ON THE ARCHIMEDEAN MULTIPLE-VALUED LOGIC ALGEBRAS

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Abstract. The Archimedean property is one of the most beautiful axioms of the classical arithmetic and some of the methods of constructing the field of real numbers are based on this property. It is well-known that every Archimedean ℓ-group is abelian and every pseudo-MV algebra is commutative. The aim of this paper is to introduce the Archimedean property for pseudo-MTL algebras and $\mathbf{FL}_w$-algebras. The main results consit of proving that there exist non-commutative Archimedean $\mathbf{FL}_w$-algebras. We also prove that any locally finite $\mathbf{FL}_w$-algebra is Archimedean.

1. Introduction

The Archimedean property was stated by Archimedes in the following form: "... the following lemma is assumed: that the excess by which the greater of (two) unequal areas exceeds the less can, by being added to itself, be made to exceed any given finite area" ([15], p.234). This is one of the most beautiful axioms of the classical arithmetic.
In the case of the field of real numbers, the Archimedean property can be formulated as follows: for any real numbers a and b such that $0 < a < b$, there exists $n \in \mathbb{N}$ such that $na > b$.
Some of the methods of constructing the field of real numbers are based on Archimedean properties (see [3]).
In the case of ℓ-groups, the Archimedean property was investigated by many authors and for the main results we refer the reader to [3]. For MV algebras this property was defined in different, but equivalent ways by Dvurečenskij ([12]) and Belluce ([2]), while in the case of pseudo-MV algebras it was defined by Dvurečenskij in [11]. In [4] and [5] there were defined Archimedean BL algebras and Archimedean pseudo-BL algebras and there were investigated some of their properties. In the same way we will define Archimedean pseudo-MTL algebras and Archimedean $\mathbf{FL}_w$-algebras. A well-known result states that every Archimedean ℓ-group is abelian (see for example [3]). Dvurečenskij proved that an Archimedean pseudo-MV algebra is commutative, i.e. an MV algebra ([10]). We will show that, generally, an Archimedean residuated lattice and an Archimedean pseudo-MTL algebra are not commutative. We also prove that any locally finite $\mathbf{FL}_w$-algebra is Archimedean.

2. On the Archimedean property in residuated structures

A *residuated lattice* is an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 1)$ of the type $(2, 2, 2, 2, 2, 0)$ satisfying the following conditions:

Date: 2007.07.15.
Key words and phrases. Archimedean residuated lattice, Archimedean pseudo-MTL algebra, Archimedean element, Hyperarchimedean residuated lattice.
2000 *Mathematics Subject Classification 06B05, 03G25, 28E15.*

(A_1) $(A, \wedge, \vee)$ is a lattice;
(A_2) $(A, \odot, 1)$ is a monoid;
(A_3) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$ (*pseudo-residuation*).

A residuated lattice with a constant 0 (which can denote any element) is called a *full Lambek algebra* or **FL**-algebra for short. The variety of all full Lambek algebras is denoted by **FL**.
An important subvariety of **FL** is that of the $\mathbf{FL}_w$-algebras, that is the **FL**-algebras A satisfying the condition $0 \leq x \leq 1$ for all $x \in A$.
An $\mathbf{FL}_w$-algebra A is *commutative* if the operation $\odot$ is commutative. It is easy to see that A is commutative iff $\to = \rightsquigarrow$. The variety of all commutative $\mathbf{FL}_w$-algebras is denoted by $\mathbf{FL}_{ew}$.
A *divisible* $\mathbf{FL}_w$*-algebra* or *Rℓ-monoid* is An $\mathbf{FL}_w$-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ satisfying the condition:
(A_4) $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (*pseudo-divisibility*).

A *pseudo-MTL algebra* or *weak pseudo-BL algebra* is an $\mathbf{FL}_w$-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ satisfying the condition:
(A_5) $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (*pseudo-prelinearity*).
Pseudo-MTL algebras were introduced in [13] and their properties were also investigated in [16].

A *pseudo-BL algebra* is an $\mathbf{FL}_w$-algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ satisfying the axioms A_4 and A_5. Pseudo-BL algebras were introduced in [14] and their properties were deeply investigated in [8] and [9].

