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ALGEBRAIC ASPECTS OF THE BACK AND FORTH METHOD

The purpose of this talk is to provide a general, abstract framework for some model-theoretic constructions, which is based on the order-oriented fixed-point theory.

The so called back and forth method is particularly useful in many branches of algebra and model theory (see e.g. Chang and Keisler [1973]). It dates back to the proof of the famous Cantor's theorem stating that any two countable linear dense orders without endpoints are isomorphic.

In a systematic way the back and forth method was studied by Fraïssé, Ehrenfeucht and others.

A plausible and general abstract formulation of the back and forth method in the context of the theory of reflexive points for ordered Kripke frames is presented.

Definition 1.

Let $(P, \leq)$ be a poset and let $\pi : P \to P$ be a mapping.

π is expansive if $x \leq \pi(x)$, for all $x \in P$.

π is monotone if $x \leq y$ implies $\pi(x) \leq \pi(y)$, for all $x, y \in P$.

π is conditionally expansive (or quasi-expansive) if

$x \leq \pi(x)$ implies $\pi(x) \leq \pi(\pi(x))$, for all $x \in P$.
□

Every expansive mapping π is quasi-expansive and every monotone mapping π is quasi-expansive.

The notion of a conditionally expansive mapping is thus a generalization of the above two types of mappings associated with posets.

The order-oriented fixed-point theory offers a variety of fixed-point theorems for monotone or expansive mappings.

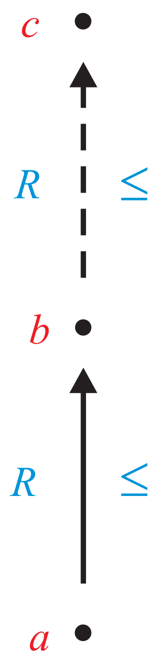
Theoretical computer science also provides fixed-point theorems (alias reflexive point theorems) for binary relations defined on posets (J. Cai and R. Paige [1992], J. Desharnais and B. Möller [2005]).

Definition 2.

A binary relation R defined on a poset $(P, \leq)$ is conditionally expansive (or quasi-expansive) if :

$$(\forall a, b \in P)[a \leq b \wedge a\,R\,b \rightarrow (\exists c \in P) b\,R\,c \wedge b \leq c]$$

(see the diagram below). □



If R is the graph of a mapping $\pi : P \to P$, then the relation R is quasi-expansive in the above sense

if and only if

the mapping π is quasi-expansive.

A poset $(P, \leq)$ is chain-σ-complete (or σ-inductive) if every chain in P of type $\leq \omega$ has a supremum.

Thus $(P, \leq)$ is σ-inductive

if and only if

every chain in P of type ω has a supremum and $(P, \leq)$ has zero $\mathbf{0}$ the supremum of the empty chain.

The case of chain-σ-complete posets

Definition 3. Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. A binary relation R on P is conditionally σ-continuous relative to P_0 if:

(1) R is conditionally expansive on P_0, i.e., for every pair $a, b \in P_0$ such that $a \leq b$ and aRb there exists $c \in P_0$ such that bRc and $b \leq c$.

(2) For every chain $C \subseteq P_0$ of type ω_0 and every monotone and expansive mapping $f : C \to P_0$, if $aRf(a)$ for all $a \in C$, then $\sup(C) R \sup(f[C])$.

($\sup(C)$ and $\sup(f[C])$ may not belong to P_0.) □

a^* is a reflexive point (or a fixed-point) of a relation R if $a^* R a^*$.

Theorem 4.

Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. Assume that a relation $R \subseteq P \times P$ is conditionally σ-continuous relative to P_0. If $\mathbf{0} \in P_0$ and the set $P_0 \cap R[\mathbf{0}]$ is non-empty, then R has a reflexive-point in P.

□

We discuss further modifications of the above definitions.

Definition 5.

Let $(P, \leq)$ be a poset and let R_1 and R_2 be two binary relations on P. Let P_0 be a subset of P.

R_1 and R_2 are adjoint on P_0 if :

