Bisimulations of descriptive frames

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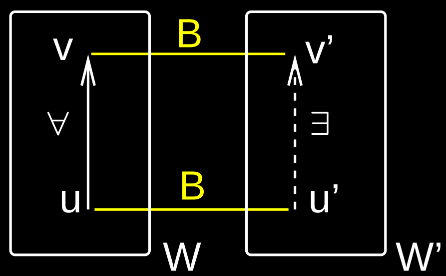
Bisimulations of descriptive frames

Frames

- Kripke frame



- descriptive frame

?

Coalgebras

- coalgebra for the powerset functor

$$\begin{array}{ccccc} A & \xleftarrow{\;\pi\;} & B & \xrightarrow{\;\pi'\;} & A' \\ \sigma\downarrow & & \downarrow\beta & & \downarrow\sigma' \\ \mathcal{P}(A) & \xleftarrow{\;\mathcal{P}(\pi)\;} & \mathcal{P}(B) & \xrightarrow{\;\mathcal{P}(\pi')\;} & \mathcal{P}(A') \end{array}$$

- coalgebra for the Vietoris functor

$$\begin{array}{ccccc} A & \xleftarrow{\;\pi\;} & B & \xrightarrow{\;\pi'\;} & A' \\ \sigma\downarrow & & \downarrow\beta & & \downarrow\sigma' \\ \mathcal{V}(A) & \xleftarrow{\;\mathcal{V}(\pi)\;} & \mathcal{V}(B) & \xrightarrow{\;\mathcal{V}(\pi')\;} & \mathcal{V}(A') \end{array}$$

Outline

Recall:

- bisimulations for Kripke frames
- descriptive frames: in the Kripke semantic and as coalgebras
- bisimulations for coalgebras

Results:

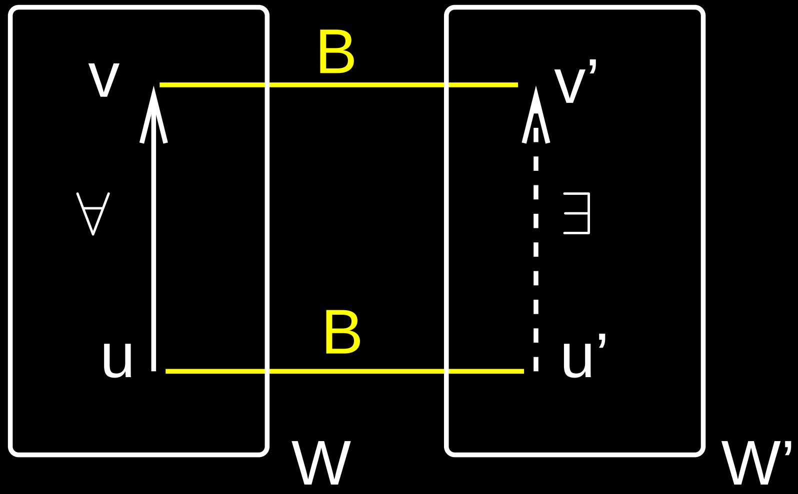
- definition of a bisimulation for descriptive frames
- properties of bisimulations for descriptive frames

