Cut-free sequent calculi for algebras with adjoint modalities

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Outline of talk

1. Algebras (here, distributive lattices) with adjoint modalities

2. Gentzen and Belnap style sequent calculi

3. Rules of our calculus, soundness

4. Cut elimination argument

5. Consequences (completeness, decidability)

6. Conclusion and future plans

Algebras with adjoint modalities

Classical algebraic modal logic

Boolean Algebra with De Morgan dual operators $(\Box, \Diamond)$

Non-classical algebraic modal logic

`weaken the base and the duality`

Heyting algebra with 'dual' operators

Complete lattice with *adjoint* operators

Distributive Lattices with Adjoint Modalities ($\mathtt{DLAM}$)

Let $\mathcal{A}$ be a set, with elements called *agents*. A $\mathtt{DLAM}$ **over** $\mathcal{A}$ is a bounded distributive lattice L with an $\mathcal{A}$-indexed family of maps $\{f_A\}_{A\in\mathcal{A}} : L \to L$, each with a right adjoint $\Box_A : L \to L$. The following are then satisfied (for finite joins and meets):

$$f_A(l) \leq l' \;\; \mathtt{iff} \;\; l \leq \Box_A l'$$

$$\begin{aligned} f_A(\bigvee_i l_i) &= \bigvee_i f_A(l_i) \\ \Box_A(\bigwedge_i l_i) &= \bigwedge_i \Box_A(l_i) \end{aligned}$$

In particular, $f_A\bot = \bot$ and $\Box_A\top = \top$; and all f_A and $\Box_A$ are order-preserving. If the lattice is complete, then the existence of the right adjoints follows routinely provided the maps f_A (exist and) preserve arbitrary joins.

Application

Reasoning about knowledge of agents $A \in \mathcal{A}$

$m, m' \in M$: logical propositions

$m \vee m', m \wedge m'$: logical disjunction, conjunction

$m \leq m'$: logical consequence

$f_A(m)$: **appearance of agent A about m**

All the propositions that appear to agent A to be true when m holds in reality.

$\Box_A m$: agent A knows that m

Gentzen-style sequent calculi

Sequent calculi for reasoning about lattices are old; there are two forms. The first [3,2] considers sequents of **“atoms”** $t \leq t'$ and is useful to decide quasi-equations in lattice theory, e.g.

$$\frac{\Gamma \Rightarrow s \leq t \quad \Gamma \Rightarrow s \leq t'}{\Gamma \Rightarrow s \leq t \wedge t'} R\wedge$$

The second kind (see [5]) considers sequents of **terms**, with the sequent arrow to represent the lattice order $\leq$, and, with

$$t = t' \text{ iff } t \Rightarrow t' \text{ and } t' \Rightarrow t,$$

decides the equational theory of **distributive** lattices, e.g. with

$$\frac{\Gamma \Rightarrow t \quad \Gamma \Rightarrow t'}{\Gamma \Rightarrow t \wedge t'} R\wedge$$

Ours is a generalisation of this second kind.

Gentzen-style sequent calculi, 2

[1] gives a Gentzen-style calculus (of the second kind) for modelling epistemic actions as resources. The calculus allows reasoning about systems, modelled by a quantale Q acting on a Q-module of epistemic propositions and facts.

The calculus includes a cut rule; this appears to be non-admissible; we seek to remedy this as a basis for automation of proof search.

Belnap-style sequent calculi

Our calculus also includes agents, whose presence forces a richer structure of sequents. Moortgat [4] attributes this to Belnap, and exploits it (in a linguistic context) for residuated lattices with modal operators. (Our work is thus a variation on Moortgat's work, distinguished by organisation of rules to allow Weakening to be admissible.)

Syntax of our calculus: Formulae, items, contexts

The set M of *formulae* m of our language is generated over a set $\mathcal{A}$ of *agents* A and a set At of *atoms* p by the following grammar:

$$m ::= \bot \mid \top \mid p \mid m \wedge m \mid m \vee m \mid \Box_A m \mid f_A(m)$$

Items $I, J, \ldots$ and *contexts* $\Gamma, \Delta, \ldots$ are generated by the following syntax:

$$\begin{array}{rcl} I & ::= & m \mid \Gamma^A \\ \Gamma & ::= & \{I, I, \cdots, I\} \end{array}$$

Thus, contexts are finite sets of items, whereas items are either formulae or contexts annotated with an agent.