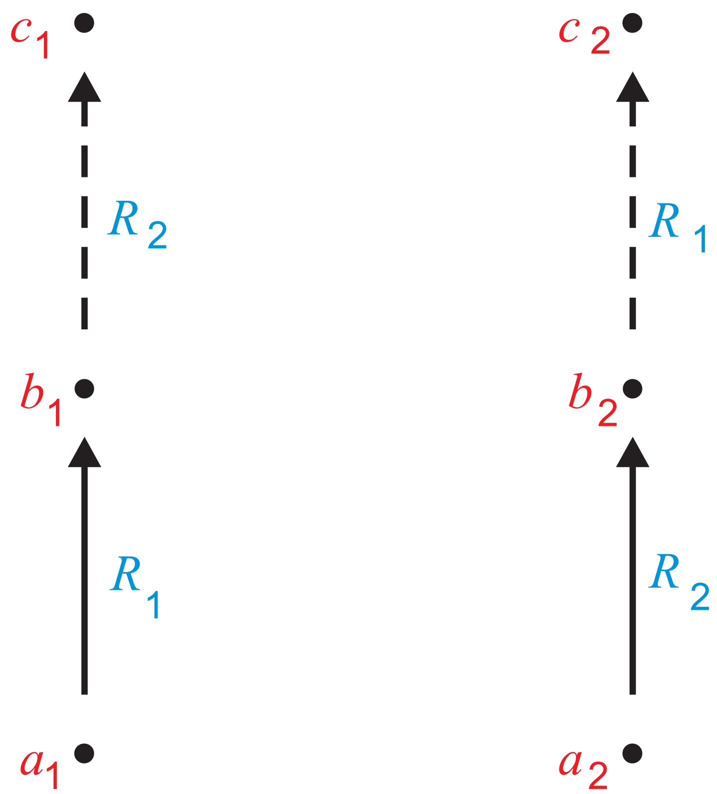
(A1) $(\forall a_1, b_1 \in P_0)[a_1 \leq b_1 \wedge a_1 R_1 b_1 \rightarrow (\exists c_1 \in P_0) b_1 R_2 c_1 \wedge b_1 \leq c_1]$,

(A2) $(\forall a_2, b_2 \in P_0)[a_2 \leq b_2 \wedge a_2 R_2 b_2 \rightarrow (\exists c_2 \in P_0) b_2 R_1 c_2 \wedge b_2 \leq c_2]$.

R_1 is then called the forth relation and R_2 is the back relation.

(see the diagram below)

R_1 **is the forth relation and** R_2 **is the back relation.**



Definition 6. Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. A pair (R_1, R_2) of binary relations on P is σ-continuously adjoint relative to P_0 if R_1 and R_2 are adjoint on P_0 and, furthermore, for every chain

$$C = \{a_n : n \in \omega\}$$

in P_0 of type $\leq \omega$ and for every monotone and expansive mapping $f : C \to P_0$ such that

$$a_{2n} R_1 f(a_{2n}) \text{ and } a_{2n+1} R_2 f(a_{2n+1}), \text{ for all } n \in \omega,$$

it is the case that:

$$\sup(C) R_1 \sup(f[C]) \text{ and } \sup(C) R_2 \sup(f[C]). \quad \square$$

(The supremums $\sup(C)$ and $\sup(f[C])$ need not belong to P_0. Furthermore, as f is expansive on C, we have that

$$\sup(C) \leq \sup(f[C]).)$$

An element $a^* \in P$ is a reflexive point (alias fixed-point) of the pair (R_1, R_2) if a^* is a reflexive point of both relations R_1 and R_2, i.e., it is the case that $a^* R_1 a^*$ and $a^* R_2 a^*$.

Theorem 7.

Let $(P, \leq)$ be a chain-σ-complete poset and let P_0 be a subset of P. Assume that a pair (R_1, R_2) of binary relations on P is σ-continuously adjoint relative to P_0.

If $\mathbf{0} \in P_0$ and the set $P_0 \cap R_1[\mathbf{0}]$ is non-empty, then the pair (R_1, R_2) has a reflexive point in P.

Proof.

We define a countable chain C (of type ω)

$$a_0 \leq a_1 \leq \ldots \leq a_n \leq a_{n+1} \leq \ldots$$

of elements of P_0. We put $a_0 := \mathbf{0}$. Let a_1 be an arbitrary element of $P_0 \cap R_1[\mathbf{0}]$.

As

$$a_0, a_1 \in P_0, a_0 \leq a_1 \text{ and } a_0 R_1 a_1,$$

there exists, by (A1), an element $a_2 \in P_0$ such that

$$a_1 \leq a_2 \text{ and } a_1 R_2 a_2.$$

Taking then the pair a_1, a_2 and applying (A2), we see that there exists an element $a_3 \in P_0$ such that

$$a_2 \leq a_3 \text{ and } a_2 R_1 a_3.$$

Then applying (A1) to the pair a_2, a_3, we find an element $a_4 \in P_0$ such that $a_3 \leq a_4$ and $a_3 R_2 a_4$. Continuing, we define an increasing chain

$$C = \{a_n : n \in \omega\} \text{ in } P_0$$

such that $a_0 R_1 a_1 R_2 a_2 R_1 a_3 R_2 a_4 \ldots a_{2n} R_1 a_{2n+1} R_2 a_{2n+2} \ldots$

The mapping $f : C \to C$ defined by $f(a_n) := a_{n+1}$, for all $n \in \omega$, is expansive and monotone. Furthermore

$$a_{2n} R_1 f(a_{2n}) \text{ and } a_{2n+1} R_2 f(a_{2n+1}), \text{ for all } n \in \omega.$$

As the pair (R_1, R_2) is σ-continuously adjoint relative to P_0, we have that

$$\sup(C) R_1 \sup(f[C]) \text{ and } \sup(C) R_2 \sup(f[C]).$$

Let $a^* := \sup(C)$. Evidently, $a^* = \sup(f[C])$. So $a^* R_1 a^*$ and $a^* R_2 a^*$. This concludes the proof of the theorem. $\Box$

As a simple (and somewhat trivial) application of Theorem 7 we give a proof of the following Cantors theorem:

Theorem 8. Every two countable linear and dense orders without end points are isomorphic.

