Ordered Cover Systems for Residuated Logics

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Aim:

Generalise the Kripke-Joyal intuitionistic semantics to non-distributive logics.

Ordered Cover Systems

Based on structures

$$\langle S, \preccurlyeq, \triangleright \rangle$$

- $\preccurlyeq$ is a pre-order on S.
  - $x \preccurlyeq y$ read "y refines x"
- $\triangleright$ is a binary "cover relation"

  from subsets $C \subseteq S$ to elements $x \in S$.
  - $C \triangleright x$ is read "C covers x"
- also write $x \triangleleft C$ for $C \triangleright x$ "x is covered by C".

axioms to follow...

Increasing Subsets

Up-sets

Definitions

- A set $X \subseteq S$ is increasing if

$$x \in X \text{ and } x \preccurlyeq y \text{ implies } y \in X.$$

- $\uparrow X = \{y : \exists x \in X(x \preccurlyeq y)\}$ the up-set generated by X
- $\uparrow x = \{y : x \preccurlyeq y\}$
- Y refines X if $Y \subseteq \uparrow X$, i.e. $(\forall y \in Y)(\exists x \in X) x \preccurlyeq y$

Example

For any topological space, take

- S = set of open sets.
- $x \preccurlyeq y$ iff $x \supseteq y$, "y refines x"
- $C \triangleright x$ iff $\bigcup C = x$.
- $\uparrow x = \{\text{open } y : y \subseteq x\}$.

Local Truth

A property holds locally if it holds of an open neighbourhood of each point.

"Pointless" example:

- A function is locally constant if its domain is covered by open sets, on each of which the function is constant.

Hence:
Abstract the relevant properties of open covers . . .

Grothendieck pretopology

- $C \triangleright x$ implies $C \subseteq \uparrow x$.

implies distributivity of $\wedge$ over $\vee$!

Axioms

- Identity:
  $x \triangleleft \{x\}$.
- Transitivity:
  if $x \triangleleft C$ and for all $y \in C$, $y \triangleleft C_y$, then $x \triangleleft \bigcup_{y \in C} C_y$.
- Refinement:
  if $x \preccurlyeq y$, then every x-cover C can be refined to a y-cover B:
  $B \subseteq \uparrow C$.

Topological case: $B = \{y \cap z : z \in C\}$.

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Kripke-Joyal Semantics

Truth-sets/Satisfaction: $\|\varphi\| = \{x : x \models \varphi\}$

$x \models \varphi \wedge \psi$ iff $x \models \varphi$ and $x \models \psi$.

$x \models \varphi \vee \psi$ iff there is an x-cover $C \subseteq \|\varphi\| \cup \|\psi\|$,

i.e. for all $z \in C$, $z \models \varphi$ or $z \models \psi$.

So $x \models \varphi \vee \psi$ iff the Boolean disjunction is locally satisfied at x.

Truth is increasing:

$x \models \varphi$ and $x \preccurlyeq y$ implies $y \models \varphi$, i.e. $\|\varphi\|$ is an up-set.

Local truth implies truth:

$\|\varphi\|$ is cover-closed, i.e.

if $x \lhd C \subseteq \|\varphi\|$, then $x \models \varphi$.

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Model Structures

$$\mathfrak{S} = \langle S, \preccurlyeq, \rhd, \cdot, \varepsilon \rangle$$

$\cdot$ is associative and $\preccurlyeq$-monotonic, with identity ε

Cover axioms:

- Existence: there exists an x-cover $C \subseteq {\uparrow}x$;
- Transitivity:
  if $x \lhd C$ and for all $y \in C$, $y \lhd C_y$, then $x \lhd \bigcup_{y \in C} C_y$.
- Refinement:
  if $x \preccurlyeq y$, then every x-cover can be refined to a y-cover.
- Stability: if $C \rhd x$ and $B \rhd y$, then $C \cdot B \rhd x \cdot y$,

  where $C \cdot B = \{c \cdot b : c \in C \text{ and } b \in B\}$.

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- Ono / Došen frames: $\langle S, \preccurlyeq, \ \cdot\ , \varepsilon \rangle$ without $\triangleright$.

- Sambin pretopology: $\langle S, \triangleright, \ \cdot\ , \varepsilon \rangle$ without $\preccurlyeq$.

The operator $j_{\triangleright}$

Definition

$$z \in j_{\triangleright} X \quad \text{iff} \quad \text{there is a } C \triangleright z \text{ with } C \subseteq X$$

$$\text{iff} \quad z \text{ locally belongs to } X.$$

Results

- jX is increasing whenever X is.
- $X \subseteq Y$ implies $jX \subseteq jY$
- $X \subseteq jX$
- $j(jX) = jX$
- $jX \bullet jY \subseteq j(X \bullet Y)$, where $X \bullet Y = \uparrow(X \cdot Y)$

Scholium: the up-sets form a quantale under $\bullet$, with j a quantic nucleus

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“Facts”

Definitions

X is cover-closed iff $jX \subseteq X$

iff $\exists C(x \lhd C \subseteq X)$ implies $x \in X$.

iff “local membership implies membership”

(cf. “local truth implies truth”)

X is a fact if it is *increasing* and *cover-closed*, i.e. $X = {\uparrow}X = jX$.

