}$
- $F_{t^n t} = \{f : \ f \ \ n\text{-ary function symbol}\}$

# Many-sorted logics

We will only consider many-sorted signatures with a distinguished sort $\phi$ (of formulas). We will call *formulas* to the elements of $T_{\Sigma,\phi}(X)$.

## Definition (Many-sorted logics)

A *many-sorted logic* is a pair $\mathcal{L} = \langle \Sigma, \vdash \rangle$ where $\Sigma$ is a many-sorted signature and $\vdash \subseteq \mathcal{P}(T_{\Sigma,\phi}(X)) \times (T_{\Sigma,\phi}(X))$ is a structural consequence relation over the set of formulas.

# How to generalize?

| | | |
|---|---|---|
| $\mathcal{L}$ | Strong representation $\Longleftrightarrow$ | **Equational logic** |
| unisorted | | unisorted |
| | Strong representation $\Longleftrightarrow$ | |
| | Strong representation $\Longleftrightarrow$ | |

# How to generalize?

| | | |
|---|---|---|
| $\mathcal{L}$<br>unisorted | Strong representation<br>$\Longleftrightarrow$ | **Equational logic**<br>unisorted |
| $\mathcal{L}$<br>many-sorted | Strong representation<br>$\Longleftrightarrow$ | **Behavioral logic**<br>many-sorted<br>same signature |
| | Strong representation<br>$\Longleftrightarrow$ | |

# How to generalize?

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| $\mathcal{L}$<br>unisorted | Strong representation<br>$\Longleftrightarrow$ | **Equational logic**<br>unisorted |
| $\mathcal{L}$<br>many-sorted | Strong representation<br>$\Longleftrightarrow$ | **Behavioral logic**<br>many-sorted<br>same signature |
| $\mathcal{L}$<br>many-sorted<br>non-congruent connectives | Strong representation<br>$\Longleftrightarrow$ | **Behavioral logic**<br>many-sorted<br>extended signature |

# Extended signature

From a many-sorted signature $\Sigma = \langle S, F \rangle$ we can consider an extended signature $\Sigma^o = \langle S^o, F^o \rangle$, such that:

- $S^o = S \biguplus \{v\}$
- $F^o = \{F^o{}_{ws}\}_{w \in (S^o)^*, s \in S^o}$ is defined as follows:
  - $F^o{}_{ws} = F_{ws}$ if $ws \in S^*$;
  - $o : \phi \to v \in F^o$.

# Behavioral logic

Consider given a subsignature $\Gamma$ of $\Sigma$.

$\Gamma$-behavioral equivalence:

Given a $\Sigma^o$-algebra $A$, two elements $a_1, a_2 \in A_s$ are $\Gamma$-behavioral equivalent, $a_1 \equiv_\Gamma a_2$, if for every formula $\varphi \in T_\Gamma(x : s)$ and every tuple $b_1, \ldots, b_n$:

$$o_A(\varphi_A(a_1, b_1, \ldots, b_n)) = o_A(\varphi_A(a_2, b_1, \ldots, b_n)).$$

# Behavioral logic

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Many-sorted behavioral equational logic $BEqn^\Sigma_{K,\Gamma}$:

$$\{t_i \approx t'_i : i \in I\} \vDash_{BEqn^\Sigma_{K,\Gamma}} t \approx t'$$

iff

for every $A \in K$ and homomorphism $h : T_{\Sigma^o}(X) \to A$, $h(t) \equiv_\Gamma h(t')$ whenever $h(t_i) \equiv_\Gamma h(t'_i)$ for every $i \in I$