Syntax of our calculus: Contexts with holes

The notion of *context-with-a-hole* $\Gamma[]$ is defined as follows:

$$\Gamma[] \;::=\; (\Gamma, []) \mid (\Gamma, \Gamma[]^A)$$

and so a context-with-a-hole is a context (i.e. a set of items) together with either a hole or an agent-annotated context-with-a-hole. **Note that a context-with-a-hole is not a context.**

The result $\Gamma[\Gamma']$ of applying $\Gamma[]$ to a context Γ', replacing the hole $[]$ by Γ', is a context, defined recursively as follows:

$$\begin{aligned} (\Gamma, [])\ \Gamma' &= \Gamma, \Gamma' \\ (\Gamma, \Gamma''[]^A)\ \Gamma' &= \Gamma, \Gamma''[\Gamma']^A \end{aligned}$$

where the commas in the right-hand sides indicate set union. This will often be applied in the particular case where Γ' is a single item (treated as a one-element set).

Rules of our calculus: Axioms (initial sequents)

$$\frac{}{\Gamma, m \vdash m}\,Id \qquad \frac{}{\Gamma[\bot] \vdash m}\,\bot L \qquad \frac{}{\Gamma \vdash \top}\,\top R$$

Note that the $\bot L$ rule allows the $\bot$ to appear anywhere (as an item) deep inside the context, whereas the Id rule requires the principal formula to appear at top level in the context. This captures the requirement that (in the lattice interpretation) $f_A(\bot) = \bot$ (this is not generally true for arbitrary elements).

Rules of our calculus: Rules for the lattice operations

$$\frac{\Gamma[m_1, m_2] \vdash m}{\Gamma[m_1 \wedge m_2] \vdash m} \wedge L \qquad \frac{\Gamma \vdash m_1 \quad \Gamma \vdash m_2}{\Gamma \vdash m_1 \wedge m_2} \wedge R$$

$$\frac{\Gamma[m_1] \vdash m \quad \Gamma[m_2] \vdash m}{\Gamma[m_1 \vee m_2] \vdash m} \vee L \qquad \frac{\Gamma \vdash m_i}{\Gamma \vdash m_1 \vee m_2} \vee R_i$$

Rules of our calculus: Rules for the modal operations

$$\frac{\Gamma[m^A] \vdash m'}{\Gamma[f_A(m)] \vdash m'}\, f_A L \qquad \frac{\Gamma \vdash m}{\Gamma'[\Gamma^A] \vdash f_A(m)}\, f_A R$$

$$\frac{\Gamma[m] \vdash m'}{\Gamma[(\Box_A m)^A] \vdash m'}\, \Box_A L \qquad \frac{\Gamma^A \vdash m}{\Gamma \vdash \Box_A\, m}\, \Box_A R$$

Rules of our calculus: Interaction between exponentiation and meet

Finally, we need a rule (named K, following [4])

$$\boxed{\frac{\Gamma[\Gamma'^{A}, \Gamma''^{A}] \vdash m}{\Gamma[(\Gamma', \Gamma'')^{A}] \vdash m}\, K}$$

Example Derivations

$$\dfrac{\dfrac{\overline{m \vdash m}}{m^A \vdash f_A(m)}\,f_A R}{m \vdash \Box_A f_A(m)}\,\Box_A R \qquad \dfrac{\dfrac{\overline{m \vdash m}}{(\Box_A m)^A \vdash m}\,\Box_A L}{f_A(\Box_A m) \vdash m}\,f_A L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{m \vdash m}}{m^A \vdash f_A(m)}\,f_A R}{m^A \vdash f_A(m) \vee f_A(m')}\,\vee R \quad \dfrac{\dfrac{\overline{m' \vdash m'}}{m'^A \vdash f_A(m')}\,f_A R}{m'^A \vdash f_A(m) \vee f_A(m')}\,\vee R}{(m \vee m')^A \vdash f_A(m) \vee f_A(m')}}{f_A(m \vee m') \vdash f_A(m) \vee f_A(m')}\,f_A L$$

(Converse of the last is easy.)

Example Derivations, 2

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{m, m'^A \vdash m}}{m^A, m'^A \vdash f_A(m)}\, f_A R \qquad \dfrac{\dfrac{}{m^A, m' \vdash m'}}{m^A, m'^A \vdash f_A(m')}\, f_A R}{m^A, m'^A \vdash f_A(m) \wedge f_A(m')}\, \wedge R}{\dfrac{(m \wedge m')^A \vdash f_A(m) \wedge f_A(m')}{f_A(m \wedge m') \vdash f_A(m) \wedge f_A(m')}\, f_A L}\, K
$$

Semantics of sequents; soundness:

Let L be a `DLAM` over $\mathcal{A}$. An **interpretation** of the set M of formulae (over the atoms At and agents $\mathcal{A}$ in L) is a map: $[\![-]\!]\colon At \to L$.