Lemma

$j{\uparrow}X$ is the smallest fact containing X.

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The Algebra of Facts

Theorem

The set $L(\mathfrak{S})$ of facts of a model structure $\mathfrak{S}$ is a residuated completely lattice ordered monoid, hence a model of the full Lambek calculus, in which

$$\begin{aligned}
\bigsqcap\mathcal{F} &= \bigcap\mathcal{F} \\
\bigsqcup\mathcal{F} &= j(\bigcup\mathcal{F}) \\
X \otimes Y &= j(X \bullet Y) = j{\uparrow}(X \cdot Y) \\
X \Rightarrow_l Y &= \{z \in S : z \cdot X \subseteq Y\} \\
X \Rightarrow_r Y &= \{z \in S : X \cdot z \subseteq Y\} \\
\mathsf{T} &= S \\
\mathsf{F} &= j\emptyset = \{x : \emptyset \rhd x\} \\
\mathbf{1} &= j{\uparrow}\varepsilon
\end{aligned}$$

Scholium: $L(\mathfrak{S})$ is the quantale of j-fixpoints

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Strong model structures:

Definition

- $X \bullet Y$ is cover-closed if X and Y are. Hence

$$X \otimes Y = X \bullet Y = \uparrow(X \cdot Y)$$

- $\uparrow\varepsilon$ is cover-closed. Hence $\mathbf{1} = \uparrow\varepsilon$

Representation

Theorem

For any residuated lattice-ordered monoid L there is a strong model structure $\mathfrak{S}_L$ and an isomorphic embedding

$$L \longrightarrow L(\mathfrak{S}_L)$$

preserving all joins and meets that exist in L.

Proof.

- Embed L into its *MacNeille completion* $\bar{L}$.
- Define a strong $\mathfrak{S}_L$ based on $\bar{L}$ with $\bar{L} \cong L(\mathfrak{S}_L)$.
- Definition of $\mathfrak{S}$ similar to topological cover systems.

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Satisfaction

$x \models \bigwedge \Phi$	iff	$x \models \varphi$ for all $\varphi \in \Phi$.
$x \models \bigvee \Phi$	iff	there exists $C \triangleright x$ such that for all $z \in C$,
		$z \models \varphi$ for some $\varphi \in \Phi$.
$x \models \varphi \& \psi$	iff	there exist y, z with $y \cdot z \preccurlyeq x$, $y \models \varphi$ and $z \models \psi$.
$x \models \varphi \to_l \psi$	iff	$y \models \varphi$ implies $x \cdot y \models \psi$.
$x \models \varphi \to_r \psi$	iff	$y \models \varphi$ implies $y \cdot x \models \psi$.
$x \models \mathsf{T}$		
$x \models \mathsf{F}$	iff	$\emptyset \triangleright x$
$x \models \mathbf{1}$	iff	$\varepsilon \preccurlyeq x$

Modelling Negation

Definition

Negation pair on a poset L = a Galois connection:

$$L \underset{-_l}{\overset{-_r}{\rightleftarrows}} L$$

$$a \leq -_l\, b \qquad \text{iff} \qquad b \leq -_r\, a$$

Equivalently:

- Antitone: $a \leq b$ implies $-_l\, b \leq -_l\, a$ and $-_r\, b \leq -_r\, a$
- Double Negation Introduction:
  - $a \leq -_l -_r\, a$
  - $a \leq -_r -_l\, a$

Orthogonality relation $\perp$ on $\mathfrak{S}$

Definition

- $\perp \subseteq S \times S$, not assumed symmetric
- $X \perp Y$ if $x \perp y$ for all $x \in X$ and $y \in Y$
- Special cases: $x \perp Y$, $X \perp y$

Axioms:

- Refinement preserves orthogonality:
  if $x \perp y$, $x \preccurlyeq x'$ and $y \preccurlyeq y'$, then $x' \perp y'$.
- Locally orthogonal implies orthogonal:
  $z \triangleleft C \perp X$ implies $z \perp X$
  $X \perp C \triangleright z$ implies $X \perp z$

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Theorem

If X is a fact, then so are

$$-_l X = \{z : z \perp X\}$$
$$-_r X = \{z : X \perp z\}$$

and these define a negation pair on the lattice $L(\mathfrak{S})$ of facts of $\mathfrak{S}$.

Theorem (Representation)

A negation pair on a lattice L is representable as the negation pair on $L(\mathfrak{S}_L)$ induced by an orthogonality relation on $\mathfrak{S}_L$.