# Behavioral algebraization

$$\mathcal{L} \underset{\Delta(x_1,x_2)}{\overset{\Theta(x)}{\rightleftarrows}} BEqn^{\Sigma}_{K,\Gamma}$$

$$T \vdash \varphi \quad \text{iff} \quad \Theta[T] \vDash_{BEqn^{\Sigma}_{K,\Gamma}} \Theta(\varphi)$$

$$\{\Delta(\delta_i, \epsilon_i) : i \in I\} \vdash \Delta(\varphi_1, \varphi_2) \quad \text{iff} \quad \{\delta_i \approx \epsilon_i : i \in I\} \vDash_{BEqn^{\Sigma}_{K,\Gamma}} \varphi_1 \approx \varphi_2$$

$$\varphi \dashv\vdash \Delta[\Theta(\varphi)] \qquad \varphi_1 \approx \varphi_2 =\mid\vDash_{BEqn^{\Sigma}_{K,\Gamma}} \Theta[\Delta(\varphi_1 \approx \varphi_2)]$$

# Behavioral Leibniz operator

## Definition (Behavioral Leibniz operator)

*Let* $\mathcal{L} = \langle \Sigma, \vdash \rangle$ *be a structural many-sorted logic and* $\Gamma$ *a subsignature of* $\Sigma$*. The* $\Gamma$*-behavioral Leibniz operator on the term algebra,*

$$\Omega^{bhv}_{\Gamma} : Th_{\mathcal{L}} \to Cong^{\Sigma}_{\Gamma}(\mathbf{T}_{\mathbf{\Sigma}}(\mathbf{X}))$$

$$T \mapsto \text{largest } \Gamma\text{-congruence compatible with } T$$

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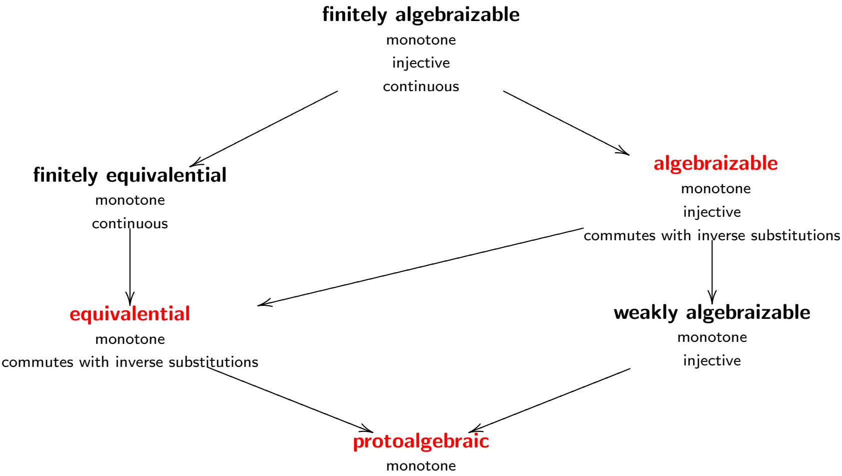
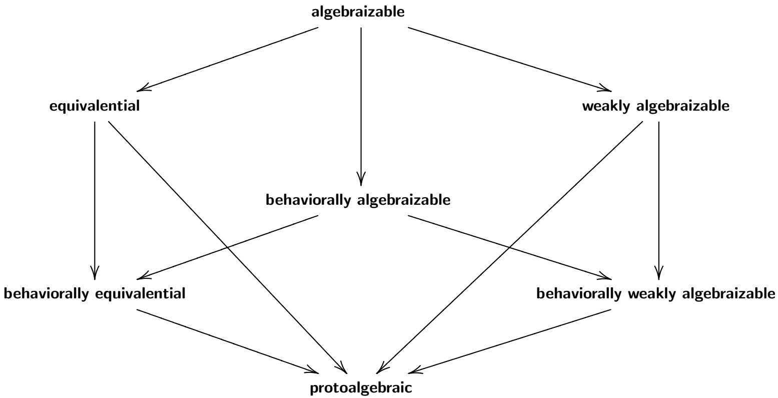
# Characterization theorems (Behaviorally)



# Leibniz Hierarchy



# Signature of $\mathcal{C}_1$

- Signature $\Sigma_{\mathcal{C}_1} = \langle \{\phi\}, F \rangle$ such that:

  $F_{\epsilon\phi} = \{\mathbf{t}, \mathbf{f}\}$, $F_{\phi\phi} = \{\neg\}$ and $F_{\phi\phi\phi} = \{\wedge, \vee, \Rightarrow\}$.

