Non-deterministic Semantics for Dynamic Topological Logic

David Fernández Duque

Overview

Dynamic Topological Logic ($\mathcal{DTL}$) is a propositional modal framework for reasoning about dynamic topological systems. Here we will present an alternative interpretation for $\mathcal{DTL}$ which uses purely relational semantics, at the cost of passing to systems where the dynamics are not deterministic. This allows us to use tools from combinatorics which are not available in the original topological setting.

Dynamical topological systems

A *dynamical topological system* is a pair $\langle X, f \rangle$, where X is a topological space and f a function acting on X. We think of points in X as *states* and f as a transformation of the system in time, taking each state x to the state $f(x)$.

We will focus on the case where f is continuous, and always assume that this is the case.

Topological semantics for $\mathcal{S}4$

The language of $\mathcal{S}4$ is obtained by extending that of propositional logic (whose formulas are built from a countably infinite set $\mathbf{Var}$ of propositional variables and the Boolean connectives $\wedge, \vee, \rightarrow, \neg$) with the modal operator $\Box$.

If we have a topological space X and a valuation

$$V : \mathbf{Var} \rightarrow \mathcal{P}(X),$$

we can extend V to all formulas of the language by setting

$$\begin{array}{lcl} V(\alpha \wedge \beta) & = & V(\alpha) \cap V(\beta) \\ V(\neg \alpha) & = & X \setminus V(\alpha) \\ V(\Box \alpha) & = & V(\alpha)^{\circ}. \end{array}$$

(Other Booleans are defined by De Morgan's rules.)

Kripke semantics for $\mathcal{S}4$

Another familiar interpretation for $\mathcal{S}4$ is given by Kripke semantics. If we have a pair

$$\langle W, \preccurlyeq \rangle$$

and a valuation V on W, Kripke semantics are defined by setting

$$w \models \Box\alpha \Leftrightarrow \forall v \, (w \preccurlyeq v \Rightarrow v \models \alpha) \, .$$

However, this can be seen as a particular case of topological semantics: we can define a topology on W by saying that a set U is open if, whenever $w \in U$ and $w \preccurlyeq v$, it follows that $v \in U$. One can then check that both interpretations of $\Box$ coincide.

This topology has the property that *arbitrary* intersections of open sets are open. Conversely, every such topology gives rise to a preorder.

Dynamic Topological Logic

Dynamic Topological Logic ($\mathcal{DTL}$) is an extension of $\mathcal{S}4$ which uses the temporal modalities $\bigcirc$ ('next') and $*$ ('henceforth'). Formulas in the language are interpreted over *dynamic topological models*, which are tuples

$$\mathfrak{M} = \langle X, f, V \rangle .$$

We define $x \models \bigcirc\alpha$ if and only if $f(x) \models \alpha$.

Likewise, $*\alpha$ holds in $x \in X$ if it holds in the entire orbit of x,

$$\left\{ x, f(x), f^2(x) ... \right\} .$$

Incompleteness of $\mathcal{DTL}$ for Kripke semantics

The formula

$$*\Box p \to \Box * p$$

is valid on all Kripke frames.

The left hand side is defined as

$$\bigcap_{n \geq 0} f^{-n}(V(p)^{\circ}),$$

which is an intersection of open sets and hence open on all Aleksandroff spaces (such as Kripke frames); therefore adding $\Box$ in front of the formula does not change its valuation.

A counterexample to $*\Box p \to \Box * p$

However, this formula is not valid in general.