Meaning of a formula: by induction on the structure, e.g.

$$[\![m_1 \wedge m_2]\!] = [\![m_1]\!] \wedge [\![m_2]\!], \quad [\![f_A(m)]\!] = f_A([\![m]\!])$$

Meaning of an item

$$[\![m]\!] = \texttt{as above}, \quad [\![\Gamma^A]\!] = f_A([\![\Gamma]\!])$$

Meaning of a context

$$[\![\{I_1, \ldots, I_n\}]\!] = [\![I_1]\!] \wedge \ldots \wedge [\![I_n]\!]$$

Semantics of sequents; soundness:

Let L be a DLAM.

Definition. *Truth*.
A sequent $\Gamma \vdash m$ is *true in an interpretation* $[\![-]\!]$ in L iff $[\![\Gamma]\!] \leq [\![m]\!]$.
A sequent $\Gamma \vdash m$ is *true in L* iff true in every interpretation in L.

Definition. *Satisfiability*.
A sequent $\Gamma \vdash m$ is *satisfiable* (in L) iff there is an interpretation (in L) in which it is true.

Definition. *Validity*.
A sequent $\Gamma \vdash m$ is valid iff it is true in all interpretations.

Theorem. *Soundness*.
Any derivable sequent is valid.

Admissible Structural Rules

Lemma. The following *Weakening* rule is admissible

$$\frac{\Gamma'[\Gamma] \vdash m}{\Gamma'[\Gamma''[\Gamma]] \vdash m} Wk$$

Proof. Induction on the height of the derivation of the premiss. As an example, we suppose the last step is by $f_A R$, with premiss $\Gamma'[\Delta] \vdash m'$, where $\Delta^A = \Gamma$ and $f_A(m') = m$. By inductive hypothesis, we can derive $\Gamma'[\Gamma''[\Delta]] \vdash m'$, whence, by $f_A R$, we can derive $\Gamma'[\Gamma''[\Delta^A]] \vdash f_A(m')$, i.e. we can derive $\Gamma'[\Gamma''[\Gamma]] \vdash m$. $\Box$

Since (in the definition of 'context') we are using sets rather than multisets or lists, there is no need to show admissibility of Contraction or Exchange.

Cut elimination

Theorem. The *Cut* rule is admissible

$$\frac{\Gamma' \vdash m \quad \Gamma[m] \vdash m'}{\Gamma[\Gamma'] \vdash m'}\,Cut$$

Proof. Strong induction on the rank of the cut, where the *rank* is given by the pair (size of cut formula m, sum of heights of derivations of premisses).

Cut elimination: example step

Suppose the cut formula is $f_A(m)$ and is principal in both premisses:

$$\dfrac{\dfrac{\Gamma \vdash m}{\Gamma''[\Gamma^A] \vdash f_A(m)}\, f_A R \qquad \dfrac{\Gamma'[m^A] \vdash m'}{\Gamma'[f_A(m)] \vdash m'}\, f_A L}{\Gamma'[\Gamma''[\Gamma^A]] \vdash m'}\, Cut$$

transforms to

$$\dfrac{\dfrac{\Gamma \vdash m \qquad \Gamma'[m^A] \vdash m'}{\Gamma'[\Gamma^A] \vdash m'}\, Cut}{\Gamma'[\Gamma''[\Gamma^A]] \vdash m'}\, Wk$$

Consequences (transitivity)

Theorem. From $m \vdash m'$ and $m' \vdash m''$ follows $m \vdash m''$.

Consequences (completeness)

Theorem. Let m, m' be formulae (in a language over the set $\mathcal{A}$ of agents. The following are equivalent:

1. $m \vdash m'$;

2. $m \leq m'$ is true in all DLAMs over $\mathcal{A}$;

3. $m \leq m'$ is true in all complete DLAMs over $\mathcal{A}$.

Thus, the sequent calculus is (w.r.t. the given semantics) sound and complete.

Consequences (proof of completeness, 3 implies 1)

Routine Lindenbaum-Tarski construction and completion.

Consequences (decidability)

Straightforward, using the sub-formula property and a loop-checker.
Is there a simple variant that allows avoidance of a loop-checker?

Conclusion and future plans

1. changing the representation, e.g. multi-sets or lists rather than sets

2. proof-theoretic and implementation issues, e.g. invertibility lemmas, termination, loop-checking

3. enriching the base: Heyting algebras, quantales (for Linear Logic), systems [1]

4. comparing with other calculi for modal logics, e.g. deep inference systems (Guglielmi, Stouppa, Stewart, Brünnler, . . .)

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