- Subsignature $\Gamma = \langle \{\phi\}, F^{\Gamma} \rangle$ of $\Sigma_{\mathcal{C}_1}$ such that:

  $F^{\Gamma}_{\phi\phi} = \emptyset$ and $F^{\Gamma}_{ws} = F_{ws}$ for every $ws \neq \phi\phi$.

# Hilbert-style axiomatization

Axioms:

- $\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_1)$
- $(\xi_1 \Rightarrow (\xi_2 \Rightarrow \xi_3)) \Rightarrow ((\xi_1 \Rightarrow \xi_2) \Rightarrow (\xi_1 \Rightarrow \xi_3))$
- $(\xi_1 \wedge \xi_2) \Rightarrow \xi_1$
- $(\xi_1 \wedge \xi_2) \Rightarrow \xi_2$
- $\xi_1 \Rightarrow (\xi_2 \Rightarrow (\xi_1 \wedge \xi_2))$
- $\xi_1 \Rightarrow (\xi_1 \vee \xi_2)$
- $\xi_2 \Rightarrow (\xi_1 \vee \xi_2)$
- $(\xi_1 \Rightarrow \xi_3) \Rightarrow ((\xi_2 \Rightarrow \xi_3) \Rightarrow ((\xi_1 \vee \xi_2) \Rightarrow \xi_3))$
- $\neg\neg\xi_1 \Rightarrow \xi_1$
- $\xi_1 \vee \neg\xi_1$
- $\xi_1^\circ \Rightarrow (\xi_1 \Rightarrow (\neg\xi_1 \Rightarrow \xi_2))$
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \wedge \xi_2)^\circ$
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \vee \xi_2)^\circ$
- $(\xi_1^\circ \wedge \xi_2^\circ) \Rightarrow (\xi_1 \Rightarrow \xi_2)^\circ$
- $\mathbf{t} \Leftrightarrow (\xi_1 \Rightarrow \xi_1)$
- $\mathbf{f} \Leftrightarrow (\xi_1^\circ \wedge (\xi_1 \wedge \neg\xi_1))$

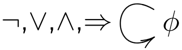
Rule:

- $\dfrac{\xi_1 \quad \xi_1 \Rightarrow \xi_2}{\xi_2}$

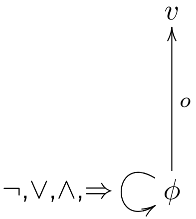
where $\varphi^\circ$ is an abbreviation of $\neg(\varphi \wedge (\neg\varphi))$.

# Extended signature

Signature $\Sigma_{\mathcal{C}_1}$:




Extended signature $\Sigma^o_{\mathcal{C}_1}$:



# $\mathcal{C}_1$ of da Costa

**Theorem**

*$\mathcal{C}_1$ is $\Gamma$-behaviorally algebraizable with $\Theta(x) = \{x \approx \mathbf{t}\}$, $\Delta(x_1, x_2) = \{x_1 \Rightarrow x_2,\ x_2 \Rightarrow x_1\}$ and the equivalent $\Gamma$-hidden quasivariety semantics is the class of two-sorted algebras introduced by Caleiro et al.*

Da Costa algebras can be recovered using behavioral reasoning over the hidden sort.

# Conclusions

- Generalization of the notion of algebraizable logic
- Strong representation of many-sorted logics in behavioral logics
- Characterization results using the Leibniz operator
- Covering many-sorted logics as well as some non-algebraizable logics (according to the old notion)

# Further work

- Other interesting examples, such as first order logic with many-sorted terms, including exogenous probabilistic and quantum logics
- Development of the full landscape of behavioral algebraization, including weakly-algebraizable, and related work, such as $k$-deductive systems
- Application of this theory to logics algebraizable according to the old notion