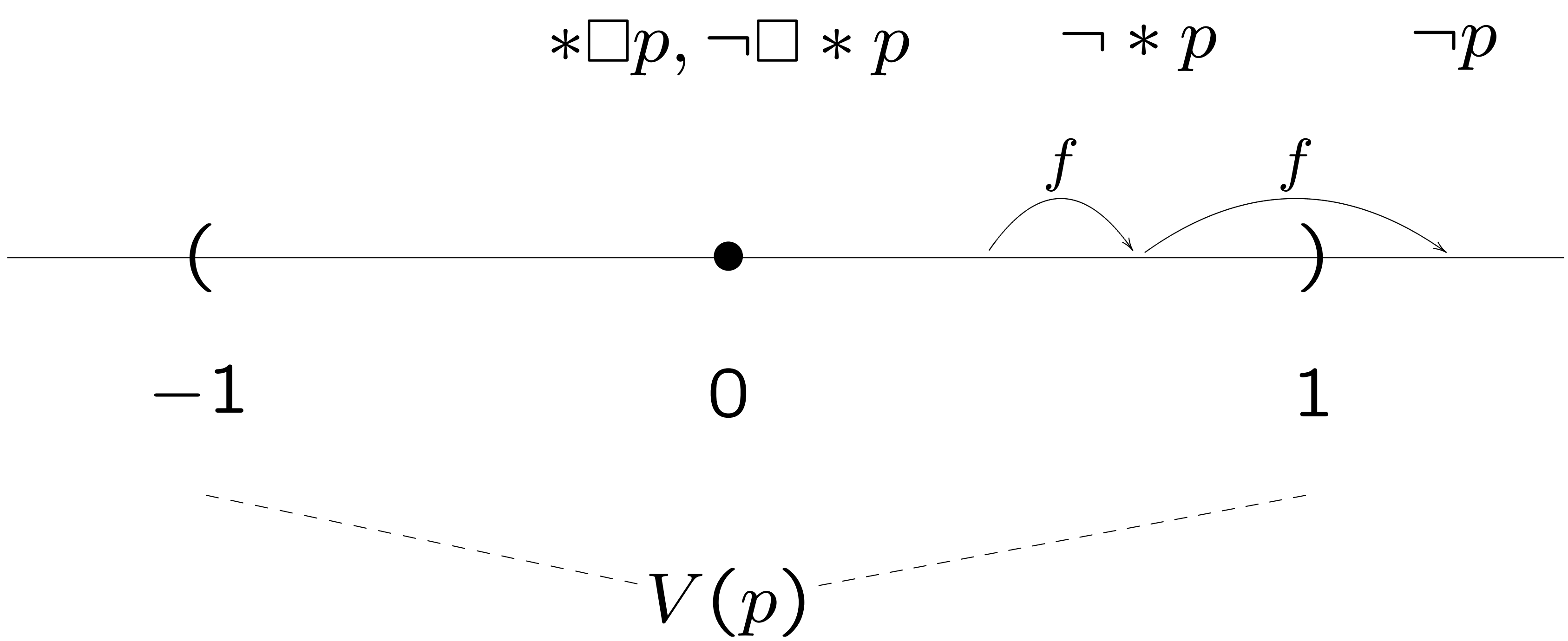
A counterexample is given by

$$X = \mathbb{R},$$

$$f(x) = 2x,$$

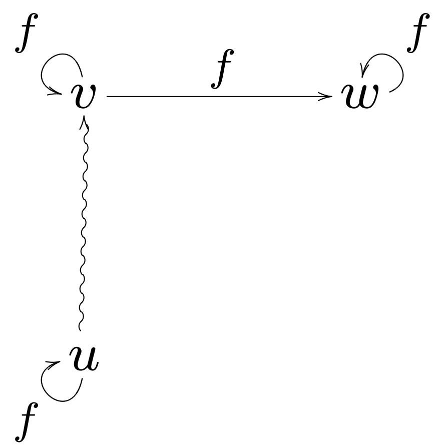
$$V(p) = (-1, 1).$$

A counterexample to $*\Box p \rightarrow \Box * p$



A counterexample to $*\Box p \rightarrow \Box * p$

This can be resumed in the following diagram:



$$u \models p, *\Box p, \neg\Box * p$$
$$v \models p, \neg * p$$
$$w \models \neg p.$$

