Algebraic and Topological Methods in Non-Classical Logics III
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Unification in some polymodal logics

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- Unification, unification types: 1, ω, ∞, 0,
- Poly/multi– modal logics: fusions, Cartesian products,
- Unification in fusions,
- Unification in logics with interacting axioms: weakly transitive, conjugated op., Local Agreement,
- Applications: Tense, Epistemic, Multiagent logic of hypercubes, DALLA,
- Non-classical case - hoops with dual normal operations.

Unification

1930 J.Herbrand - automated deduction 1^{st} ord. logic,
1965 J.A.Robinson - resolution, unification algor., mgu,

E-unification

E - equational theory, t_1, t_2 - terms, in var. $\underline{x} = \{x_1, ...x_n\}$.
$t_1 =^? t_2$ - a unification problem, a solution is a substitution $\sigma : \underline{x} \rightarrow T$
$\vdash_E \ \sigma t_1 = \sigma t_2$ - is called a *unifier* for t_1 and t_2;
t_1, t_2 - *unifiable* if such σ exists,

σ is *more general then* τ, $\tau \preceq \sigma$, if
$\vdash_E \delta(\sigma(x)) = \tau(x)$, for some substitution δ; $\preceq$ a pre-order,
a *mgu* - a most general unifier, not unique.

$E = \emptyset$ -syntactic unification (Robinson): unifiable - mgu, $E \neq \emptyset$ - four types: 1 - unitary (“best"), ω - finitary, ∞ - infinitary, 0 - nullary, according to a number of $\preceq$- maximal unifiers of ,,the worst case" of t_1, t_2.

Ex. unitary - Boolean alg., discriminator var.(Burris 89); finitary - Heyting alg. (Ghilardi 99), Abelian groups (Lankford 79);
infinitary - groups, rings (Lawrence 89);
nullary - lattices, distributive lattices (Willard 91, 89).

Unification is basic to Resolution Theorem Provers (McCunne solution to the Robbins problem) and Term Rewriting Systems; categorial grammars, generalizing proofs.

Unification in logic

A logic L; a unifier for a formula $A(\underline{x})$ in L - a substitution $\sigma : \underline{x} \to F$ such that

$$\vdash_L \sigma(A);$$

σ - a unifier, A - unifiable; $\preceq$ related to $\vdash_L$.
Unification is *filtering*: τ_1, τ_2 unif for A there exists σ unif for A such that $\tau_1 \preceq \sigma$ and $\tau_2 \preceq \sigma$; unitary or nullary.

Ex. CL - classical propositional logic - unitary (best),
Modal: K4, S4, GL - finitary, Ghilardi (2000),
S5 -unitary; K4.2, S4.2 - unitary, Ghilardi, Sacchetti (2004).
Unification - not preserved under extensions/weakenings and expansions/reducts.
Applications: admissibility of rules (decidability).

Poly/multi-modal logic; m unary connectives: $\Box_1, \ldots, \Box_m$, "necessity", $\Diamond_1, \ldots, \Diamond_m$, "possibility", where $\Diamond_i A = \neg \Box_i \neg A, i \leq m$, classical connectives $\neg, \vee, \wedge, \rightarrow$.

A *normal poly/multimodal logic* is a set of formulas from F containing all classical tautologies, the axioms:

K_i : $\Box_i(A \rightarrow B) \rightarrow (\Box_i A \rightarrow \Box_i B)$, $i \leq m$

closed under: Modus Ponens, Necessitation: $A/\Box_i A$, $i \leq m$ and substitution.

m-frame $\mathfrak{F} = \langle W, R_1, \ldots, R_m \rangle$, $W \neq \emptyset$, $R_1, \ldots, R_m$, $R_i \subseteq W^2$

BAO's: $\mathfrak{B} = \langle B, \vee, \wedge, -, \{\mathfrak{m}_i\}_{i \leq m} \rangle$, $\mathfrak{m}_i(a \vee b) = \mathfrak{m}_i a \vee \mathfrak{m}_i b$ (additive) and $\mathfrak{m}_i 0 = 0$ (normal); $\mathfrak{m}_i$ corresponds to $\Diamond_i$.
Jónsson – Tarski.

Fusions of unimodal logics

$L_1, \dots, L_m$ with m distinct $\Box_1, \dots, \Box_m$, resp., the *fusion* $L_1 \otimes \cdots \otimes L_m$ is the smallest m-modal logic containing $L_1 \cup \cdots \cup L_m$; $\mathsf{K_m}, \mathsf{S5_m}$, etc. Kripke Cpl, dec., unif.interpol.

