

Modal logics for region-based theories of space

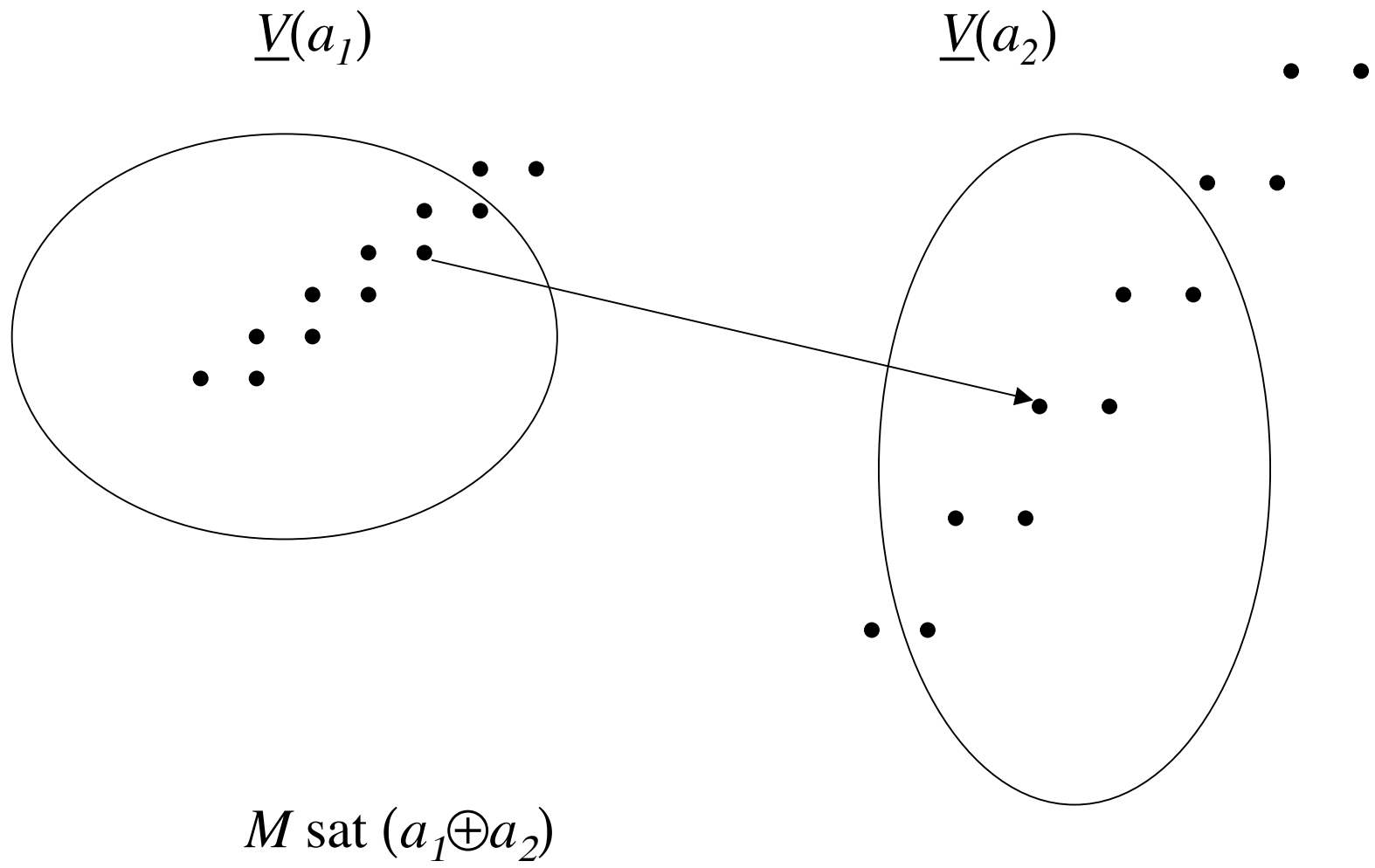
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Syntax