Typed Kripke frames

A φ-*type* is a set of formulas $\mathfrak{t} \subseteq \mathrm{sub}(\varphi)$ satisfying

$$\alpha \wedge \beta \in t(w) \Rightarrow \alpha \in t(w) \text{ and } \beta \in t(w)$$

and

$$\neg\alpha \in t(w) \Leftrightarrow \alpha \notin t(w).$$

Definition 1 (typed Kripke frame)**.** *Let* φ *be a formula in the language of* $\mathcal{DTL}$*. A* φ-typed Kripke frame *is a triple*

$$\mathfrak{F} = \langle W, \preccurlyeq, t \rangle$$

where W *is a finite set,* $\preccurlyeq$ *a preorder on* W *and* t *a function assigning a* φ*-type* $t(w)$ *to each* $w \in W$ *satisfying*

$$\Box\psi \in t(w) \Leftrightarrow \forall v \, (w \preccurlyeq v \Rightarrow \psi \in t(v)) \, .$$

Sensible relations

Definition 2 (sensible relation). *A continuous relation*

$$g \subseteq W \times W$$

such that $g(w) \neq \varnothing$ *for all* $w \in W$ *is* sensible *if*

1. *for all* $\bigcirc\psi \in \mathrm{sub}(\varphi)$ *and* gwv, $\bigcirc\psi \in t(w) \Leftrightarrow \psi \in t(v)$ *and*

2. *for all* $*\psi \in \mathrm{sub}(\varphi)$ *and* gwv,

   $$*\psi \in t(w) \Leftrightarrow \psi \in t(w) \text{ and } *\psi \in t(v).$$

   Further, g *is* ω*-sensible if*

3. *for all* $*\psi \in \mathrm{sub}(\varphi)$, $\neg * \psi \in t(w) \Leftrightarrow \exists v \in W$ *and* $N \geq 0$ *such that* $\neg\psi \in t(v)$ *and* $g^N wv$.

Non-deterministic semantics

Non-deterministic semantics are given by typed Kripke frames equipped with an ω-sensible relation. A *non-deterministic quasi-model* for φ is a tuple

$$\mathfrak{D} = \langle W, \preccurlyeq, g, t \rangle ,$$

where $\langle W, \preccurlyeq, t \rangle$ is a φ-typed Kripke frame and g is an ω-sensible relation on W.

Soundness and completeness

Theorem 1. *$\mathcal{DTL}$ is sound and complete for the class of non-deterministic quasimodels; that is, a formula φ of $\mathcal{DTL}$ is satisfiable in a dynamic topological model if and only if it is satisfiable in a non-deterministic quasimodel.*

Soundness of $\mathcal{DTL}$ for non-deterministic semantics

Given a non-deterministic quasimodel

$$\mathfrak{D} = \langle W, \preccurlyeq, g, t \rangle ,$$

we can extract a dynamic topological model from it, which satisfies the same set of formulas. This implies that $\mathcal{DTL}$ is sound for non-deterministic semantics.

We will do this by considering infinite sequences of worlds as points of a topological space. Only those sequences which respect the temporal modalities will be part of the model; the rest will be discarded.

Definition 3 (realizing sequence)**.** *An infinite sequence*

$$w_0, w_1, ..., w_n, ...$$

is realizing *if* gw_nw_{n+1} *for all* n *and, whenever* $\neg * \psi \in t(w_n)$*, there exists* $m \geq n$ *such that* $\neg\psi \in t(w_m)$*. We will denote the set of all realizing sequences by* W^g*.*

Define the 'shift' operator, σ, by

$$\sigma \langle w_n \rangle_{n \geq 0} = \left\langle w_{n+1} \right\rangle_{n \geq 0}.$$

Clearly, W^g is closed under σ.

If p is a propositional variable, we will set

$$\vec{w} \models p \Leftrightarrow p \in t(w_0).$$

By the definition of a sensible relation, we have that

$$\vec{w} \models \bigcirc \alpha \Leftrightarrow \sigma(\vec{w}) \models \alpha,$$

and, because all sequences are realizing,

$$\vec{w} \models * \alpha \Leftrightarrow \sigma^n(\vec{w}) \models \alpha$$

for all $n \geq 0$.