Lemma 1. *Let $L_i, i \leq m$ be unimodal logics from the set* $\{\mathsf{K}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K4.2}, \mathsf{K4.3}, \mathsf{KD4}, \mathsf{KD45}, \mathsf{S4}, \mathsf{S4.2}, \mathsf{S4.3}\}$. *Then the fusion $L = L_1 \otimes ... \otimes L_m$ have the following property, for any $A_i \in F, i \leq m$:*

$\vdash_L \Box_1 A_1 \vee ... \vee \Box_m A_m \;\; \Rightarrow \; \vdash_L A_i$, *for some $i \leq m$.*

Hence unification in L is not unitary.

Lemma 2. *Unification in* $\mathsf{S5_m} = \mathsf{S5} \otimes ... \otimes \mathsf{S5}$, $m \geq 2$, *is not unitary.*

Unitary unification is not preserved under fusions.

Ex. S4.2, K4.2, S5

Cartesian products $L_1 \times L_2$ of modal logics.

L_i determined by $\mathcal{C}_i$, $i \leq 2$, $L_1 \times L_2$ - det. by all $\mathcal{F}_1 \times \mathcal{F}_2$, $\mathcal{F}_1 \in \mathcal{C}_1$, $\mathcal{F}_2 \in \mathcal{C}_2$, where for $\mathcal{F}_1 = \langle W_1, R_1 \rangle$, $\mathcal{F}_2 = \langle W_2, R_2 \rangle$ the *Cartesian product* $\mathcal{F}_1 \times \mathcal{F}_2$ *of frames* $\mathcal{F}_1$ and $\mathcal{F}_2$ is the frame: $\mathcal{F}_1 \times \mathcal{F}_2 = \langle W_1 \times W_2, R_h, R_v \rangle$, where

$(u_1, v_1)\ R_h\ (u_2, v_2)$ iff $u_1\ R_1\ u_2$ and $v_1 = v_2$,
$(u_1, v_1)\ R_v\ (u_2, v_2)$ iff $v_1\ R_2\ v_2$ and $u_1 = u_2$.

$L_1 \otimes L_2 \subset L_1 \times L_2$, true in $L_1 \times L_2$ but not in $L_1 \otimes L_2$:
$comm$: $\Box_1 \Box_2 x \leftrightarrow \Box_2 \Box_1 x$ (commutativity),
$con(1,2)$: $\Diamond_1 \Box_2 x \rightarrow \Box_2 \Diamond_1 x$ (confluence, or Church-Rosser property), also $con(2,1) : \Diamond_2 \Box_1 x \rightarrow \Box_1 \Diamond_2 x$.
Generalize to product of n-copies.
$con(i,j)$: $\Diamond_i \Box_j x \rightarrow \Box_j \Diamond_i x$.

Weakly transitive m-modal logics

let $\Box A = \Box_1 A \wedge \cdots \wedge \Box_m A$, $\quad \Diamond A = \Diamond_1 A \vee \cdots \vee \Diamond_m A$.

$\Box^{(0)} A = A$, $\quad \Box^{(n+1)} A = \Box^{(n)} A \wedge \Box\Box^{(n)} A$;

$\Box^{(n)} A = A \wedge \Box A \wedge \Box\Box A \wedge \cdots \wedge \Box^n A$, $\quad \Diamond^{(n)} A$ dually,

$4^{(n)}$ $\qquad \Box^{(n)} x \to \Box^{(n+1)} x$ $\qquad$ or $\qquad \Diamond^{(n+1)} A \to \Diamond^{(n)} A$.

m-modal logic L is n-*transitive* if $\vdash_L \Box^{(n)} x \to \Box^{(n+1)} x$.
L is *weakly transitive* if it is n-transitive for some n.

L is weakly transitive iff the corresponding class of BAO's has Equationally Definable Principal Congruences, EDPC (Blok, Pigozzi).

n-confluence axiom or *n-Church-Rosser axiom*

$$2^{(n)}: \qquad \Diamond^{(n)}\Box^{(n)}x \to \Box^{(n)}\Diamond^{(n)}x.$$

Theorem 3. *If a m-modal logic L is n-transitive and n-confluent for some n, then unification in L is filtering, i.e. unitary or nullary (generalization of Ghilardi-Sachetti).*

Corollary 4. *Let L_i, $i \leq m$, be Kripke complete unimodal logics containing* S4 *and having filtering unification. Then $L_1 \times \cdots \times L_m$ has filtering unification too.*

Conjugate operations (J-T) Operations f and g on a Boolean algebra $\mathcal{B}$, g is a *conjugate of* f if, for $x, y \in B$:

$$x \wedge f(y) = 0 \quad \text{iff} \quad g(x) \wedge y = 0.$$

f is *conjugate* if there is g such that g is a conjugate of f.