Proposition 2.1. *(*[16], [13]*), In any pseudo-MTL algebra A the following rules of calculus hold:*
(1) $z \odot (x \wedge y) = (z \odot x) \wedge (z \odot y)$ *and* $(x \wedge y) \odot z = (x \odot z) \wedge y \odot z)$;
(2) $(x \wedge y)^{-} = x^{-} \vee y^{-}$ *and* $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$;
(3) $(x \vee y)^{-\sim} = x^{-\sim} \vee y^{-\sim}$ *and* $(x \vee y)^{\sim-} = x^{\sim-} \vee y^{\sim-}$;
(4) $z \odot (x_1 \wedge x_2 \wedge \cdots \wedge x_n) = (z \odot x_1) \wedge (z \odot x_2) \wedge \cdots \wedge (z \odot x_n)$ *and*
$(x_1 \wedge x_2 \wedge \cdots \wedge x_n) \odot z = (x_1 \odot z) \wedge (x_2 \odot z) \wedge \cdots \wedge (x_n \odot z)$;
(5) $x \vee y = [(x \to y) \rightsquigarrow y] \wedge [(y \to x) \rightsquigarrow x]$ *and*
$x \vee y = [(x \rightsquigarrow y) \to y] \wedge [(y \rightsquigarrow x) \to x]$;
(6) $(x \to y) \to z \leq ((y \to x) \to z) \to z$ *and*
$(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$;
(7) $(x \to y) \to z \leq ((y \to x) \to z) \rightsquigarrow z$ *and*
$(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \to z$;
(8) $(x \to y)^n \vee (y \to x)^n = 1$ *and* $(x \rightsquigarrow y)^n \vee (y \rightsquigarrow x)^n = 1$,
for all $x, y \in A$ *and* $n \in \mathbb{N}$, $n \geq 1$.

Definition 2.2. Let A be an $\mathbf{FL}_w$-algebra. A nonempty set F of A is called *filter* of L if the following conditions hold:
(F_1) If $x, y \in F$, then $x \odot y \in F$;
(F_2) If $x \in F$, $y \in A$, $x \leq y$ then $y \in F$.

Proposition 2.3. *(*[8]*) If F is a filter of A then:*
(F_3) $1 \in F$;

(F_4) *If* $x \in F$, $y \in A$ *then* $y \to x \in F$, $y \rightsquigarrow x \in F$;
(F_5) *If* $x, y \in F$, *then* $x \wedge y \in F$.

Proposition 2.4. *For a subset* F *of* A *the following are equivalent:*
(a) F *is a filter;*
(b) $1 \in F$ *and if* $x, x \to y \in F$, *then* $y \in F$;
(c) $1 \in F$ *and if* $x, x \rightsquigarrow y \in F$, *then* $y \in F$.

Definition 2.5. A proper filter of A is called *maximal* or *ultrafilter* if it is not strictly contained in any other proper filter of A.
Denote $Max(A) = \{F \mid F \, \text{is maximal filter of} \, A\}$.

Theorem 2.6. *([9]) If* H *is a proper normal filter of* A *then the following are equivalent:*
(1) $H \in Max(A)$;
(2) *For any* $x \in A$, $x \notin H$ *iff* $(x^n)^- \in H$ *for some* $n \in \mathbb{N}$;
(3) *For any* $x \in A$, $x \notin H$ *iff* $(x^n)^\sim \in H$ *for some* $n \in \mathbb{N}$.

Definition 2.7. Let A be an $\mathbf{FL}_w$-algebra. A filter H of A is called *normal* if for any $x, y \in A$, $\quad x \to y \in H$ iff $\mathrm{x \rightsquigarrow y \in H}$.
We denote by $\mathcal{F}_n(A)$ the set of all normal filters of A.
We also denote be $Max_n(A)$ the set of all maximal and normal filters of A.

Definition 2.8. An element a of the $\mathbf{FL}_w$-algebra A is called *co-atom* if $a \leq x < 1$ implies $x = a$.
For any $a \in A$, the set ${}^\perp a = \{x \in A \mid x \vee a = 1\}$ is called the *co-annihilator* of a.
If $X \subseteq A$, then the set ${}^\perp X = \{a \in A \mid x \vee a = 1 \text{ for any } x \in X\}$ is called the *co-annihilator* of X.