Proof. Let $(X_1, \leq_1)$ and $(X_2, \leq_2)$ be two such orders. By a partial isomorphism from $(X_1, \leq_1)$ to $(X_2, \leq_2)$ we mean any partial function $f : X_1 \to X_2$ such that f is injective on its domain Dom(f) and, furthermore, for any elements $x, y \in$ Dom(f),

$$x \leq_1 y \text{ iff } f(x) \leq_1 f(y).$$

A partial isomorphism $f : X_1 \to X_2$ is finite if its domain Dom(f) is a finite set.

0 denotes the empty partial isomorphism.

A (partial) isomorphism f is total if $\mathrm{Dom}(f) = X_1$ and the co-domain $\mathrm{CDom}(f)$ is equal to X_2.

Denote by P the set of all partial isomorphisms from $(X_1, \leq_1)$ to $(X_2, \leq_2)$. P is partially ordered by the inclusion relation $\subseteq$ between partial isomorphisms. (Each partial isomorphism is a subset of the product $X_1 \times X_2$.)

The poset $(P, \subseteq)$ is chain-σ-complete because the union of any ω-chain of partial isomorphisms is a partial isomorphism. Furthermore, the empty isomorphism 0 is the least element in $(P, \subseteq)$.

We define two relations R_1 and R_2 on P. As X_1 and X_2 are countably infinite, we can write

$$X_1 = \{a_n : n \in \omega\} \text{ and } X_2 = \{b_n : n \in \omega\}.$$

Given partial isomorphisms f and g, we put :

$f R_1 g$ iff either f is a total isomorphism and $g = f$ or f is a finite isomorphism and $g = f \cup \{(a_m, b_n)\}$, where

(1) m is the smallest i such that $a_i \notin \mathrm{Dom}(f)$,

(2) n is the smallest j such that $b_j \notin \mathrm{CDom}(f)$ and $f \cup \{(a_m, b_j)\}$ is a partial isomorphism.

(Note that the choice of n depends on the definition of m.)

$f R_2 g$ iff either f is a total isomorphism and $g = f$ or f is a finite isomorphism and $g = f \cup \{(a_m, b_n)\}$, where

(3) n is the smallest j such that $b_j \notin \mathrm{CDom}(f)$,

(4) m is the smallest i such that $a_i \notin \mathrm{Dom}(f)$ and $f \cup \{(a_i, b_n)\}$ is a partial isomorphism.

(The choice of m depends on the definition of n.)

Let $P_0 \subseteq P$ be the set of all finite isomorphisms. Using the fact that the orders $(X_1, \leq_1)$ and $(X_2, \leq_2)$ are linear, dense, and without endponts, it is easy to verify that

> **(R_1, R_2) is a back and forth pair of relations relative to P_0**

and

> **the pair (R_1, R_2) is σ-continuously adjoint relative to P_0.**

Evidently, the set $P_0 \cap R_1[\mathbf{0}]$ is non-empty. Hence, applying Theorem 7, we obtain that the pair (R_1, R_2) has a fixed-point in $(P, \subseteq)$, say f^*. It follows from the definition of R_1 and R_2 that f^* is a total (bijective) isomorphism between $(X_1, \leq_1)$ and $(X_2, \leq_2)$. $\square$

The case of directed-complete posets

Definition 9. Let $(P, \leq)$ be a poset. Two relations R_1 and R_2 defined on P are adjoint if they are adjoint on the whole of P. The pair (R_1, R_2) is then called an adjoint pair of relations.

R_1 is called the forth relation and R_2 is the back relation. □

If $R_1 = R_2 = R$, then it is easy to see that the pair (R, R) is adjoint if and only if R is conditionally expansive.

Note. The definition of a back and forth pair of relations can be expressed in terms of one relation (together with its inverse !), but defined on a poset having a more complicated set-theoretic structure than $(P, \leq)$, viz. the direct power $P \times P$. However, for didactical and conceptual reasons, it is easier to work with posets having two binary relations defined on them. □

Before introducing the next definition we note the following simple fact.