The limit model of a non-deterministic quasimodel

Theorem 2. *There exist a topology $\mathcal{T}$ on W^g such that σ is continuous under $\mathcal{T}$ such that, for all $\psi \in \mathrm{sub}(\varphi)$,*

$$\vec{w} \models \psi \Leftrightarrow \psi \in t(w_0).$$

Thus we can define a dynamic topological model given by

$$\lim \mathfrak{D} = \langle W^g, \sigma, V \rangle ,$$

which satisfies the same subformulas of φ that $\mathfrak{D}$ does.

Recursive enumerability

In *Dynamic topological logics over spaces with continuous functions*, Konev, Kontchakov, Wolter and Zakharyaschev prove that a fragment of $\mathcal{DTL}$, $\mathcal{DTL}_1$, is recursively enumerable.

$\mathcal{DTL}_1$ uses all three modalities but $*$ is not allowed to appear in the scope of $\Box$. This fragment is complete for Kripke frames.

The proof uses a model-search procedure, and uses Kruskal's Tree Theorem to show that the procedure terminates whenever it is applied to a valid formula.

A recursive enumeration of $\mathcal{DTL}_1$

Definition 4 (path of Kripke frames)**.** *A* path *of typed Kripke frames is a finite or infinite sequence of typed Kripke frames $\langle K_n \rangle$ such that there exist continuous functions*

$$f_n : K_n \to K_{n+1}$$

which respect the temporal modalities.

A recursive enumeration of $\mathcal{DTL}_1$

The procedure attempts to construct a model by searching through all finite paths of Kipke frames.

The main obstacle is that there is no way to tell how long we must wait before a formula of the form $\neg * \alpha$ is realized in a path; that is, if $w \models \neg * \alpha$, how big should N be so that $f^N(x) \models \neg\alpha$?

However, this can be resolved by noting that, if we have indices $m < n$ and an embedding

$$e : K_m \to K_n,$$

we can use this to shorten the path. We will say m and n form a *loop*.

A recursive enumeration of $\mathcal{DTL}_1$

$$K_0 \xrightarrow{f} K_1 \xrightarrow{f} K_2 \xrightarrow{f} K_3 \xrightarrow{f} K_4 \xrightarrow{f} K_5 \xrightarrow{f} \cdots$$

$$K_1 \xrightarrow[e]{} K_4$$

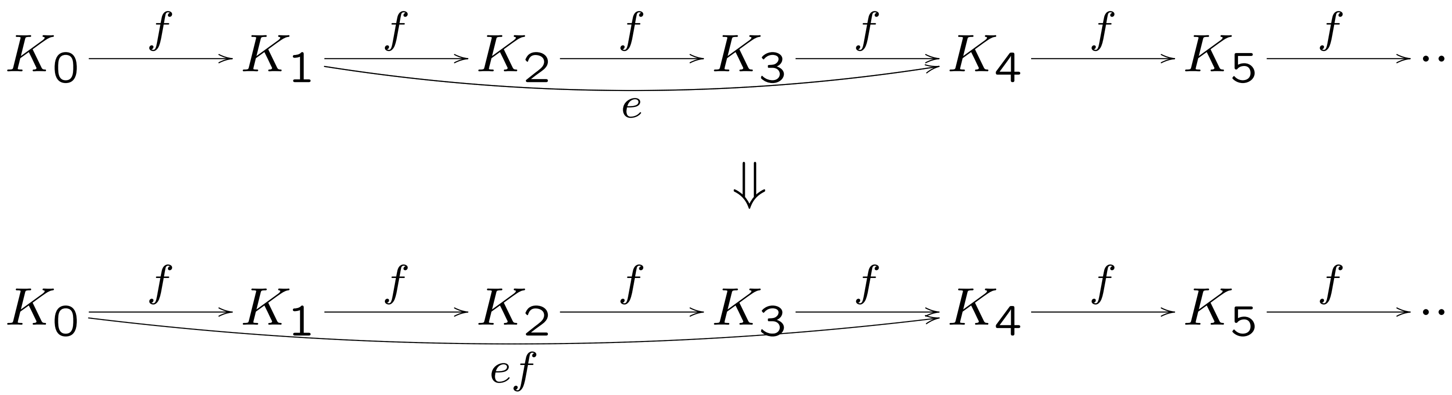
A recursive enumeration of $\mathcal{DTL}_1$

$$K_0 \xrightarrow{f} K_1 \xrightarrow{f} K_2 \xrightarrow{f} K_3 \xrightarrow{f} K_4 \xrightarrow{f} K_5 \xrightarrow{f} \cdots$$