Remark 2.9. It is obvious that:
(1) ${}^\perp\{1\} = A$, ${}^\perp A = \{1\}$;
(2) For any $a \in A$ and $X \subseteq A$, ${}^\perp a$ and ${}^\perp X$ are filters of A.

Proposition 2.10. *If* A *is a pseudo-MTL algebra and* $a \in A$, *then* ${}^\perp a$ *is a normal filter of* A.

Proof. Let's suppose that $x \to y \in^\perp a$, that is $(x \to y) \vee a = 1$. We have:
$y \to x = (y \to x) \vee 1 = (y \to x) \vee (x \to y) \vee a = 1 \vee a = 1$, so $y \leq x$.
It follows that $y \rightsquigarrow x = 1$ and taking into consideration that

$$(x \rightsquigarrow y) \vee (y \rightsquigarrow x) \vee a = 1 \vee a = 1,$$

we get $(x \rightsquigarrow y) \vee a = 1$, that is $x \rightsquigarrow y \in^\perp a$. Similarly $x \rightsquigarrow y \in^\perp a$ implies $x \to y \in^\perp a$.
Thus, ${}^\perp a$ is a normal filter of A. □

Let A be an $\mathbf{FL}_w$-algebra. For any $n \in \mathbb{N}$, $x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \odot x = x \odot x^n$. The order of $x \in A$, denoted $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$.
If there is no such n, then $ord(x) = \infty$.
An $\mathbf{FL}_w$-algebra A is *locally finite* if for any $x \in A$, $x \neq 1$ implies $ord(x) < \infty$.
Let $B(A)$ be the set of all complemented elements of the lattice $L(A) = (A, \wedge, \vee, 0, 1)$.

Proposition 2.11. *([6]) If* A *be an* $\boldsymbol{FL}_w$*-algebra and* H *is a proper normal filter of* A, *then the following are equivalent:*
(1) $H \in Max_n(A)$;
(2) A/H *is locally finite.*

Lemma 2.12. *([7]) Let A be an $\boldsymbol{FL}_w$-algebra. Then, the following are equivalent:*
(a) $x \in B(A)$;
(b) $x \vee x^- = 1$ and $x \wedge x^- = 0$;
(c) $x \vee x^\sim = 1$ and $x \vee x^\sim = 0$.

Proposition 2.13. *([7]) Let A be an $\boldsymbol{FL}_w$-algebra, $x \in B(A)$ and $n \in \mathbb{N}$, $n \geq 1$. Then, the following are equivalent:*
(a) $x^n \in B(A)$;
(b) $x \vee (x^n)^- = 1$ and $x \vee (x^n)^\sim = 1$.

Proposition 2.14. *([7]) Let A be an $\boldsymbol{FL}_w$-algebra. If $x \in A$, $n \in \mathbb{N}$, $n \geq 1$ such that $x^n \in B(A)$ and $x^n \geq x^- \vee x^\sim$, then $x = 1$.*

Proposition 2.15. *In any $\boldsymbol{FL}_w$-algebra the following are equivalent:*
(a) $x^n \geq x^- \vee x^\sim$ for any $n \in \mathbb{N}$ implies $x = 1$;
(b) $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$ implies $x \vee y = 1$.

Proof. $(a) \Rightarrow (b)$ Take $x, y \in A$ such that $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$.
By the properties of $\mathbf{FL}_w$-algebras and by the hypothesis we have:

$$(x \vee y)^- = x^- \wedge y^- \leq y^- \leq y^- \vee y^\sim \leq x^n \leq (x \vee y)^n$$
$$(x \vee y)^\sim = x^\sim \wedge y^\sim \leq y^\sim \leq y^- \vee y^\sim \leq x^n \leq (x \vee y)^n,$$

hence $(x \vee y)^n \geq (x \vee y)^- \vee (x \vee y)^\sim$ for any $n \in \mathbb{N}$. Thus, by the hypothesis we get $x \vee y = 1$.
$(b) \Rightarrow (a)$ Consider $x \in A$ such that $x^n \geq x^- \vee x^\sim$ for any $n \in \mathbb{N}$.
Taking $y = x$ in (b) we get $x \vee x = 1$, hence $x = 1$. □

Definition 2.16. an $\mathbf{FL}_w$-algebra is called *Archimedean* if one of the equivalent conditions from the above proposition is satisfied.

