Quantifiers on m-semilattices

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Contents

(The aim is not to algebraize some strange first-order logic, but to find out how weak an algebraic structure may be to admit a reasonable theory of an existential quantifier.)

BOOLEAN QUANTIFIERS:
Basic facts

• A: Definition

A *quantifier* on a Boolean algebra A is an operation ∇ such that

$\nabla 0 = 0$,
$a \leq \nabla a$,
$\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$.

• B: Algebraic properties

▪ Every quantifier is a completely additive closure operator.

▪ An operation ∇ on A is a quantifier iff it is an additive closure operator whose range is a subalgebra of A.

▪ There is a bijection (*algebraic duality*) between quantifiers on A and relatively complete subalgebras A_0 of A (where $\min\{x \in A_0\colon a \leq x\}$ exists for every $a \in A$):

$\nabla^* := \text{ran}\ \nabla \quad (A_0)^*(a) := \min\{x \in A_0\colon a \leq x\}$.

• Quantifiers and equivalence relations I

▪ There is a bijection (*Kripke duality*) between the set of binary relations on $Y \times X$ and the set of completely additive mappings of $\mathcal{P}(X) \to \mathcal{P}(Y)$: if

R is a binary relation on $Y \to X$,

h is a completely additive selfmap of

$$\mathcal{P}(X) \to \mathcal{P}(Y),$$

then

$$v\, h^*\, u :\equiv v \in h(\{u\}),$$
$$v \in R^*(M) :\equiv v\, R\, u \text{ for some } u \in M.$$

▪ h is a quantifier iff h^* is an equivalence.

• Quantifiers and equivalence relations II

▪ There is a bijection (*Halmos duality*) between $(\vee, 0)$-preserving mappings $h\colon A \to B$ and certain topologically well-behaved relations R on $S(B) \times S(A)$ (Stone spaces of B an of A).

▪ h is a quantifier iff the corresponding relation is an equivalence.

- Functional representation:

▪ A *monadic algebra* is a pair (A, ∇), where A is a Boolean algebra and ∇ is quantifier on A.

▪ B – a Boolean algebra, X – a set.
A – a Boolean subalgebra of B_X such that
for every $p \in A$ there is $q \in A$ satisfying
$q(x) = \bigvee(p(y) \colon y \in X)$.
The *standard quantifier* on A:
$\exists p := q$;
$(A, \exists)$ – a *functional monadic algebra*.

▪ Every monadic Boolean algebra is embeddable into a monadic functional algebra.

QUANTIFIERS ON DISTRIBUTIVE LATTICES

- A: Definition

A *quantifier* on a distributive lattice A is an operation ∇ such that

$\nabla(a \vee b) = \nabla a \vee \nabla b$,

$\nabla 0 = 0$,

$a \leq \nabla a$,

$\nabla(a \wedge \nabla b) = \nabla a \wedge \nabla b$.

- B: Algebraic properties

▪ A quantifier is an additive closure operator, but need not be completely additive.

▪ The range of a quantifier is a sublattice of A, which need not be relatively complete: *the algebraic duality fails*.

▪ It still holds for Heyting algebras.

• C: Quantifiers and equivalences I

▪ Kripke duality holds.

• D: Quantifiers and equivalences II

Halmos duality (modified) holds:

▪ There is a bijection between quantifiers on a distributive lattice and certain topologically well-behaved equivalences on its Priestley space.

• E: Functional representation

▪ No representation theorem for monadic, or Q-distributive, lattices is known to the author.

▪ There is a representation theorem for monadic Heyting algebras with both existential and universal quantifiers.

(Halmos techniques does not work in full.)

MULTIPLICATIVE SEMILATTICES

- *M-semilattice*:
  algebra $(A, \vee, \cdot, 0)$, where
  ▪ $(A, \vee, 0)$ is a join semilattice with the least element,
  ▪ $(A, \cdot, 0)$ is a groupoid with absorbing zero,
  ▪ multiplication $\cdot$ is left and right distributive over $\vee$.

- An m-semilattice A is
  ▪ *commutative* or *associative* if multiplication is commutative, resp., associative,
  ▪ *unital* if it has two-side multiplicative unit 1,
  ▪ *integral* if it is unital and 1 is the greatest element in A,
  ▪ an *idempotent semiring* or *dioid* if it is unital and associative,
  ▪ *infinitely distributive* if multiplication is left and right distributive over all existing joins,
  ▪ *complete* if arbitrary joins exist,

▪ *(left) residuated* if, for all y and z, the subset $\{x\colon xy \leq z\}$ has the maximal element $y \to z$,
▪ *biresiduated* if it is left and right residuated.

• A complete and infinitely distributive semilattice monoid is known as a quantale.

A complete m-semilattice is left residuated iff it is infinitely right distributive.

Example 1. A bounded distributive lattice is a commutative integral dioid.

Example 2. In an MV-algebra $(A, \oplus, \odot, \neg, 0, 1)$, the reduct $(A, \vee, \odot, 0, 1)$ is a commutative integral dioid. It is even residuated with

$x \to y := \neg(x \odot \neg y)$

▪ MV-algebra is a particular residuated dioid: $(A, \vee, \odot, \to, 0, 1)$.

