Covariety and quasi-covariety lattices

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1. Introduction
   - Basic definitions and properties
   - Covarieties

2. Lattices
   - Covariety lattices: basics
   - quasi-covariety lattices: basics
   - Covariety lattices

Coalgebras

Definition

Let $F : \mathsf{Set} \to \mathsf{Set}$ be an endofunctor. An F-*coalgebra* is a pair (A, α), where A is a set and $\alpha : A \to F(A)$ is a mapping.

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The definitions of a *homomorphic image* and *subcoalgebra* are clear.

Coalgebraic operators

Definition

Given a family of F-coalgebras $\{(A_i, \alpha_i)\}_{i\in I}$ we define the *disjoint sum* F-coalgebra $\Sigma_{i\in I}(A_i, \alpha_i)$.

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Given a class K of F-coalgebras we define the following:

- $\mathcal{H}(\mathsf{K})$ - homomorphic images of K,
- $\mathcal{S}(\mathsf{K})$ - subcoalgebras of K,
- $\Sigma(\mathsf{K})$ - disjoint sums of K.

Covarieties and quasi-covarieties

Definition

A class K is called *covariety* if $\mathcal{H}(\mathsf{K}) \subseteq \mathsf{K}$, $\mathcal{S}(\mathsf{K}) \subseteq \mathsf{K}$ and $\Sigma(\mathsf{K}) \subseteq \mathsf{K}$.

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Theorem [1]

The smallest covariety (quasi-coavariety) containing a class K is $\mathcal{SH}\Sigma(\mathsf{K})$ (resp. $\mathcal{H}\Sigma(\mathsf{K})$).

Covarieties and quasi-covarieties

Assume that F is bounded. This guarantees that the family of covarieties (quasi-covarieties) of F-coalgebras is a set.

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It is possible to define a *coequation* (*coimplication*). To satisfy a coequation (resp. coimplication) is to omit some "behaviour".

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Coalgebraic Birkhoff Theorem [1]

Covarieties (quasi-covarieties) are exactly the classes defined by the satifaction of some set of coequations (resp. coimplications).

Covarieties and quasi-covarieties

Theorem [1]

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Notation

Let K be a covariety (quasi-covariety) of F-coalgebras. The lattice of subcovarieties (quasi-subcovarieties) is denoted by $L_{\mathcal{CV}}(\mathsf{K})$ (resp. $L_{\mathcal{QCV}}(\mathsf{K})$).

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Theorem

Let K be a covariety of F-coalgebras. Then $L_{\mathcal{CV}}(\mathsf{K})$ is a distributive lattice.

Quasi-covariety lattices

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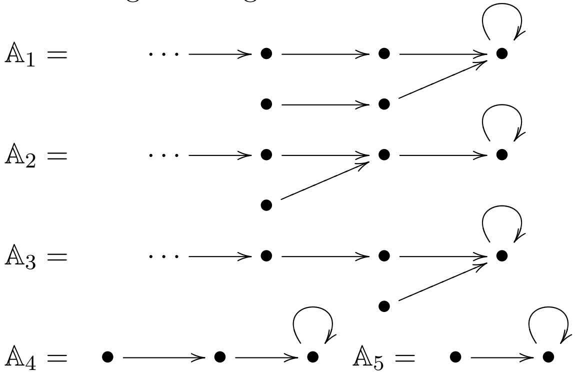

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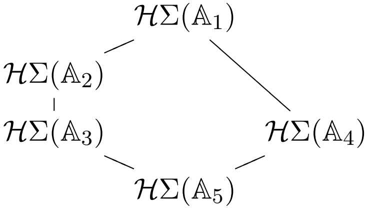

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Theorem

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Conjecture

The lattice $L_{\mathcal{QCV}}(\mathsf{Set}_F)$ is distributive iff $F \cong \mathcal{C}_M$.

Strongly simple coalgebras and their properties

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Theorem

Let $\mathbb{A}$ be a strongly simple F-coalgebra. Then

$$L_{\mathcal{CV}}(\mathcal{SH}\Sigma(\mathbb{A})) \cong (\mathtt{S}(\mathbb{A}), \cup, \cap).$$

Construction of covariety lattices

Lemma

Let (A, α) be an F-coalgebra and B be a set such that $A \subseteq B$. Then $B \times F$-coalgebra $(A, (\alpha, \subseteq))$ is strongly simple and $(\mathtt{S}((A, \alpha)), \cup, \cap) = (\mathtt{S}((A, (\alpha, \subseteq))), \cup, \cap)$

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Example

Let X be a set and τ a topology on X. Then there exists an $\mathcal{F}$-coalgebra $\mathbb{X}$ such that $(\mathtt{S}(\mathbb{X}), \cup, \cap) \cong (\tau, \cup, \cap)$.

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Theorem

Let (X, τ) be a topological space. There exists a bounded functor $F : \mathsf{Set} \to \mathsf{Set}$ and a covariety K of F-coalgebras such that $L_{\mathcal{CV}}(\mathsf{K}) \cong (\tau, \cup, \cap)$.

Functors preserving arbitrary intersections

Definition

An F-coalgebra $\mathbb{A}$ is called *rooted* if there exists $a \in A$ such that $\mathbb{A}$ is the smallest subcoalgebra of $\mathbb{A}$ containg a.

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Define:

$$\mathcal{D}(\mathfrak{R}_K) := \{U \subseteq \mathfrak{R}_K \mid \mathfrak{R}_K \cap \mathcal{SH}(U) = U\}.$$

Functors preserving arbitrary intersections

Theorem

If $F : \mathsf{Set} \to \mathsf{Set}$ preserves arbitrary intersections, then

$$L_{\mathcal{CV}}(\mathsf{Set}_F) \cong (\mathcal{D}(\mathfrak{R}_{\mathsf{Set}_F}), \cup, \cap).$$

Example

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$$\mathcal{I}d\text{-coalgebras} = \text{mono-unary algebras}$$

We can speak of an *index* and a *period* of a rooted $\mathcal{I}d$-coalgebra.

Example

The following theorem holds:

Theorem

$$L_{\mathcal{CV}}(\mathsf{Set}_{\mathcal{I}d}) \cong (\mathcal{O}(\mathsf{N}_0 \times \mathsf{N} \cup \{(\infty, 0)\}), \cup, \cap),$$

where $\mathsf{N}_0 \times \mathsf{N} \cup \{(\infty, 0)\}$ denotes the poset in which $(i, p) \leq (i', p') :\iff i \leq i'$ and $p | p'$.

Characterization

Theorem

The lattice $L_{\mathcal{CV}}(\mathsf{K})$ for a functor F preserving arbitrary intersections is isomorphic to a lattice of subcoalgebras of some $\mathcal{P}_\kappa$-coalgebra.

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Theorem

Conversely, any lattice of subcoalgebras of a $\mathcal{P}_\kappa$-coalgebra is isomorphic to a lattice $L_{\mathcal{CV}}(\mathsf{K})$ of subcovarieties of some covariety K of F-coalgebras for a bounded functor F preserving arbitrary intersections.

Bibliography

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