Proposition 2.17. *If in an Archimedean $\boldsymbol{FL}_w$-algebra A, $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$, then $x \rightarrow y = x \rightsquigarrow y = y$.*

Proof. By the properties of an $\mathbf{FL}_w$-algebra, if $x, y \in A$ we have (see [6]):

$$(x \vee y) \leq [(x \rightarrow y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \rightarrow x]$$
$$(x \vee y) \leq [(x \rightsquigarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightsquigarrow x].$$

Since $x \vee y = 1$, it follows that:

$$[(x \rightarrow y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \rightarrow x] = 1$$
$$[(x \rightsquigarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightsquigarrow x] = 1,$$

hence $(x \rightarrow y) \rightsquigarrow y = 1$ and $(x \rightsquigarrow y) \rightarrow y = 1$.
From $(x \rightarrow y) \rightsquigarrow y = 1$ we have $x \rightarrow y \leq y$ and taking into consideration that $y \leq x \rightarrow y$, we obtain $x \rightarrow y = y$.
Similarly, $x \rightsquigarrow y = y$. □

Example 2.18. ([17]) Let's take $A = \{0, a_1, a_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < s < a$, $b < n < c$, $d < m < 1$. Consider the operations $\odot, \rightarrow, \rightsquigarrow$ given by the following tables:

$\odot$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	0	0	0	0	0	0	0	0	0	0	0
a_1	0	0	0	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	0	a_1	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2	a_2
s	0	a_1	a_2	s	s	s	s	s	s	s	s
a	0	a_1	a_2	s	s	s	s	s	s	s	a
b	0	a_1	a_2	s	s	s	s	s	s	s	b
n	0	a_1	a_2	s	s	s	s	s	s	s	n
c	0	a_1	a_2	s	s	s	s	s	s	s	c
d	0	a_1	a_2	s	s	s	s	s	s	s	d
m	0	a_1	a_2	s	s	s	s	s	s	s	m
1	0	a_1	a_2	s	a	b	n	c	d	m	1

$\rightarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_1	1	1	1	1	1	1	1	1	1	1
a_2	a_1	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

$\rightsquigarrow$	0	a_1	a_2	s	a	b	n	c	d	m	1
0	1	1	1	1	1	1	1	1	1	1	1
a_1	a_2	1	1	1	1	1	1	1	1	1	1
a_2	0	a_1	1	1	1	1	1	1	1	1	1
s	0	a_1	a_2	1	1	1	1	1	1	1	1
a	0	a_1	a_2	m	1	m	1	1	1	1	1
b	0	a_1	a_2	m	m	1	1	1	1	1	1
n	0	a_1	a_2	m	m	m	1	1	1	1	1
c	0	a_1	a_2	m	m	m	m	1	m	1	1
d	0	a_1	a_2	m	m	m	m	m	1	1	1
m	0	a_1	a_2	m	m	m	m	m	m	1	1
1	0	a_1	a_2	s	a	b	n	c	d	m	1

Then, $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is an $\mathbf{FL}_w$-algebra. Since $s^n = s \geq s^- \vee s^\sim = 0$ for all $n \in \mathbb{N}$, it follows that A is not an Archimedean $\mathbf{FL}_w$-algebra.

Definition 2.19. An element $x \in A$ is called *Archimedean* if there is $n \in \mathbb{N}$, $n \geq 1$ such that $x^n \in B(A)$.
An $\mathbf{FL}_w$-algebra A is called *hyperarchimedean* if all its elements are Archimedean.