QUANTIFIERS ON M-SEMILATTICES

A − an integral m-semilattice.

• A: Definition

A *(left) quantifier* on A is an operation ∇ on A such that

$$\nabla(x \vee y) = \nabla x \vee \nabla y,$$
$$\nabla 0 = 0,$$
$$x \leq \nabla x,$$
$$\nabla(x \nabla y) = \nabla x \nabla y.$$

∇ is *completely additive* if it preserves all exixting joins.

B: Algebraic properties

▪ A quantifier is an additive closure operator, but need not be completely additive.

▪ The range of a quantifier is a subalgebra of A, which need not be relatively complete: *the algebraic duality fails.*

▪ Theorem 1. In a residuated m-semilattice A, an operation ∇ is a quantifier iff it is an additive closure operation whose range is a subalgebra of A.
If it is the case, then ∇ is completely additive iff the subalgebra is complete, i.e., closed under all existing joins.

▪ So, algebraic duality holds true for residuated m-semilattices.

• C: Equivalence and quantifiers I
Kripke duality holds in a modified form for function algebras instead of powersets (see the next slide).

• D: Quantifiers and equivalences II
A programm for future (fuzzification) (see slide 14).

Functional m-semilattices

B – an integral m-semilattice,
U – a non-empty set.

Then B^U is an integral m-semilattice with pointwise defined operations.
If B is (bi)residuated, then so is B^U.
Functions in B^U admit left and right multiplication by elements of B.

Kripke duality in algebras of functions

B – complete.
U, V – sets.

▪ A *hemimorphism* from B^U to B^V is a $(\vee, 0)$-preserving map $h : B^U \to B^V$.
A hemimorphism $h : B^U \to B^V$ is *(right) linear* if $h(p) \cdot b = h(p \cdot b)$.
▪ A completely additive quantifier on B^U is linear iff its range contains all constant functions.

- A *B-relation* on $V \times U$ is a mapping
  $\mu\colon V \times U \to B$
identified with the family of functions $\mu_v \in B^U$:
  $\mu_v(u) := \mu(v, u)$.
- The *B-identity* on U is a B-relation η defined by
  $\eta_u(u') := 1$ if $u' = u$,
  $\eta_u(u') := 0$ otherwise.
- The *full* B-relation $\omega\colon V \times U$ is defined by
  $\omega_v(u) := 1$.

- The *dual* of h – the relation h^* on $V \to U$ such that
  $h^*(v, u) = h(\eta_u)(v)$.
- The *dual* of μ – the mapping $\mu^*\colon B^U \to B^V$ such that
  $\mu^*(p)(v) = \bigvee(\mu_v(u)p(u)\colon u \in U)$.
- Theorem 2. This duality is a bijection between linear completely additive hemimorphisms $B^U \to B^V$ and B-relations on $V \times U$.

- A B-relation λ on U is an *B-equivalence* if it satisfies the conditions
  (E1) $\lambda(u,u)=1$,
  (E2) $\lambda(u,u')\lambda(u',u'')=\lambda(u,u')\lambda(u,u'')$.
- A B-equivalence is symmetric and transitive:
  (E3) $\lambda(u,u')=\lambda(u',u)$,
  (E4) $\lambda(u,u')\lambda(u',u'')\leq\lambda(u,u'')$.
- λ is *strong*, if instead of (E2), a stronger condition
  $\lambda(u,u')b_1\cdot\lambda(u',u'')b_2=\lambda(u,u')b_1\cdot\lambda(u,u'')b_2$
  is fulfilled for all $b_1,b_2\in B$.

- B-identity η is a strong equivalence.
- The full relation ω is a strong equivalence.
- If B is commutative and associative, then every equivalence is strong.

- Theorem 3. If B is infinitely distributive, then the dual of a competely additive linear quantifier is a strong B-equivalence, and conversely.
- Corollary. If B is a quantale, then the dual of a linear complete quantifier is a B-equivalence, and conversely.

Towards a fuzzy Halmos-style duality

(an outline)

Generally, an m-semilattice cannot be represented as an algebra of subsets of any space.

- A – an integral dioid,
- $A \subset \prod(B_i\colon i \in U)$ – A is a subdirect product of subdirectly irreducible integral dioids B_i,
- B – a minimal (in a certain strict sense) quantale in which all B_i are embedded.

(If A is a distributive lattice, then B is the two-element lattice $\mathbf{2}$.)

U replaces the prime ideal space of A.

- A quantifier on A induces a strong B-equivalence on U.
- Problem. Is this transformation injective?

- A subset τ of B_U is a *fuzzy topology* on U if it is closed under $\cdot$ and arbitrary joins, and contains all constant functions. Thus, τ is a sub-quantale of B^U.

- A is embedded in B^X, and induces a subbase of some fuzy topology τ on U.
- Problem. Is the B-relation induced by a quantifier on A topologically well-behaved w.r.t. τ?