Lemma 10. Let $(P, \leq)$ be a directed-complete poset and let D_1 and D_2 be non-empty directed subsets of $(P, \leq)$. Furthermore, let $f_1 : D_1 \to D_2$ and $f_2 : D_2 \to D_1$ be monotone mappings such that

(1) $x \leq f_1(x)$ for all $x \in D_1$,

(2) $y \leq f_2(y)$ for all $y \in D_2$.

Then

$$\sup(D_1) = \sup(D_2) = \sup(f_1[D_1]) = \sup(f_2[D_2]). \quad \square$$

The above lemma gives rise to the following definition.

Definition 11. Let $(P, \leq)$ be a directed-complete poset and let R_1 and R_2 be binary relations on P. The system $(P, \leq, R_1, R_2)$ is said to have the back and forth property if :

(1) (R_1, R_2) is an adjoint pair;

(2) For every pair (D_1, D_2) of non-empty $\leq$-directed subsets of P and for every pair (f_1, f_2) of monotone mappings $f_1 : D_1 \to D_2$, $f_2 : D_2 \to D_1$ such that

(a) $xR_1f_1(x)$ and $x \leq f_1(x)$ for all $x \in D_1$,

(b) $yR_2f_2(y)$ and $y \leq f_2(y)$ for all $y \in D_2$

it is the case that

$$R_1[a] \cap \uparrow a \neq \emptyset \text{ and } R_2[a] \cap \uparrow a \neq \emptyset,$$

where $a := \sup(D_1)$

$(= \sup(D_2) = \sup(f_1[D_1]) = \sup(f_2[D_2])$ by Lemma 1 1). $\square$

Since in (2), f_1 and f_2 are monotone, the above lemma applies to the above situation. Hence a satisfies the above equations. (2) states that there exist elements $b_1, b_2 \in P$ such that

$$a \leq b_1, aR_1b_1 \text{ and } a \leq b_2, aR_2b_2.$$

Theorem 12. Let $(P, \leq)$ be a directed-complete poset and let R_1 and R_2 be binary relations on P such that

$$(R_1[\mathbf{0}] \cap \uparrow \mathbf{0}) \cup (R_2[\mathbf{0}] \cap \uparrow \mathbf{0}) \neq \emptyset.$$

If the system $(P, \leq, R_1, R_2)$ has the back and forth property, then the pair (R_1, R_2) has a fixed-point a^* in P. $\square$

The basic idea of this and of many other proofs of fixed-point theorems is based on the same cumulative scheme: one constructs a specialized well-ordered chain of elements of a directed-complete poset and proves the supremum of the chain is the required fixed- point.

Back and forth mappings.

Definition 13. If $F : P \to P$ and $G : P \to P$ are mappings, then F and G are said to be adjoint if the graphs of F and G form an adjoint pair of relations. In this case F is the forth function and G is the back function. □

It is easy to see that (F, G) is an adjoint pair of mappings if and only if

(A1)$_{\text{funct}}$ $\quad (\forall a \in P)[a \leq F(a)$ implies $F(a) \leq G(F(a))]$

and

(A2)$_{\text{funct}}$ $\quad (\forall b \in P)[b \leq G(b)$ implies $G(b) \leq F(G(b))]$.

In particular, if F and G coincide, then (F, F) is an adjoint pair if and only if F is conditionally expansive.

If R_1 and R_2 are graphs of adjoint mappings

$$F_1 : P \to P,\ F_2 : P \to P,$$

then condition (2) of Definition 11 says that for any non-empty $\leq$-directed sets $D_1, D_2 \subseteq P$ such that

$$F_1\lceil D_1 : D_1 \to D_2,\ F_2\lceil D_2 : D_2 \to D_1,$$

if $F_1\lceil D_1$ and $F_2\lceil D_2$ are monotone,

$$x \leq F_1(x) \text{ for all } x \in D_1, \text{ and } y \leq F_2(y) \text{ for all } y \in D_2,$$

then for $a := \sup(D_1)$ ($= \sup(D_2) = \sup(F_1[D_1]) = \sup(F_2[D_2])$) it is the case that:

$$a \leq F_1(a) \text{ and } a \leq F_2(a).$$

Definition 14. If $(P, \leq)$ be a directed-complete poset and

$$F_1 : P \to P \text{ and } F_2 : P \to P$$

are mappings, then we say that the system $(P, \leq, F_1, F_2)$ has the back and forth property if and only if the system over $(P, \leq)$ formed from the graphs of the above mappings has the back and forth property. □

Corollary 15.

Let $(P, \leq)$ be a directed-complete poset. Let $F : P \to P$ and $G : P \to P$ be mappings such that the system $(P, \leq, F, G)$ has the back and forth property. Then π has a fixed-point, i.e., there exists a^* in P such that $F(a^*) = G(a^*) = a^*$.

□

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