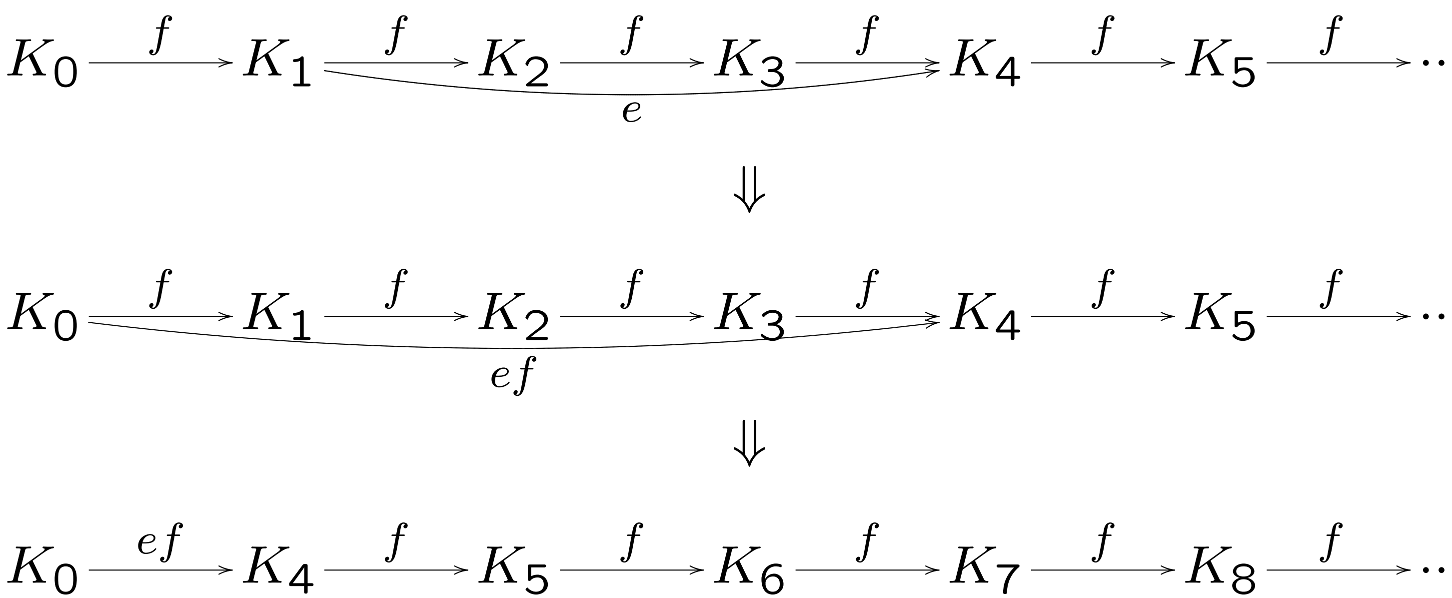
$$K_1 \xrightarrow[e]{} K_4$$

$\neg * \gamma$ (at K_0) $\qquad \neg\gamma$ (at K_5)