- **Boolean terms**
 - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
- **Modal formulas**
 - $\phi ::= (a_1 \oplus a_2) \mid 0 \mid \neg \phi \mid (\phi_1 \vee \phi_2) \mid (a_1 = a_2)$
- Abbreviations (Boolean terms)
 - $1 ::= \neg 0, (a_1 \wedge a_2) ::= \neg(\neg a_1 \vee \neg a_2)$
- Abbreviations (modal formulas)
 - $1 ::= \neg 0, (\phi_1 \wedge \phi_2) ::= \neg(\neg \phi_1 \vee \neg \phi_2)$
- $(a_1 \oplus a_2)$ is equivalent to $\langle U \rangle (a_1 \wedge \langle R \rangle a_2)$ and $(a_1 = a_2)$ is equivalent to $[U](a_1 \leftrightarrow a_2)$

Semantics

- A **model** is a structure of the form $M = \langle W, R, V \rangle$ where
 - W is a nonempty set
 - R is a binary relation on W
 - V associates a subset $V(p)$ of W to each Boolean variable p
- \underline{V} associates a subset $\underline{V}(a)$ of W to each Boolean term a
 - $\underline{V}(p) = V(p)$
 - $\underline{V}(0) = \emptyset$, $\underline{V}(\neg a) = W \setminus \underline{V}(a)$, $\underline{V}(a_1 \vee a_2) = \underline{V}(a_1) \cup \underline{V}(a_2)$
- Remark that
 - $\underline{V}(1) = W$, $\underline{V}(a_1 \wedge a_2) = \underline{V}(a_1) \cap \underline{V}(a_2)$



Semantics

- **Satisfiability** of ϕ in $M = \langle W, R, V \rangle$ is defined by:
 - $M \text{ sat } (a_1 \oplus a_2)$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
 - Not $M \text{ sat } 0$, $M \text{ sat } \neg\phi$ iff not $M \text{ sat } \phi$, $M \text{ sat } (\phi_1 \vee \phi_2)$ iff $M \text{ sat } \phi_1$ or $M \text{ sat } \phi_2$
 - $M \text{ sat } (a_1 = a_2)$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- Remark that
 - $M \text{ sat } 1$, $M \text{ sat } (\phi_1 \wedge \phi_2)$ iff $M \text{ sat } \phi_1$ and $M \text{ sat } \phi_2$
- **Validity** of modal formula ϕ in frame $F = \langle W, R \rangle$ is defined by:
 - $F \text{ val } \phi$ iff for all models $M = \langle W, R, V \rangle$ based on F , $M \text{ sat } \phi$

Semantics

- **Correspondence theory**
 - $F \text{ val } (p \neq 0) \rightarrow (p \oplus p)$ iff $\forall w \in W (w R w)$
 - $F \text{ val } (p \oplus q) \rightarrow (q \oplus p)$ iff $\forall w, w' \in W (w R w' \rightarrow w' R w)$
 - $F \text{ val } (p \neq 0) \rightarrow (p \oplus 1)$ iff $\forall w \in W \exists w' \in W (w R w')$
 - $F \text{ val } (p \neq 0) \rightarrow (1 \oplus p)$ iff $\forall w \in W \exists w' \in W (w' R w)$
 - $F \text{ val } (p \neq 0) \rightarrow (p \oplus 1) \vee (1 \oplus p)$ iff $\forall w \in W \exists w' \in W (w R w' \vee w' R w)$
 - $F \text{ val } (1 \oplus 1)$ iff $\exists w, w' \in W (w R w')$
 - $F \text{ val } (p \neq 0) \wedge (q \neq 0) \rightarrow (p \oplus q)$ iff $\forall w, w' \in W (w R w')$
 - $F \text{ val } (p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)$ iff R is connected

Bisimulation

- Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be models
- A **bisimulation** between M and M' is a binary relation Z between W and W' such that
 - $\forall w \in W \exists w' \in W' (w Z w')$
 - $\forall w' \in W' \exists w \in W (w Z w')$
 - $\forall w_1, w_2 \in W \exists w_1', w_2' \in W' (w_1 R w_2 \rightarrow w_1' R' w_2')$
 - $\forall w_1', w_2' \in W' \exists w_1, w_2 \in W (w_1' R' w_2' \rightarrow w_1 R w_2)$
 - $\forall w \in W \forall w' \in W' (w Z w' \rightarrow (w \in V(p) \leftrightarrow w' \in V'(p)))$

