A dimension theory for Heyting algebras

Luck Darnière*, Markus Junker**

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1 Dimension and codimension

Let $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge, \leq\}$ be the language of bounded lattices. Let $\mathcal{L}_{\text{HA}} = \mathcal{L}_{\text{lat}} \cup \{\rightarrow\}$. **Heyting algebras**, the algebraic models of intuitionistic propositional calculus, are the $\mathcal{L}_{\text{HA}}$-structures L whose $\mathcal{L}_{\text{lat}}$-reduct is a distributive bounded lattice and:

$$b \rightarrow a = \sup\{c \in L \,/\, c \wedge b \leq a\}$$

We denote by L^* the **dual** of L, that is the same set with the reverse order. Let $\mathcal{L}_{\text{HA}^*} = \mathcal{L}_{\text{lat}} \cup \{-\}$. **Dual Heyting algebras** are $\mathcal{L}_{\text{HA}^*}$-structures which are duals of Heyting algebras, that is their $\mathcal{L}_{\text{lat}}$-reduct is a distributive bounded lattice and:

$$a - b = \inf\{c \in L \,/\, a \leq b \vee c\}$$

For any given distributive bounded lattice the **spectrum** of L, denoted $\text{Spec}(L)$, is the set of all prime filters of L, and for every a in L:

$$P_L(a) = \{\mathbf{p} \in \text{Spec}(L) \,/\, a \in \mathbf{p}\}$$

The smallest and largest elements of L are denoted $\mathbf{0}$ and $\mathbf{1}$ respectively, so $P_L(\mathbf{0})$ is the empty set and $P_L(\mathbf{1})$ is the spectrum of L. As a ranges over L, $P_L(a)$ form a basis of closed sets for the so-called Zariski's topology on $\text{Spec}(L)$ which turns $\text{Spec}(L)$ into a spectral space, that is a topological space homeomorphic to the spectrum of a ring. The following definitions for an element a and a prime filter $\mathbf{p}$ of L therefore come directly from algebraic geometry:

- $\text{height}_L\, \mathbf{p}$ is the foundation rank of $\mathbf{p}$ in $\text{Spec}(L)$ for '$\subset$'.
- $\text{coheight}_L\, \mathbf{p}$ is the foundation rank of $\mathbf{p}$ in $\text{Spec}(L)$ for '$\supset$'.
- $\dim_L a = \sup\{\text{coheight}_L\, \mathbf{q} \,/\, \mathbf{q} \in P_L(a)\}$ if $a \neq \mathbf{0}$ ($\dim_L \mathbf{0} = -\infty$).

*Dép. de mathématiques, UFR sciences, 2 Bd Lavoisier, 49045 Angers Cedex 01 (France) luck.darniere_at_univ-angers.fr

**Mathematisches Institut, Universität Freiburg, Eckerstrasse 1, 79104 Freiburg (Germany) junker_at_mathematik.uni-freiburg.de

- $\operatorname{codim}_L a = \inf\{\operatorname{height}_L \mathbf{q} \,/\, \mathbf{q} \in P_L(a)\}$ if $a \neq \mathbf{0}$ ($\operatorname{codim}_L \mathbf{0} = +\infty$).

Note that height and coheight, hence dim and codim, when they exist are ordinal-valued. However we will consider only finite values (an infinite dimension will mean that the dimension does not exist in the ordinals).

Example 1.1 Let L be the lattice of all closed subsets of an algebraic variety X. This is a dual Heyting algebra with $A - B$ the topological closure of $A \setminus B$ (for $A, B \in L$). Then $\dim_L A$ (resp. $\operatorname{codim}_L A$) is the usual geometric dimension of A (resp. codimension of A in X).

It follows immediately from the definition that the (co)dimension has at least the following property in any distributive bounded lattice L:

$$\dim_L a \vee b = \max(\dim_L a, \dim_L b)$$

$$\operatorname{codim}_L a \vee b = \min(\operatorname{codim}_L a, \operatorname{codim}_L b)$$

However, from a model-theoretic point of view, these notions have a nicer behaviour in dual Heyting algebras. Indeed the above definitions are second order, but can easily be made first order in the class of dual Heyting algebras. Let us define:

$$b \ll a \iff \forall c \in L(a \leq b \vee c \Rightarrow a \leq c)$$

Proposition 1.2 *For any element a of a dual Heyting algebra L and any positive integer n:*

1. $\dim_L a \geq n$ *iff there exists* $a_0, \dots, a_n \in L$ *such that:*

$$\mathbf{0} \neq a_0 \ll \cdots \ll a_n \leq a$$

2. $\operatorname{codim}_L a \geq n$ *iff there exists* $a_0, \dots, a_n \in L$ *such that:*

$$a \leq a_n \ll \cdots \ll a_0$$

This proposition reflects the fact that the dimension was initially defined geometrically in terms of length of chains of *closed sets*, and that any dual Heyting algebras L is canonically represented (via the map $a \mapsto P_L(a)$) as an $\mathcal{L}_{\mathrm{HA}^*}$-substructure of the dual Heyting algebra of all *closed subsets* of Spec$L$.