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Negation from a fixed fact

Fix a fact 0 in $\mathfrak{S}$, and define

$$x \perp y \qquad \text{iff} \qquad x \cdot y \in 0.$$

Then we get residuated negation:

$$\{z : z \perp X\} = X \Rightarrow_l 0$$
$$\{z : X \perp z\} = X \Rightarrow_r 0.$$

Lemma

In this case, $z \perp X$ implies $z \perp jX$.

This implies
$-_l -_r X = -_l -_r jX, \quad -_r -_l X = -_r -_l jX.$

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Double Negation Elimination (DNE)

$$-_l -_r a = a, \quad -_r -_l a = a$$

Theorem

In any model structure $\mathfrak{S}$ with residuated negation, the following are equivalent:

1. *The lattice $L(\mathfrak{S})$ of facts satisfies DNE.*
2. $jX = -_l -_r X = -_r -_l X$, *if X increasing.*
3. $jX = -_l -_r X = -_r -_l X$, *if X is a fact.*

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Corollary

If $\perp$ *is symmetric (e.g. if* $x \cdot y = y \cdot x$*)*

then the following are equivalent for any up-set X*:*

- X *is a* j*-fact:* $X = jX$
- X *is a Girardian fact:* $X = --X$

The Girard Modality $!\varphi$

"Of Course φ"

Weakening

$$\frac{\Gamma \vdash \Delta}{\Gamma,\ !\varphi \vdash \Delta}$$

Contraction

$$\frac{\Gamma,\ !\varphi,\ !\varphi \vdash \Delta}{\Gamma,\ !\varphi \vdash \Delta}$$

Set-theoretic semantics

- Girard: $!\,X = -\,-\,(X \cap Id)$

  where Id is the set of idempotent ($x \cdot x = x$) members of $\mathbf{1} = -\,-\,\{\varepsilon\}$.

- Here: $!\,X = j{\uparrow}(X \cap I)$

  where $I \subseteq j{\uparrow}\varepsilon$ has
  - $x \in I$ implies $x \cdot x \preccurlyeq x$
  - I is closed under $\cdot$
  - $I \rhd \varepsilon$

Note: I is a set of idempotents if $\uparrow\varepsilon$ is a fact and $\preccurlyeq$ is antisymmetric.

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  - $I \triangleright \varepsilon$

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Theorem

This operation

$$!\,X = j{\uparrow}(X \cap I)$$

is a *modality* *over the residuated lattice* $L(\mathfrak{S})$ *of facts, i.e.*

- $!\,X \leq X$
- $!\,!\,X = !\,X$
- $X \leq Y$ *implies* $!\,X \leq !\,Y$
- $!\,\mathsf{T} = \mathbf{1}$
- $!\,X \leq !\,X \otimes !\,X$
- $!\,X \otimes !\,Y \leq !(!\,X \otimes !\,Y)$

Theorem (Representation)

Any modality on a residuated lattice L is representable as the modality $j{\uparrow}(X \cap I)$ on $L(\mathfrak{S}_L)$ induced by a suitable $I \subseteq {\uparrow}\varepsilon$ on $\mathfrak{S}_L$.

Yet another lattice representation?

Let L be any lattice. Suppose

- S is the set of proper filters of L.
  (or the set of intersection irreducible filters)
- $\phi(a) = \{F \in S : a \in F\}$ where $a \in L$.

Theorem (Birkhoff)

The map $a \mapsto \phi(a)$ is

- *order invariant: $a \leq b$ iff $\phi(a) \subseteq \phi(b)$;*
- *hence injective, and*
- *represents meets as intersections:*

$$\phi(a \wedge b) = \phi(a) \cap \phi(b).$$

Now on $\langle S, \subseteq \rangle$ define

- $C \triangleright F$ iff $\bigcap C \subseteq F$.
- $T =$ the topology with sub-base $\{-\phi(a) : a \in L\}$.

Theorem

- *$\phi(a \vee b) = j_{\triangleright}(\phi(a) \cup \phi(b))$.*
- *$\langle S, T \rangle$ is compact.*
- *Each $\phi(a)$ is closed, cocompact and a j-fact.*
- *Any closed cocompact j-fact is equal to $\phi(a)$ for some $a \in L$.*

Another Heyting algebra representation

Let L be any Heyting algebra.

- S = all principal filters of L.
- $C \triangleright F$ iff $\bigcap C = F$. Hence $C \triangleright F$ implies $C \subseteq \uparrow F$.
- Then $L(\mathfrak{S})$ is a Heyting algebra.
- Use distributivity of L to show

$$\phi(a \vee b) = j_{\triangleright}(\phi(a) \cup \phi(b))$$

- Can make this a Grothendieck topology, by requiring covers themselves to be up-sets.