Proposition 2.20. *If $a \in A$ is an Archimedean co-atom of the $\boldsymbol{FL}_w$-algebra A, then $^{\perp}a$ is a maximal filter of A.*

Proof. Since a is Archimedean, then there is $n \in \mathbb{N}$, $n \geq 1$ such that $a^n \in B(A)$. According to Proposition 2.13 we have $a \vee (a^n)^- = 1$, so $(a^n)^- \in^{\perp} a$.
Consider $x \notin^{\perp} a$, that is $x \vee a \neq 1$, so $a \leq x \vee a < 1$.
Since a is a co-atom, we get $a = x \vee a \geq x$. It follows that $a^n \geq x^n$, so $(x^n)^- \geq (a^n)^-$, so $(x^n)^- \in^{\perp} a$. Applying Theorem 2.6 we conclude that $^{\perp}a$ is a maximal filter of A. □

Example 2.21. Consider $\mathbf{FL}_w$-algebra A from Example 2.18.
Since $a_1^2 = 0 \in B(A)$, it follows that a_1 is Archimedean.
By contrast, $a_2^n = a_2 \notin B(A)$ for all $n \in \mathbb{N}$, $n \geq 1$, so a_2 is not Archimedean.
Thus, A is not a hyperarchimedean $\mathbf{FL}_w$-algebra.

Proposition 2.22. *Every locally finite $\boldsymbol{FL}_w$-algebra is hyperarchimedean.*

Proof. Let A be a locally finite $\mathbf{FL}_w$-algebra algebra and $x \in A$. Hence, there exists $n \in \mathbb{N}$ such that $x^n = 0 \in B(A)$. It follows that any element x of A is Archimedean, so A is hyperarchimedean. □

Corollary 2.23. *Every hyperarchimedean $\boldsymbol{FL}_w$-algebra is Archimedean.*

Proof. Let A be a hyperarchimedean $\mathbf{FL}_w$-algebra and $x \in A$ such that $x^n \geq x^- \vee x^\sim$ for any $n \in \mathbb{N}$. Since A is hyperarchimedean, there exists $m \in \mathbb{N}$, $m \geq 1$ such that $x^m \in B(A)$. According to Proposition 2.14 it follows that $x = 1$, so A is Archimedean. □

Corollary 2.24. *Every locally finite $\boldsymbol{FL}_w$-algebra is Archimedean.*

Proof. It follows from Proposition 2.22 and Corollary 2.23. □

Proposition 2.25. *For any commutative $\boldsymbol{FL}_w$-algebra A the following properties are equivalent:*
(a) A is Archimedean;
(b) $x^n \geq y^-$ for any $n \in \mathbb{N}$ implies $x \to y = y$ and $y \to x = x$.

Proof. $(a) \Rightarrow (b)$. Let $x, y \in A$ such that $x^n \geq y^-$ for any $n \in \mathbb{N}$.
By the hypothesis we have:

$$(x \vee y)^- = x^- \wedge y^- \leq y^- \leq x^n \leq (x \vee y)^n,$$

hence, by the hypothesis, we get $x \vee y = 1$.
Since in any $\mathbf{FL}_w$-algebra we have $x \vee y \leq [(x \to y) \to y] \wedge [(y \to x) \to x]$ (see [6]) and taking in consideration that $x \vee y = 1$, it follows that $[(x \to y) \to y] \wedge [(y \to x) \to x] = 1$, hence $(x \to y) \to y = 1$ and $(y \to x) \to x = 1$.
From $(x \to y) \to y = 1$, we have $x \to y \leq y$ and considering that $y \leq x \to y$ we obtain $x \to y = y$. Similarly, $y \to x = x$.
$(b) \Rightarrow (a)$: Consider $x \in A$ such that $x^n \geq x^-$, for any $n \in \mathbb{N}$.
By the hypothesis we obtain $x \to x = x$, hence $x = 1$.
Thus, A is Archimedean. □

Proposition 2.26. *For any pseudo-MTL algebra A the following are equivalent:*
(a) A is Archimedean;
(b) $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$ implies $x \to y = x \rightsquigarrow y = y$.

Proof. $(a) \Rightarrow (b)$ Assume that $x^n \geq y^- \vee y^\sim$ for any $n \in \mathbb{N}$.
Since A is Archimedean, it follows that $x \vee y = 1$. For $x, y \in A$ we have (see [16]):

$$(x \vee y) = [(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x]$$

$$(x \vee y) = [(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x].$$

Since $x \vee y = 1$, it follows that:

$$[(x \to y) \rightsquigarrow y] \wedge [(y \rightsquigarrow x) \to x] = 1,$$

$$[(x \rightsquigarrow y) \to y] \wedge [(y \to x) \rightsquigarrow x] = 1,$$

hence $(x \to y) \rightsquigarrow y = 1$ and $(x \rightsquigarrow y) \to y = 1$.
From $(x \to y) \rightsquigarrow y = 1$ we have $x \to y \leq y$ and taking into consideration that $y \leq x \to y$, we obtain $x \to y = y$. Similarly, $x \rightsquigarrow y = y$.
$(b) \Rightarrow (a)$ Consider $x \in A$ such that $x^n \geq x^- \vee x^\sim$ for any $n \in \mathbb{N}$.
Taking $y = x$ in (b) we get $x \to x = x$, hence $x = 1$. Thus, A is Archimedean. $\square$