Remark. restriction to frames (instead of models) for simplicity

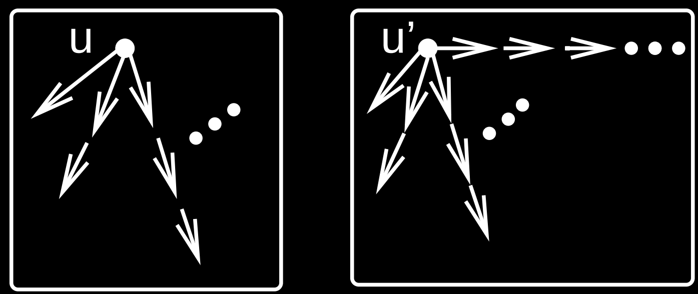
Bisimulation for Kripke frames

$B \subseteq W \times W'$ is a bisimulation between (W,R) and (W',R') if

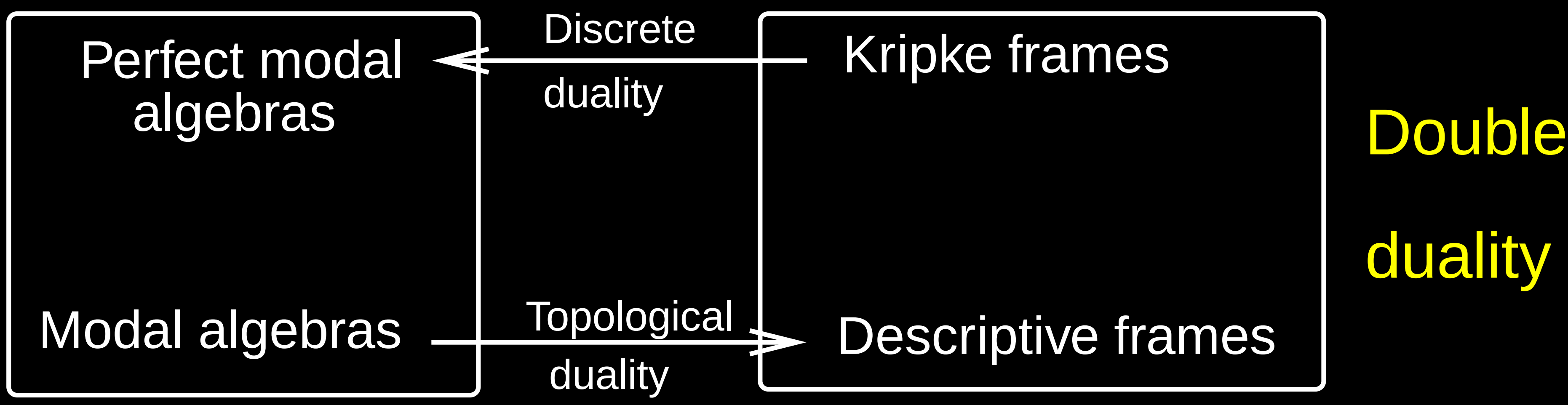
- **back condition:** if (u,u') in B and uRv, there exists v' s.t. (v,v') in B and u'R'v'
- **forth condition:** if (u,u') in B and u'R'v', there exists v s.t. (v,v') in B and uRv



Remark: modal equivalence $\neq$ bisimilarity



Descriptive frames



- Descriptive frame: (Stone space X, relation R) s.t.
  - for all $x \in X$, $\{ y \mid xRy \}$ is closed
  - if U is clopen, then $\{ x \mid \exists y \in U \text{ s.t. } xRy \}$ is clopen
- Result: every modal logic is complete with respect to a class of descriptive frames

Descriptive frames as coalgebras

A coalgebra for a functor F in a category C is a pair (A, σ: A $\rightarrow$ F(A))

Frames

- Kripke frame:
  (set W, relation R)

- descriptive frame:
  (Stone space X, relation R)

Coalgebras

- coalgebra for the powerset functor $\mathcal{P}$ in Set:
  (W, R[.]: W $\rightarrow$ $\mathcal{P}$ (W))
  R[.]: x $\mapsto$ { y | xRy }

- coalgebra for the Vietoris functor $\mathcal{V}$ in Stone Spaces:
  (X, R[.]: X $\rightarrow$ $\mathcal{V}$ (X))
  R[.]: x $\mapsto$ { y | xRy }

Descriptive frames as coalgebras

Frames

- Kripke frame
- descriptive frame:
  (Stone space X, relation R)

Coalgebras

- coalgebra for the powerset functor $\mathcal{P}$ in Set
- coalgebra for the Vietoris functor $\mathcal{V}$ in Stone Spaces:
  (X, R[.]: X $\rightarrow \mathcal{V}$ (X))
  R[.]: x $\mapsto$ { y | xRy }

