On Many-Valued Modal Logics over finite residuated lattices

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Outline

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  We can think of A as our set of truth values, which is the same one in every world.

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Definition

A formula φ is valid in this Kripke model in case that $e(\varphi, w) = 1$ for every world $w \in W$.

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Example

Let us a consider **A** as the standard Łukasiewicz algebra and the Kripke model with only one point $\bullet$. This point satisfies $R(\bullet, \bullet) = \frac{1}{2}$, $e(p, \bullet) = \frac{1}{2}$ and $e(q, \bullet) = 0$. Then,

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  The same translation embeds our many-valued modal language into the many-valued first-order logic given by (truth values in) **A**.

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- In our approach chains are neither enough.

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- Now we have at least two possible options:
  - “if a formula is 1-satisfied in a Kripke model of the logic then it is also 1-satisfied in a finite Kripke model of the logic” [this is what we will call finite model property]
  - “if a formula is positively-satisfied in a Kripke model of the logic then it is also positively-satisfied in a finite Kripke model of the logic” [this is the one helpful for decidability issues in case that negation is involutive]

Another Remark

Remark

Canonicity does not seem to work in a lot of cases (we remember that we are considering minimal logics; the parameter is **A**).

Two types of semantical structures

Let $\mathbf{A} = \langle A, 0, 1, \wedge, \vee, \odot, \rightarrow \rangle$ be a residuated lattice.
We can consider another kind of **A**-valued Kripke models:

$\mathcal{M} = \langle W, R, e \rangle$	$\mathcal{M}_* = \langle W, \{R_\alpha\}_{\alpha \in A}, e \rangle$
$R : W \times W \mapsto A$	$\forall \alpha \in A : R_\alpha \subseteq W \times W$ satisfying: $R_0 = W \times W$ $R_\alpha \cap R_\beta \subseteq R_{\alpha \vee \beta}$
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If **A** is **finite** then there is a bijection between both families of structures:

$$\forall \alpha \in \mathcal{A} : R_\alpha = \{\langle x, y\rangle : R(x, y) \geq \alpha\}$$

$$\forall \langle x, y\rangle \in W \times W : R(x, y) = \bigvee_{\alpha\in A} \alpha \wedge R_\alpha(x, y)$$

Languages to Consider

Given a *finite residuated lattice* **A**, we consider two languages:

- $\mathcal{L}_{\mathbf{A}}^{\Box}$ is defined from a set *Var* of propositional variables, logical connectives $\odot, \wedge, \vee, \rightarrow, \neg$, a truth constant $\overline{a}$ for each element $a \in A$ and a modality $\Box$.

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- $\mathcal{L}_{\mathbf{A}}^{\Box *}$ is also defined from a set *Var* of propositional variables, logical connectives $\odot, \wedge, \vee, \rightarrow, \neg$, a truth constant $\overline{a}$ for each element $a \in A$, but in this case we have a family of modalities $\{\Box_\alpha\}_{\alpha \in A \setminus \{0\}}$.

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The sets of formulas of the resulting languages are denoted by $Fm(\mathcal{L}^{\Box}_{\mathbf{A}})$ and $Fm(\mathcal{L}^{\Box *}_{\mathbf{A}})$, respectively.

Semantics

Given an **A**-valued Kripke model $\mathcal{M} = \langle W, R, e \rangle$, the map $e : Var \times W \to A$ is uniquely extended to a map $\bar{e}$:

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We define $Log_\Box(\mathbf{A}) \subseteq \mathcal{L}_{\mathbf{A}}^{\Box}$ as the set of formulas valid in every **A**-valued Kripke model.

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We define $Log_\Box(\mathbf{A}) \subseteq \mathcal{L}_{\mathbf{A}}^{\Box}$ as the set of formulas valid in every **A**-valued Kripke model. Similarly, $Log_{\Box_*}(\mathbf{A}) \subseteq \mathcal{L}_{\mathbf{A}}^{\Box_*}$.

Interdefinability Issues

For every *finite residuated lattice* **A**, the following formulas are valid in every **A**-valued Kripke model:

$$\Box_\alpha(\varphi \to \psi) \to (\Box_\alpha \varphi \to \Box_\alpha \psi)$$

$$(\Box\varphi) \leftrightarrow (\bigwedge \{\bar{\alpha} \to \Box_\alpha \varphi : \alpha \in A, \alpha \neq 0\}).$$

This tells us that the modalities $\Box_\alpha$ satisfy the *normality* axiom K, and that $\Box$ is definable using the modalities $\Box_\alpha$.

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Taking into account that the modalities $\Box_\alpha$ are normal and that $\Box$ is definable using them, we try first to axiomatize the logic $Log_{\Box_*}(\mathbf{A})$ in order to reach an axiomatization for the logic $Log_{\Box}(\mathbf{A})$.

Main Result

For any finite residuated lattice **A** it is well-known there exists a (non necessarily recursively enumerable) Hilbert-style calculus axiomatizing the **A**-logic. If **A** is a BL-chain, it is always finitely axiomatizable (EGM, 2001).

Main Result

For any finite residuated lattice **A** it is well-known there exists a (non necessarily recursively enumerable) Hilbert-style calculus axiomatizing the **A**-logic. If **A** is a BL-chain, it is always finitely axiomatizable (EGM, 2001).