Example 2.27. Let's consider $A = \{0, a, b, c, 1\}$ where $0 < a < b < c < 1$ and the operations $\odot, \to, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	a	a	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	b	c	c	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

Then, $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL algebra and we have:

$$ord(0) = 1, \ ord(a) = 2, \ ord(b) = 2, \ ord(c) = 3.$$

Thus, A is a locally finite pseudo-MTL algebra, so it is Archimedean and hyperarchimedean.

Example 2.28. Let's consider $A = \{0, a, b, c, 1\}$ with $0 < a < b, c < 1$, but b, c are incomparable. Consider also the operations $\odot, \to, \rightsquigarrow$ given by the following tables:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	a	0	a
b	0	0	b	0	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	0	c	1	c	1
c	b	b	b	1	1
1	0	a	b	c	1

$\rightsquigarrow$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	c	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

Then, $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$ is a pseudo-MTL algebra.
Since $a^2 = 0 \in B(A)$, it follows that a is Archimedean. By contrast, $b^n = b \notin B(A)$ for all $n \in \mathbb{N}$, $n \geq 1$, so b is not Archimedean. Thus, A is not a hyperarchimedean pseudo-MTL algebra.

We give an example of Archimedean pseudo-MTL algebra which is not a chain and is not a hyperarchimedean pseudo-MTL algebra.

Example 2.29. (Archimedean, but not hyperarchimedean pseudo-MTL algebra).
Let's consider the pseudo-MTL algebra A from Example 2.28 . We have:

$$0^n = 0 \ngeq 0^- \vee 0^\sim = 1 \vee 1 = 1, \ n \geq 1$$

$$a^n = 0 \ngeq a^- \vee a^\sim = b \vee c = 1,\ n \geq 2$$

$$b^n = b \ngeq b^- \vee b^\sim = 0 \vee c = c,\ n \geq 1$$

$$c^n = c \ngeq c^- \vee c^\sim = b \vee 0 = b,\ n \geq 1$$

$$1^n = 1 \geq 1^- \vee 1^\sim = 0 \vee 0 = 0,\ n \geq 1.$$

We conclude that, if $x^n \geq x^- \vee x^\sim$ for all $n \in \mathbb{N}$, $n \geq 1$, then $x = 1$.
Hence, A is an Archimedean pseudo-MTL algebra.
In Example 2.28 we showed that A is not a hyperarchimedean pseudo-MTL algebra.

Remark 2.30. By Examples 2.27 and 2.29 we proved that, generally, an Archimedean pseudo-MTL algebra is not commutative. Obviously, this result is also valid in the case of $\mathbf{FL}_w$-algebras.
This result seems to be important, taking in consideration the known results in the case of other structures: any Archimedean ℓ-group is abelian ([3]) and any Archimedean pseudo MV-algebra is an MV-algebra, so it is commutative ([10])).

Open problem 2.31. Find an Archimedean pseudo-BL algebra which is not commutative.

Theorem 2.32. *If A is a pseudo-MTL algebra and $a \in A$ an Archimedean co-atom of A, then $A/^{\perp}a$ is Archimedean and hyperarchimedean.*

Proof. According to Propositions 2.10 and 2.20, $^{\perp}a \in Max_n(A)$. Applying Proposition 2.11 if follows that $A/^{\perp}a$ is a locally finite pseudo-MTL algebra. Finally, by Proposition 2.22 and Corollary 2.24 we conclude that $A/^{\perp}a$ is Archimedean and hyperarchimedean. □

Let's consider the case of a $\mathbf{FL}_{ew}$-algebra.