Def. If X Stone space, the Vietoris space of X is the space (V(X), τ), where

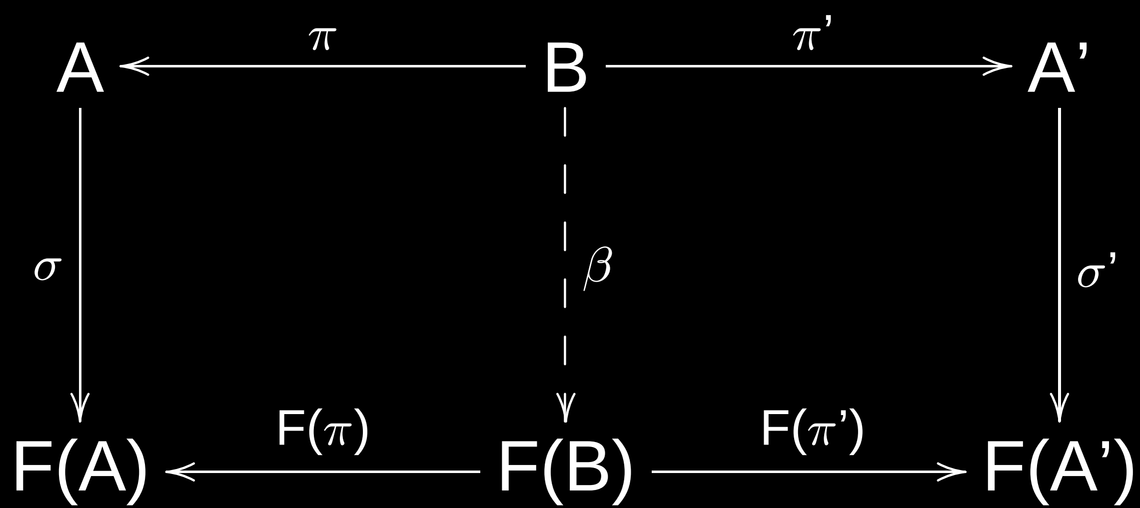
- V(X) = { F $\subseteq$ X | F closed }
- τ has as subbasis the sets

$$\begin{cases} [\ni]\ U = \{\ F \text{ closed} \mid F \subseteq U\ \}, \\ \langle\ni\rangle\ U = \{\ F \text{ closed} \mid F \cap U \neq \emptyset\ \}, \end{cases}$$

where U ranges over open subsets of X.

Bisimulation for coalgebras

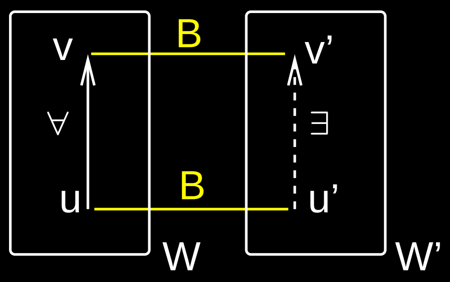
Definition. B bisimulation between (A,σ) and (A',σ') if there exists β: B $\rightarrow$ F(B) s.t. the diagram commutes:



Bisimulations for descriptive frames

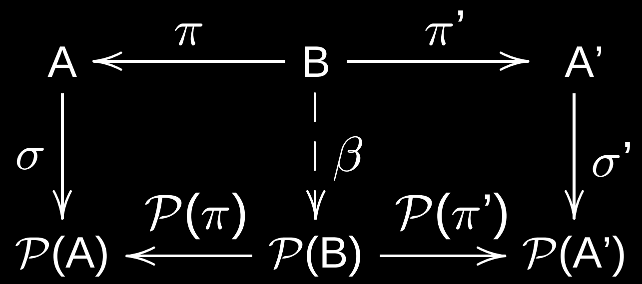
Frames

- Kripke frame



Coalgebras

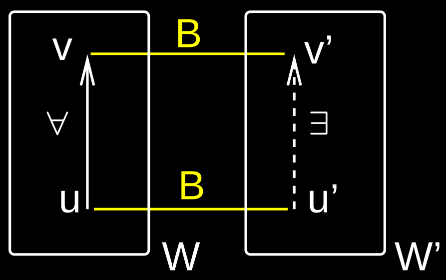
- $\mathcal{P}$-coalgebra



Bisimulations for descriptive frames

Frames

- Kripke frame

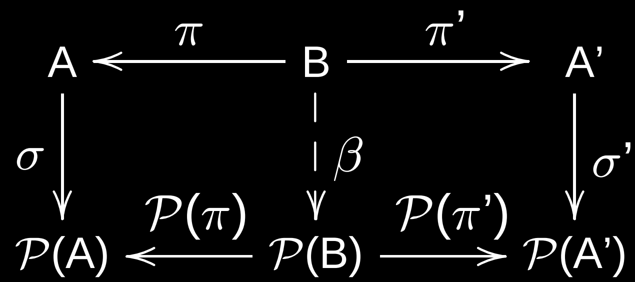


- descriptive frame

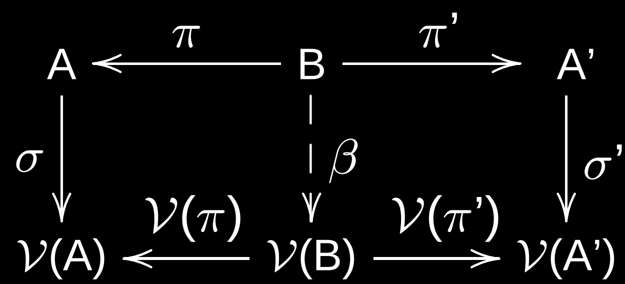
?

Coalgebras

- $\mathcal{P}$-coalgebra



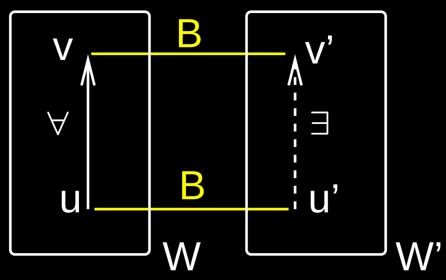
- $\mathcal{V}$-coalgebra



Bisimulations for descriptive frames

Frames

- Kripke frame



- descriptive frame

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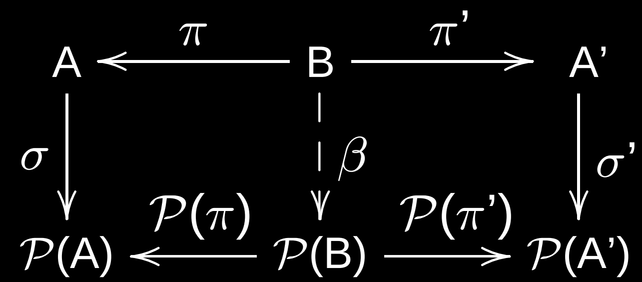
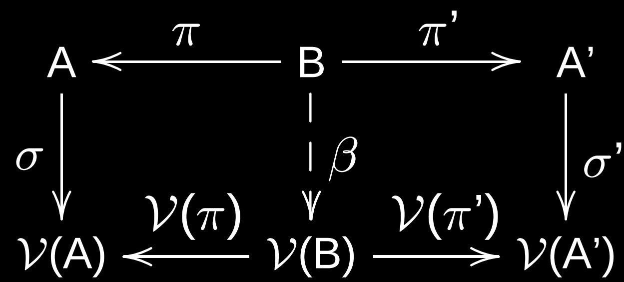
Coalgebras

- $\mathcal{P}$-coalgebra



- $\mathcal{V}$-coalgebra



Theorem. B $\subseteq$ W x W' is a bisimulation between descriptive frames (W,R,τ) and (W',R',τ') iff

- B Kripke bisimulation between (W,R) and (W',R')
- B $\subseteq$ (W,τ) x (W',τ') is closed

Results

Remind. bisimulation for descriptive frames = closed Kripke bisimulation

- Hennessy-Milner property: bisimilarity = modal equivalence

Results

Remind. bisimulation for descriptive frames = closed Kripke bisimulation

- Hennessy-Milner property: bisimilarity = modal equivalence
- Link between Kripke bisimulation and bisimulation for descriptive frames?
  - If B bisimulation for descriptive frames, then B Kripke bisimulation
  - Main result:

    Theorem. Let (W,R,τ) and (W',R',τ') be descriptive frames.
    If $B \subseteq W \times W'$ is a Kripke bisimulation between (W,R) and (W',R'), its closure $\overline{B}$ is a Vietoris bisimulation between (W,R,τ) and (W',R',τ').

Corollary. Kripke bisimilarity = bisimilarity for descriptive frames

Conclusion

- use coalgebras to get a notion of bisimulation for descriptive frames (bisimulation = closed Kripke bisimulation)

- nice link between Kripke bisimulations and bisimulations for descriptive frames

Further work

- generalize the link between $\mathcal{P}$ on Set and $\mathcal{V}$ on Stone Spaces to arbitrary F on Set and G on Stone Spaces?