Bisimulation

- **Bisimulation theorem**
 - If M and M' are bisimilar then they are modally equivalent
- **Hennesy-Milner theorem**
 - If M and M' are finite and modally equivalent then they are bisimilar
- **Van Benthem characterization theorem**
 - For all 1st-order sentences A like
 - $A ::= R(x_1, x_2) \mid P(x) \mid 0 \mid \neg\phi \mid (\phi_1 \vee \phi_2) \mid \forall xA$
 - A is invariant for bisimulations iff A is equivalent to the standard translation of a modal formula

Axiomatization/completeness

- **Axioms of L_{min}**
 - Identity axioms:
 - $(a=a), (a_1=a_2) \rightarrow (a_2=a_1), (a_1=a_2) \wedge (a_2=a_3) \rightarrow (a_1=a_3)$
 - Congruence axioms:
 - $(a=b) \rightarrow (\neg a = \neg b), (a_1=b_1) \wedge (a_2=b_2) \rightarrow ((a_1 \vee a_2) = (b_1 \vee b_2))$
 - Boolean axioms:
 - $(a=b)$ if a and b are equivalent Boolean terms
 - $(0 \neq 1)$

Axiomatization/completeness

- **Axioms of L_{min}**
 - Proximity axioms:
 - $(a \oplus b) \rightarrow (a \neq 0) \wedge (b \neq 0)$
 - $((a \vee b) \oplus c) \leftrightarrow (a \oplus c) \vee (b \oplus c)$
 - $(a \oplus (b \vee c)) \leftrightarrow (a \oplus b) \vee (a \oplus c)$
- **Completeness of L_{min}** : For all modal formulas ϕ , ϕ is provable from the axioms of L_{min} iff ϕ is valid in the class of all frames $F = \langle W, R \rangle$

Axiomatization/completeness

- Let Σ be a set of modal formulas
- **Axioms of L_Σ** are those of L_{min} plus the following
 - Σ -axioms: Every modal formula $\psi(a_1, \dots, a_n)$ which can be obtained from a modal formula $\psi(p_1, \dots, p_n)$ of Σ by uniformly substituting the Boolean terms a_1, \dots, a_n for the Boolean variables p_1, \dots, p_n
- **Completeness of L_Σ** : If Σ is finite then for all modal formulas ϕ , ϕ is provable from the axioms of L_Σ iff ϕ is valid in the nonempty class of all frames $F = \langle W, R \rangle$ such that $\langle W, R \rangle \text{ val } \Sigma$

Axiomatization/completeness

- A first extension of L_{min}
 - Let Σ_{sym} be $\{(p \oplus q) \rightarrow (q \oplus p)\}$
 - The **axioms of $L_{\Sigma_{sym}}$** are those of L_{min} plus every modal formula like $(a \oplus b) \rightarrow (b \oplus a)$
- **Completeness of $L_{\Sigma_{sym}}$** : For all modal formulas ϕ , ϕ is provable from the axioms of $L_{\Sigma_{sym}}$ iff ϕ is valid in the nonempty class of all frames $F = \langle W, R \rangle$ such that $\forall w, w' \in W (w R w' \rightarrow w' R w)$

Axiomatization/completeness

- A second extension of L_{min}
 - Let Σ_{con} be $\{(p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)\}$
 - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(a \neq 0) \wedge (a \neq 1) \rightarrow (a \oplus \neg a)$
- **Completeness of $L_{\Sigma con}$** : For all modal formulas ϕ , ϕ is provable from the axioms of $L_{\Sigma con}$ iff ϕ is valid in the nonempty class of all frames $F = \langle W, R \rangle$ such that **R is connected**

Axiomatization/completeness

- **Open problems**

- Find a set Σ of modal formulas such that L_Σ is **not complete** with respect to the nonempty class of all frames $F = \langle W, R \rangle$ such that $\langle W, R \rangle \vDash \Sigma$
- Find a set Σ of modal formulas such that **the class of all frames $F = \langle W, R \rangle$ such that $\langle W, R \rangle \vDash \Sigma$ is empty**