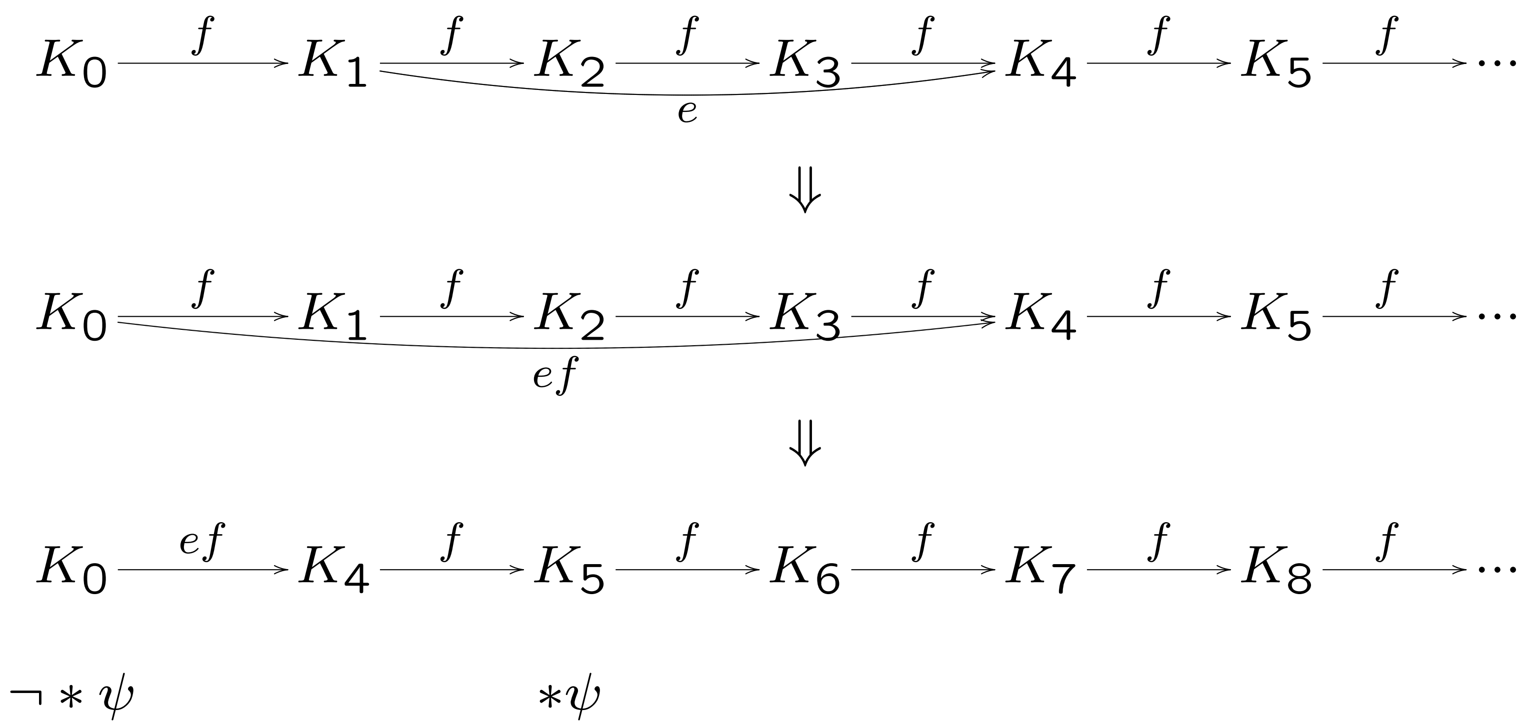
A recursive enumeration of $\mathcal{DTL}_1$