2 The finitely presented case

To any element a of a distributive bounded lattice L one can attach four magnitudes: its dimension and co-dimension as an element of L but also as an element of L^*. In the case of a finitely presented Heyting algebra L, all the four notions are first order definable because L and L^* are both dual Heyting algebras (L is bi-Heyting). Let us first consider the case of H_n, the free Heyting algebra with n generators.

Proposition 2.1 *Let a be any element of $H_n \setminus \{\mathbf{1}\}$.*

1. $\operatorname{codim}_{H_n} a$ *and* $\dim_{H_n^*} a$ *are infinite.*
2. *If a is $\wedge$-irreducible then* $\dim_{H_n} a$ *is finite.*
3. $\operatorname{codim}_{H_n^*} a$ *is finite.*

So the dual codimension seems the most interesting one while dealing with H_n. However the dimension carries also some information on the set K_n of $\wedge$-irreducible elements of H_n. This set can be canonically identified with the underlying set of the Kripke model constructed in [Bel86]. Our stratification of K_n by the dimension corresponds exactly to the 'levels' of [Bel86].

This remark generalises to all the results of this section: most of them are folklore (see [Bel86], [EK90], [Bez06] among many others) but we expect our interpretation in terms of (co)dimension to shed new light on them.

For any positive integer p let pL denote the set of elements of L such that $\operatorname{codim}_{L^*} a \geq p$. This is a filter of L (uniformly definable in the class of Heyting algebras) hence the quotient L/pL is a Heyting algebra.

Proposition 2.2 *Let L be a finitely generated Heyting algebra and p a positive integer.*

1. *There are finitely many $\wedge$-irreducible elements a of L which do not belong to pL.*
2. *L/pL is finite, it is order-isomorphic (via the projection map) to the lower semi-lattice generated by the $\wedge$-irreducible elements of $L \setminus pL$.*
3. *pL is a principal filter whose generator is uniformly definable (in the class of Heyting algebras with a fixed number of generators).*

As p ranges over the positive integers, the quotients L/pL form a projective system of finite Heyting algebra. We call **profinite completion** of L the projective limit $\hat{L}$ of this system. This is a Heyting algebra which is $\vee$-complete and $\wedge$-complete, as a profinite lattice. The universal property of projective limits gives a canonical map from L to $\hat{L}$.

Proposition 2.3 *Let L be a finitely presented Heyting algebra.*

1. $\operatorname{codim}_{L^*} a$ *is finite for every $a \neq \mathbf{1}$, in other words $\bigcap_{p\geq 0} pL = \{\mathbf{1}\}$ hence the canonical map from L the $\hat{L}$ is an embedding.*
2. *$d(a,b) = 2^{-\operatorname{codim}_{L^*}(a\to b)\wedge(b\to a)}$ defines an ultra-metric distance on L, and $\hat{L}$ is the completion of L for this distance.*
3. *The $\wedge$-irreducible elements of L and $\hat{L}$ are the same. They are completely $\wedge$-irreducible.*

This proposition suggests an analogy between the structure (and the model-theory) of the profinite completion of a finitely presented Heyting algebra and the ring $\mathbb{Z}_p$ of p-adically integers, which is both the completion of $\mathbb{Z}$ for the p-adic distance and the projective limit of its finite quotients $\mathbb{Z}/p^l\mathbb{Z}$ (for all positive integers l). This analogy is sometimes misleading: while the ring $\mathbb{Z}_p$ has a model-complete and decidable theory, the free Heyting algebra with l generators is undecidable for every $l \geq 2$. Nevertheless it leads us to the following questions, for every finitely presented Heyting algebra L, or at least for H_n.