$A = \langle A, \leq \rangle$, $B = \langle B, \leq' \rangle$, $f : A \to B$, $h : B \to A$, the pair (f, h) is called *residuated* if $f(a) \leq' b$ iff $a \leq h(b)$.
Given f, if g, h exist, then: $g(y) = h^d(y)$, $h(y) = g^d(y)$.

1-variable formulas M_1 and M_2 *are conjugate* in L if, $\vdash_L M_1 M_2^d x \to x$ and $\vdash_L M_2 M_1^d x \to x$;
where $M^d x = \neg M \neg x$.

Theorem 5. *If L is a n-transitive normal m-modal logic containing* $\mathsf{D_i} : \Diamond\top$, $i \leq m$ *and if either, (A) every* $\Diamond_i$ *is conjugate, or (B) every* $\Diamond_i$ *occurs in some self-conjugate formula* $\Diamond_{i_1}^{n_1} \dots \Diamond_{i_k}^{n_r} x \vee \dots \vee \Diamond_{j_1}^{m_1} \dots \Diamond_{j_p}^{m_s} x$, *then unification in* L *is unitary.*

$\mathsf{B^{(n)}}$: $\quad \Diamond^{(n)} \Box^{(n)} x \to x, \quad n$-symmetry.

Corollary 6. *If a m-modal logic L contains* $\mathsf{D_i}$ *and* $\mathsf{4^{(n)} B^{(n)}}$, *i.e. it is n-transitive and n-symmetric, for some n, in particular, if L contains* $\mathsf{S5}^m = \mathsf{S5} \times \dots \times \mathsf{S5}$, *then unification in L is unitary.*

APPLICATIONS - unitary unification:

- Tense logics: linear, weakly future (past) connected, tense logic of $\mathbf{Q}, \mathbf{R}, \mathbf{Z}$ (with infinite time),

- Logic w. Local Agreement: $\Box_i x \wedge \Box_j x \leftrightarrow \Box_i \Box_j x \wedge \Box_j \Box_i x$, DALLA = $\mathsf{S5}_{\mathsf{m}}$ + (LA) (Demri, Orłowska),

- Multiagent Logic of Hypercubes (Lomuscio, Ryan),

- Logic of inaccessible worlds of Humberstone,

Here mgu is explicit, effective.

Wolter et al., K, K4 + univ. modality - unif. undecidable

Hoops with dual normal operators

A **hoop** $\mathbf{A} = (A, \cdot, \to, 1)$, $(A, \cdot, 1)$ is a commutative *monoid* with the unit 1 satisfying: (1) $x \to x = 1$, (2) $x \to (y \to z) = (x \cdot y) \to z$, (3) $(x \to y) \cdot x = (y \to x) \cdot y$. $x^0 = 1, x^{k+1} = x^k \cdot x$; *k- potent* hoop : $\quad x^{k+1} = x^k$

A hoop with dual normal operators in the sense of Jónsson and Tarski , see Blok and Pigozzi, $(A, \cdot, \to, 1, \Box_i)_{i \leq m}$, a hoop $(A, \cdot, \to, 1)$ expanded with $\{\Box_i\}_{i \leq m}$ such that $\Box_i(1) = 1$; $\quad \Box_i(a \cdot b) = \Box_i(a) \cdot \Box_i(b), \quad i \leq m$.

Theorem 7. *Let L be a weakly transitive logic of k-potent hoops with dual normal operators $\Box_i$, $i \leq m$, containing the Beth axiom:* $\quad \Box_i a \to \Box_i b \leq \Box_i(\Box_i a \to b), \quad i \leq m$. *Then L has unitary unification.*

Summary:

- in fusions unification is bad, unification is not preserved under fusions,

- n-transitive and n-confluent logics have filtering unification (unitary or nullary),

- n-transitive and n-symmetric logics have unitary unification,

- filtering unification of unimodal logics over S4 is preserved under Cartesian products.

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