A recursive enumeration of $\mathcal{DTL}_1$



A recursive enumeration of $\mathcal{DTL}_1$



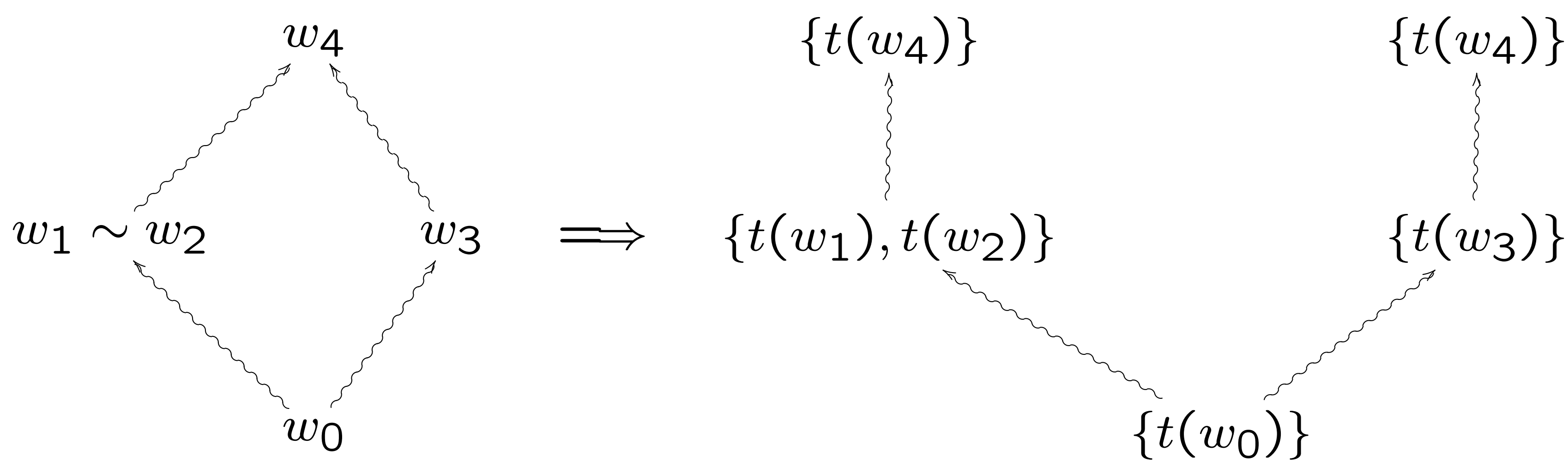
A recursive enumeration of $\mathcal{DTL}_1$

Theorem 3 (Kruskal). *If $\langle T_n \rangle_{n \geq 0}$ is an infinite sequence of trees with labels on a finite set Λ, then there exist $m < n$ such that T_m embeds into T_n.*

We can use this to guarantee that any infinite path of Kripke frames contains loops.

A recursive enumeration of $\mathcal{DTL}_1$

To see this, note that we can represent finite Kripke frames by trees labeled by sets of φ-types.



A recursive enumeration of $\mathcal{DTL}_1$

To check if a formula φ is valid, one can enumerate all finite paths of Kripke frames which contain no unnecessary loops and begin at a frame satisfying $\neg\varphi$. Explicit bounds can be given for the k-th term of the sequence.

If the enumeration does not terminate we can use König's Lemma to extract an infinite sequence which gives us a countermodel for φ.

Otherwise, every path runs into a 'dead end' and thus there exists no countermodel for φ.

Recursive enumerability of full $\mathcal{DTL}$

In principle, the techniques we described above can be applied only to Kripke frames and not to arbitrary topological spaces.

However, non-deterministic semantics allows us to run a very similar procedure on quasimodels to obtain the following theorem:

Theorem 4. *The set of valid formulas of* $\mathcal{DTL}$ *is recursively enumerable.*

There are a few modifications which must be made to the verification algorithm for $\mathcal{DTL}_1$. Because sensible relations may assign multiple temporal successors to each world, now rather than searching through paths of Kripke frames we must consider a type of tree of frames. However, such trees can be dealt with as before, and Kruskal's theorem can be used in much the same way.

References

D. Fernández; **Non-deterministic semantics for Dynamic Topological Logic**; submitted.

D. Fernández; **Dynamic Topological Completeness for** $\mathbb{R}^2$; Logic Journal of IGPL 2007; doi: 10.1093/jigpal/jzl036.

B. Konev, R. Kontchakov, F. Wolter and M. Zakharyaschev; **Dynamic topological logics over spaces with continuous functions**, forthcoming.

P. Kremer, G. Mints; **Dynamic Topological Logic**, Annals of Pure and Applied Logic, vol. 131, pp. 133-158, 2005.