Question 2.4 *Is the inclusion of L into $\hat{L}$ an existentially closed embedding?*

This is essentially asking whether or not, when an equation $t(x) = \mathbf{1}$ (with t an $\mathcal{L}_{\mathrm{HA}}$-term) has a solution in $\hat{L}$, then it has a solution in L. By analogy with Hensel's lemma for p-adic integers, one can sharpen this question:

Question 2.5 *Is it possible to compute, for each $\mathcal{L}_{\mathrm{HA}}$-term t, a positive integer p such that if $t(x) = \mathbf{1}$ has a solution in L/pL (possibly satisfying some additional conditions) then it has a solution in L?*

Passing from existential formulas to arbitrary formulas we can even ask whether the inclusion of L into $\hat{L}$ is an elementary embedding. In this direction one can prove:

Proposition 2.6 *The set of generators of H_n is definable in H_n and in $\hat{H}_n$ by the same formula.*

3 The finite dimensional case

In order to use a geometric intuition it is more convenient in this section to dualize once for all, hence to work in dual Heyting algebras. A positive integer N is given, and a language $\mathcal{L}_{\mathrm{SC}_N} = \mathcal{L}_{\mathrm{HA}^*} \cup \{\mathrm{C}^i\}_{0\leq i\leq N}$ where the C^i's are unary function symbols. An N**-scaled lattice** (see [Dar06]) is an $\mathcal{L}_{\mathrm{SC}_N}$-structure whose $\mathcal{L}_{\mathrm{HA}^*}$-reduct is a dual Heyting algebra and for every a, b in L:

- $a = \mathrm{C}^0(a) \vee \mathrm{C}^1(a) \vee \cdots \vee \mathrm{C}^N(a)$.
- For every i, if $\mathrm{C}^i(a) \not\leq b$ then $\dim_L \mathrm{C}^i(a) - b = i$.
- For every $i \neq j$, $\dim_L \mathrm{C}^i(a) \wedge \mathrm{C}^j(a) < \min(i, j)$.

Example 3.1 Let L be the lattice of all closed subsets of an algebraic variety X of dimension N. For any $A \in L$ let $\mathrm{C}^i(A)$ be its i-th pure dimensional component, that is the (possibly empty) union of all the irreducible components of A of dimension i. This turns L into an N-scaled lattice. Notice that in this motivating example as well as in every N-scaled lattice, the $\mathcal{L}_{\mathrm{SC}_N}$-structure is definable in the lattice structure.

Proposition 3.2 *Every finitely generated substructure of an N-scaled lattice is finite.*

This result, which contrasts with the situation in (dual) Heyting algebras, holds essentially because in N-scaled lattices the dimension of the elements is uniformly bounded (by N).

Theorem 3.3 *The theory of N-scaled lattices admits a decidable model completion, axiomatized by the following two properties:*

- **Catenarity.** *For every $r < q < p$, if $\mathrm{C}^r(c) < \mathrm{C}^p(a)$ then there exists $b \in L$ such that $\mathrm{C}^r(c) < \mathrm{C}^q(b) < \mathrm{C}^p(a)$.*
- **Splitting.** *If $a - (b_1 \vee b_2) = a \neq \mathbf{0}$ then $a = a_1 \vee a_2$ for some a_1, a_2 in L such that $b_1 < a_1 = a - a_2$, $b_2 < a_2 = a - a_1$ and $b_1 \wedge b_2 = a_1 \wedge a_2$.*

It is worthwhile to notice that this last axiom only concerns the $\mathcal{L}_{\mathrm{HA}^*}$-structure of the N-scaled lattice. It asserts that in a very strong sense there is no 'connected' elements in L, hence in particular no atom.

4 The general case?

Using a theorem of A. Pitts in [Pit92], S. Ghilardi and M. Zawadowski proved in [GZ97] the existence of a model-completion for the theory of Heyting algebras. The proof of Pitts being purely proof-theoretic, little is known about this model-completion, in particular no enlightening axiomatisation is given. They only noticed that the strict order $\ll$ (see section 1) in the dual of any existentially closed Heyting algebra is dense. We can prove something more:

Proposition 4.1 *The dual of any existentially closed Heyting algebra satisfies the splitting property of theorem 3.3.*

This and other partial results then lead us to the following conjecture:

Conjecture 4.2 *The theory of Heyting algebras whose dual has a dense $\ll$ order and satisfy the splitting property, is the model-completion of the theory of Heyting algebras.*

References

[Bel86] Fabio Bellissima. Finitely generated free Heyting algebras. *J. Symbolic Logic*, 51(1):152–165, 1986.

[Bez06] Nick Bezhanishvili. *Lattices of intermediate and cylindric modal logics.* Phd thesis, Universiteit van Amserdam, January 2006.

[Dar06] Luck Darnière. Model-completion of scaled lattices. arXiv math 0606792, June 2006.

[EK90] Marcel Erné and Ralph Kopperman. Natural continuity space structures on dual Heyting algebras. *Fund. Math.*, 136(3):157–177, 1990.

[GZ97] Silvio Ghilardi and Marek Zawadowski. Model completions and r-Heyting categories. *Ann. Pure Appl. Logic*, 88(1):27–46, 1997.

[Pit92] Andrew M. Pitts. On an interpretation of second-order quantification in first-order intuitionistic propositional logic. *J. Symbolic Logic*, 57(1):33–52, 1992.