If we add to this calculus the following:

Axioms

$\Box_\alpha(\varphi \rightarrow \psi) \rightarrow (\Box_\alpha\varphi \rightarrow \Box_\alpha\psi)$

$\Box_{\alpha_i}\varphi \rightarrow \Box_{\alpha_j}\varphi$, if $\alpha_i \leq \alpha_j$

$\Box_{\alpha_i}(\overline{\alpha_j} \rightarrow \varphi) \leftrightarrow (\overline{\alpha_j} \rightarrow \Box_{\alpha_i}\varphi)$

New rule

From φ derive $\Box_\alpha\varphi$, for each $\alpha \in A \setminus \{0\}$

We obtain a *sound and complete axiomatization of the logic* $Log_{\Box_*}(\mathbf{A})$ by using a standard technique of the canonical model construction

A Problem

From the axiomatization of $Log_{\Box_*}(\mathbf{A})$ we do not directly obtain an axiomatization of $Log_{\Box}(\mathbf{A})$ because we cannot in general define the $\Box_\alpha$'s in the language $\mathcal{L}^{\Box}_{A}$.

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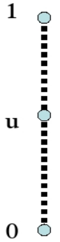
In particular, if **A** is a finite BL-algebra different to an $Ł_n$ (i.e., a finite Łukasiewicz chain) then we know that this is impossible.

Counterexample

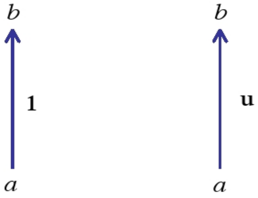
$W = \{a, b\}$

$e(p, w) = 1$, for all variable p and $w \in W$

A



$M_1 = (W, R_1, e)$ $\quad$ $M_2 = (W, R_2, e)$



$u * u = u$

$e(\varphi, w) = e(\varphi, w) \quad \varphi \in L_\Box(A)$

$e(\Box_1 \bot, a) = 0 \qquad e(\Box_1 \bot, a) = 1$

The particular case of finite MV-chains

Remark

However, we have succeded in the particular case of **A** being a finite MV-chain, i.e. the case of modal logics over $Ł_n$. In this case, the formula

$$\Box_\alpha \varphi \leftrightarrow \bigwedge \{ (\bar{\alpha} \to \neg \Box \neg ((\varphi \leftrightarrow \bar{\beta})^{n-1}))^{n-1} \to \bar{\beta} : \beta \in Ł_n \}$$

is valid in all $Ł_n$-valued Kripke models.
Using this validity, we can check that the level-cuts of the accessibility relation of the canonical model for $Log_\Box(\mathbf{A})$ correspond to the relations of the canonical model defined in order to prove the completeness of $Log_{\Box_*}(\mathbf{A})$.

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Remark

In addition, in this logic we can define the dual possibility operator $\Diamond = \neg\Box\neg$ with the usual semantics.

Notation

For $a \neq 0$, $\Box_a\varphi$ stands for $\bigwedge\left\{\left(\bar{a} \to \neg\Box\neg((\varphi \leftrightarrow \bar{b})^{n-1})\right)^{n-1} \to \bar{b} : b \in Ł_n\right\}$

$$m.\varphi := \varphi \oplus \overset{m}{\ldots} \oplus \varphi$$

$$\varphi^m := \varphi \odot \overset{m}{\ldots} \odot \varphi$$

Axioms

$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

$$\varphi \to (\psi \to \varphi)$$

$$((\varphi \to \psi) \to \psi) \to ((\psi \to \varphi) \to \varphi)$$

$$(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$$

$$(\varphi \wedge \psi) \leftrightarrow (\varphi \odot (\varphi \to \psi))$$

$$(\varphi \vee \psi) \leftrightarrow (((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi))$$

$$(\varphi \odot \psi) \leftrightarrow \neg(\varphi \to \neg\psi)$$

$$n.\varphi \to (n-1).\varphi$$

$$(m.\varphi^{m-1})^n \leftrightarrow (n.\varphi^m), \quad 2 \leq m \leq n-2 \text{ and } m \nmid (n-1)$$

$$(a_i \to a_j) \leftrightarrow a_k, \quad \text{if } a_k = a_i \to a_j$$

$$\Box_a(\varphi \to \psi) \to (\Box_a\varphi \to \Box_a\psi)$$

$$\Box_{a_i}\varphi \to \Box_{a_j}\varphi, \quad \text{if } a_i \leq a_j$$

$$\Box_{a_i}(a_j \to \varphi) \leftrightarrow (a_j \to \Box_{a_i}\varphi)$$

$$(\Box\varphi) \leftrightarrow \left(\bigwedge\left\{a \to \Box_a\varphi : a \in Ł_n, a \neq 0\right\}\right)$$

Rules

If $\emptyset \vdash \varphi$ then $\emptyset \vdash \Box_a\varphi$

$$\varphi, \varphi \to \psi \vdash \psi$$

Some Results

Theorem

The minimal modal many-valued logic given by $Ł_n$ is decidable.

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Remark

G. HANSOUL and B. TEHEUX have recently given an axiomatization over Ł$_n$ for the case where the accesibility relation is classical (like in the α-cuts). There they do not need constants in the language.

The standard Łukasiewicz algebra Ł

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Some Open Problems

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Thanks.