Proposition 2.33. *For any $\mathbf{FL}_{ew}$-algebra A the following properties are equivalent:*
(a) A is Archimedean;
(b) $x^n \geq y^-$ for any $n \in \mathbb{N}$ implies $x \to y = y$ and $y \to x = x$.

Proof. $(a) \Rightarrow (b)$. Let $x, y \in A$ such that $x^n \geq y^-$ for any $n \in \mathbb{N}$.
By $(rl - c_{10})$ and by the hypothesis we have:

$$(x \vee y)^- = x^- \wedge y^- \leq y^- \leq x^n \leq (x \vee y)^n,$$

hence, by the hypothesis, we get $x \vee y = 1$.
By $(rl - c_{18})$ we have $x \vee y \leq [(x \to y) \to y] \wedge [(y \to x) \to x]$.
Since $x \vee y = 1$, it follows that $[(x \to y) \to y] \wedge [(y \to x) \to x] = 1$,
hence $(x \to y) \to y = 1$ and $(y \to x) \to x = 1$.
From $(x \to y) \to y = 1$, we have $x \to y \leq y$ and considering that $y \leq x \to y$ we obtain $x \to y = y$. Similarly, $y \to x = x$.
$(b) \Rightarrow (a)$: Consider $x \in A$ such that $x^n \geq x^-$, for any $n \in \mathbb{N}$.
By the hypothesis we obtain $x \to x = x$, hence $x = 1$.
Thus, A is Archimedean. □

We will give bellow one example of Archimedean $\mathbf{FL}_{ew}$-algebra.

Example 2.34. ([18]) Let's consider $A = \{0, a, b, c, d, 1\}$ with $0 < a < b, c < d < 1$ and b, c incomparable. Define the operations $\odot, \rightarrow$ by the following tables:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	0	d
1	0	a	b	c	1	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	1	1	1	1
b	d	d	1	d	1	1
c	d	d	d	1	1	1
d	d	d	d	d	1	1
1	0	a	b	c	d	1

.

Then, $\mathcal{A} = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a proper $\mathbf{FL}_{ew}$-algebra.
Indeed, since $(b \rightarrow c) \odot b = 0 \neq a = b \wedge c$, it follows that the condition (B_4) is not satisfied, so A is neither a BL algebra nor a divisible residuated lattice.
Moreover, $(b \rightarrow c) \vee (c \rightarrow b) = d \neq 1$, so A is not an MTL algebra.
(In fact, A is a $\mathbf{FL}_{ew}$-algebra with *weak nilpotent minimum*(WNM) and $(C_\vee)$ conditions:

(WNM): $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$

$(C_\vee)$: $x \vee y = [(x \rightarrow y) \rightarrow y] \wedge [(y \rightarrow x) \rightarrow x]$).

We have:

$$0^n = 0 \ngeq 0^- = 1,\ n \geq 1$$
$$a^n = 0 \ngeq a^- = d,\ n \geq 2$$
$$b^n = 0 \ngeq b^- = d,\ n \geq 2$$
$$c^n = 0 \ngeq c^- = d,\ n \geq 2$$
$$d^n = 0 \ngeq d^- = d,\ n \geq 2$$
$$1^n = 1 \geq 1^- = 0,\ n \geq 1.$$

We conclude that, if $x^n \geq x^-$ for all $n \in \mathbb{N}$, $n \geq 1$, then $x = 1$. Hence, A is an Archimedean $\mathbf{FL}_{ew}$-algebra.

We will give bellow an example of not Archimedean $\mathbf{FL}_{ew}$-algebra.

Example 2.35. Consider the $\mathbf{FL}_{ew}$-algebra $A = (A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ defined on the unit interval $A = [0, 1]$ with the operations (see [21]):

$$x \odot y = \begin{cases} 0, \text{ if } x + y \leq \frac{1}{2} \\ x \wedge y, \text{ otherwise} \end{cases}$$

$$x \rightarrow y = \begin{cases} 1, \text{ if } x \leq y \\ max\{\frac{1}{2} - x, y\}, \text{ otherwise} \end{cases}$$

Since $(\frac{1}{3})^n = \frac{1}{3} > \frac{1}{6} = (\frac{1}{3})^-$ for al $n \in \mathbb{N}$ and $\frac{1}{3} \neq 1$, it follows that A is not Archimedean.

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