ALGEBRAIC AND TOPOLOGICAL MODELS OF SOLOVAY'S MODAL SYSTEM

LEO ESAKIA & DAVID GABELAIA

RAZMADZE MATHEMATICAL INSTITUTE, TBILISI, GEORGIA

Provability Interpretations of Modal Logic

$$(\cdot)^* : \underbrace{\mathrm{ML}}_{\text{Modal Language}} \longrightarrow \underbrace{\mathrm{ZFL}}_{\text{Language of ZF set theory}}$$

$$p^* = \text{a sentence of ZFL}$$

$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$$

$$(\neg \varphi)^* = \neg \varphi^*$$

$$(\Box \varphi)^* = \mathrm{Bew}(\ulcorner \varphi^* \urcorner) = \text{“}\varphi^* \text{ is true in all models of ZF”}$$

$$\boxed{\mathrm{GL} = \{ \varphi \in \mathrm{ML} \mid \text{for all } *,\ \varphi^* \in \mathrm{ZF} \}}$$

PROVABILITY INTERPRETATIONS (2)

$$(\cdot)^* : ML \longrightarrow ZFL$$

$p^* \quad = \text{ a sentence of } ZFL$

$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*$

$(\neg \varphi)^* = \neg \varphi^*$

$(\Box \varphi)^* =$ "φ^* holds in all <u>transitive</u> models of ZF"

$$\boxed{GL.S = \{ \varphi \in ML \mid \text{for all } *,\ \varphi^* \in ZF \}}$$

Modal System GL: Kripke Semantics

$GL = K + \Box(\Box p \to p) \to \Box p$

(W,R) is a GL-frame iff R is
- transitive
- irreflexive
- conversely well-founded

GL is the modal logic of:

- Finite transitive irreflexive frames
- Finite irreflexive trees

MODAL SYSTEM GL: TOPOLOGICAL SEMANTICS

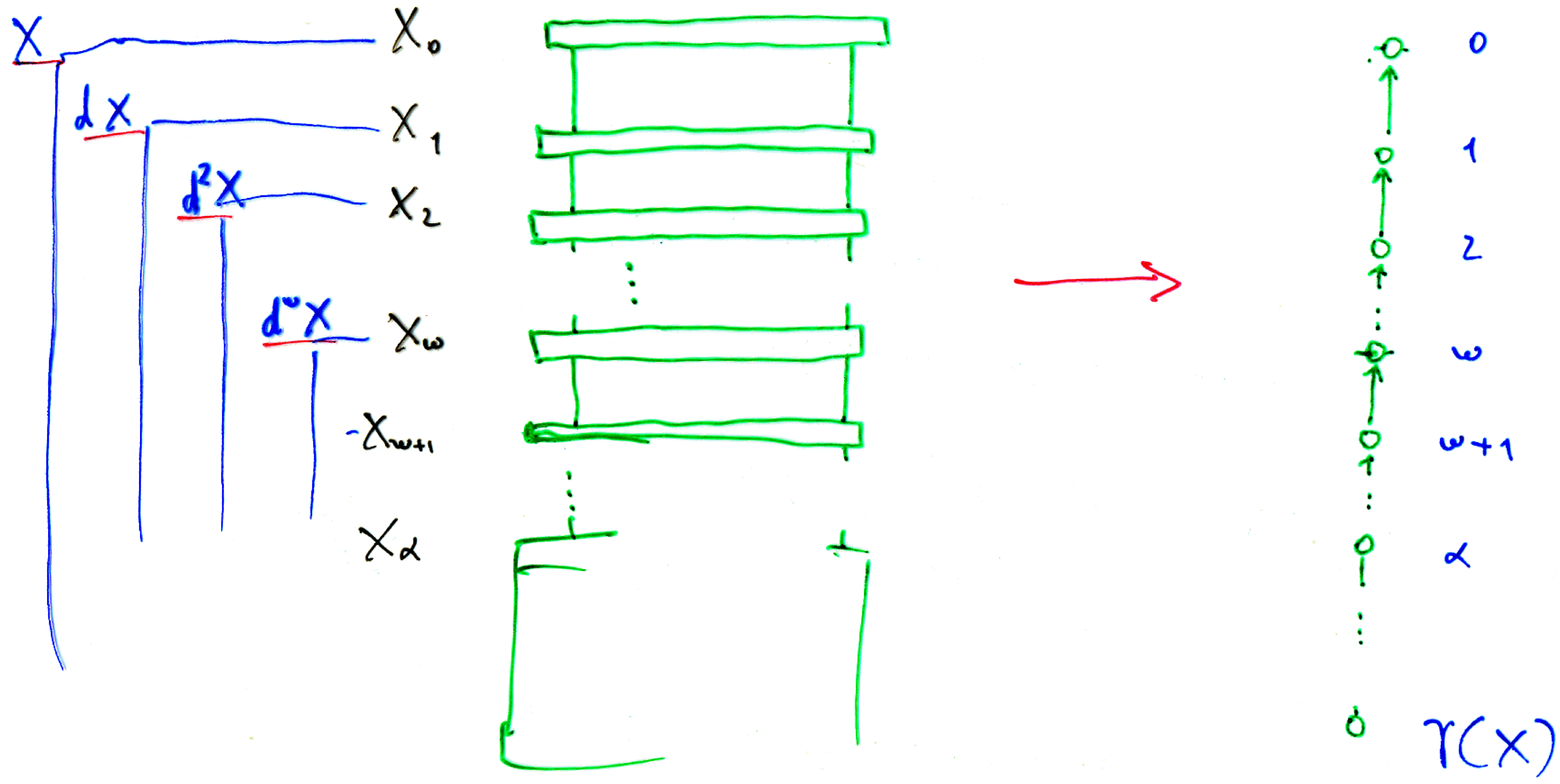
Interpret $\Diamond$ as a limit operator d over a topological space

$$\boxed{\mathfrak{M}, x \models \Diamond A \quad \text{iff} \quad x \in dA}$$

X is a GL-space iff $\forall A \subseteq X.\ (dA = d(A - dA))$

GL is the modal logic of:

- Scattered spaces
- Ordinals (with interval topology)
- ω^ω



X
dX
d²X
d^ωX
X₀
X₁
X₂
X_ω
X_{ω+1}
X_α
0
1
2
ω
ω+1
α
γ(X)

$$\neg[\Diamond(\Diamond p \wedge \neg\Diamond q) \wedge \Diamond(\Diamond q \wedge \neg\Diamond p))$$

MODAL SYSTEM GL.S: KRIPKE SEMANTICS

$$\boxed{GL.S = GL + \Box(\Box p \to \Box q) \to (\Box q \to \Box p)}$$

Correction:

$$GL.S = GL + \Box(\Box p \to \Box q) \vee \Box(\Box q \to p)$$

(W,R) is a GL.S-frame

iff

$\forall x,y,z \in W$: (1) $xRy \Rightarrow y \not R x$

(2) $x \not R y \wedge y \not R z \Rightarrow x \not R z$ – "Antitransitivity

---

(2') $xRy \Rightarrow xRz \vee zRy$

Transitivity follows

(1) and (2') are von Wright's conditions on Preference relations

MODAL SYSTEM GL.S: TOPOLOGICAL SEMANTICS

LET US CALL GL.S-SPACES SOLOVAY SPACES

A SCATTERED SPACE X IS A SOLOVAY SPACE

iff

$$\forall A, B \subseteq X \,.\, \left(\underline{d(A - dB) \cap d(dB - dA) = \emptyset} \right)$$

TOPOLOGY OF SOLOVAY SPACES

X

$(\wp(X), \cup, -, d)$

DERIVATIVE ALGEBRA

$(Op(X), \cup, \cap, \subseteq)$

HEYTING ALGEBRA

SOLOVAY SPACES

A VARIETY OF DERIVATIVE ALGEBRAS

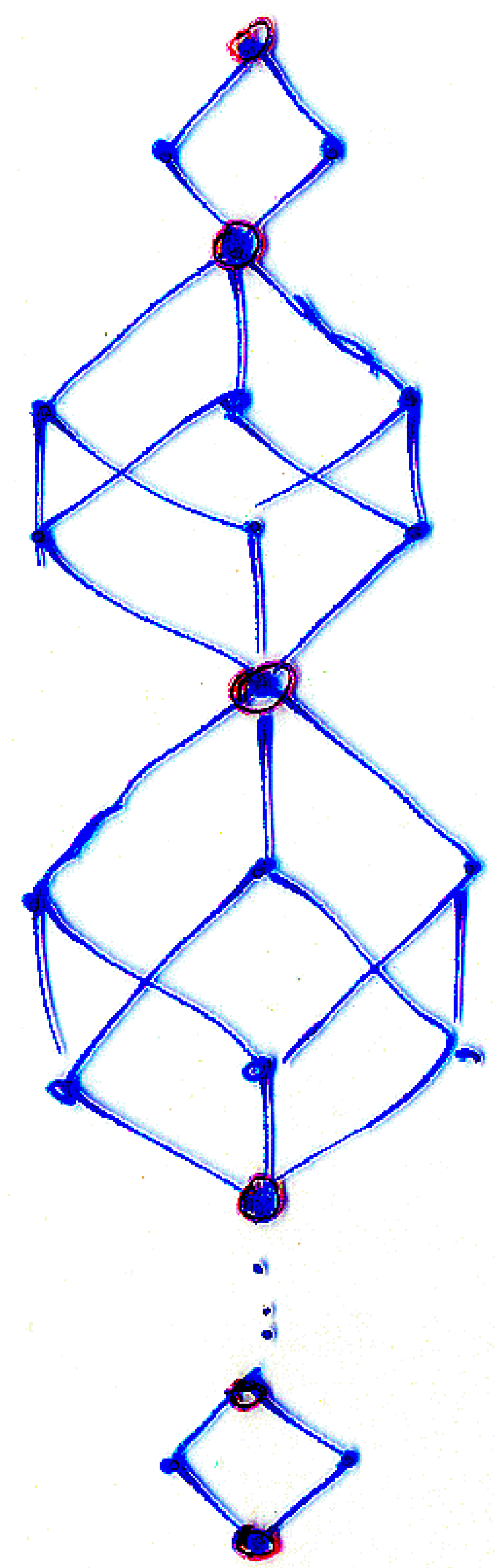
$$\delta(a - \delta b) \wedge \delta(\delta b - \delta a) = 0$$

A VARIETY OF HEYTING ALGEBRAS

$$\boxed{(q \to p) \vee [((p \to q) \to p) \to p]}$$

CASCADE HEYTING ALGEBRAS

$$\underbrace{(p \to q) \vee (q \to p)}_{\text{linearity}} \vee \underbrace{[((p \to q) \to p) \to p]}_{\text{Pierce Law}}$$



$\Pi B_i \quad B_i$ - Boolean

$\underline{d}$

- FIND A TOPOLOGICAL CHARACTERIZATION OF SOLOVAY SPACES

- FIND NATURAL EXAMPLES OF NON-ALEXANDROFF SOLOVAY SPACES.