Canonicity

- Given a set Σ of modal formulas and a maximal L_Σ -consistent set S of modal formulas, the **canonical frame of L_Σ** defined by S is the structure $F_S = \langle W_S, R_S \rangle$ defined as follows
 - W_S is the set of all maximal consistent sets w of Boolean terms such that for all Boolean terms $a \in w$, $(a \neq 0) \in S$
 - R_S is the binary relation on W_S such that for all $w, w' \in W_S$, $w R_S w'$ iff for all Boolean terms $a \in w$ and for all Boolean terms $a' \in w'$, $(a \oplus a') \in S$

Canonicity

- Let Σ be a set of modal formulas
 - L_Σ is **strongly canonical** iff for all maximal L_Σ -consistent sets S of modal formulas, the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_Σ defined by S validates L_Σ
 - L_Σ is **weakly canonical** iff there exists a maximal L_Σ -consistent set S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_Σ defined by S validates L_Σ

Canonicity

- **Strong canonicity of L_{min} :** For all maximal L_{min} -consistent sets S of modal formulas, the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_{min} defined by S validates L_{min}
- **Weak canonicity of L_{min} :** There exists a maximal L_{min} -consistent set S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of L_{min} defined by S validates L_{min}

Canonicity

- A first extension of L_{min}
 - Let Σ_{sym} be $\{(p \oplus q) \rightarrow (q \oplus p)\}$
 - The **axioms of $L_{\Sigma_{sym}}$** are those of L_{min} plus every modal formula like $(a \oplus b) \rightarrow (b \oplus a)$
- **Strong canonicity of $L_{\Sigma_{sym}}$** : For all maximal $L_{\Sigma_{sym}}$ -consistent sets S of modal formulas, the canonical frame $F_S = \langle W_S, R_S \rangle$ of $L_{\Sigma_{sym}}$ defined by S validates $L_{\Sigma_{sym}}$

Canonicity

- A second extension of L_{min}
 - Let Σ_{con} be $\{(p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)\}$
 - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(a \neq 0) \wedge (a \neq 1) \rightarrow (a \oplus \neg a)$
- **Non strong canonicity of $L_{\Sigma con}$** : There exists a maximal $L_{\Sigma con}$ -consistent sets S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of $L_{\Sigma con}$ defined by S does not validate $L_{\Sigma con}$
- **Weak canonicity of $L_{\Sigma con}$** : There exists a maximal $L_{\Sigma con}$ -consistent sets S of modal formulas such that the canonical frame $F_S = \langle W_S, R_S \rangle$ of $L_{\Sigma con}$ defined by S validates $L_{\Sigma con}$

Canonicity

- **Open problems**
 - Find a set Σ of modal formulas such that L_Σ is **not weakly canonical**
 - Find a syntactic condition on the sets Σ of modal formulas implying that L_Σ is **strongly canonical**
 - Find a syntactic condition on the sets Σ of modal formulas implying that L_Σ is **weakly canonical**

Decidability/complexity

- Let **sat- L_{min}** be the following decision problem
 - Input: A modal formula ϕ
 - Output: Determine if there exists a model $M = \langle W, R, V \rangle$ such that $M \text{ sat } \phi$
- **Complexity of sat- L_{min}** : $\text{sat-}L_{min}$ is NP-complete

Decidability/complexity

- Let Σ be a set of modal formulas
- Let **sat- L_Σ** be the following decision problem
 - Input: A modal formula ϕ
 - Output: Determine if there exists a model $M = \langle W, R, V \rangle$ such that $M \text{ sat } \phi$ and $\langle W, R \rangle \text{ val } \Sigma$
- **Complexity of sat- L_Σ (upper bound)**: If Σ is finite then sat- L_Σ is in NEXPTIME
- **Complexity of sat- L_Σ (lower bound)**: If the class of all frames $F = \langle W, R \rangle$ such that $\langle W, R \rangle \text{ val } \Sigma$ is nonempty then sat- L_Σ is NP-hard

Decidability/complexity

- A first extension of L_{min}
 - Let Σ_{sym} be $\{(p \oplus q) \rightarrow (q \oplus p)\}$
 - The **axioms of $L_{\Sigma_{sym}}$** are those of L_{min} plus every modal formula like $(a \oplus b) \rightarrow (b \oplus a)$
- **Complexity of sat- $L_{\Sigma_{sym}}$** : sat- $L_{\Sigma_{sym}}$ is NP-complete

Decidability/complexity

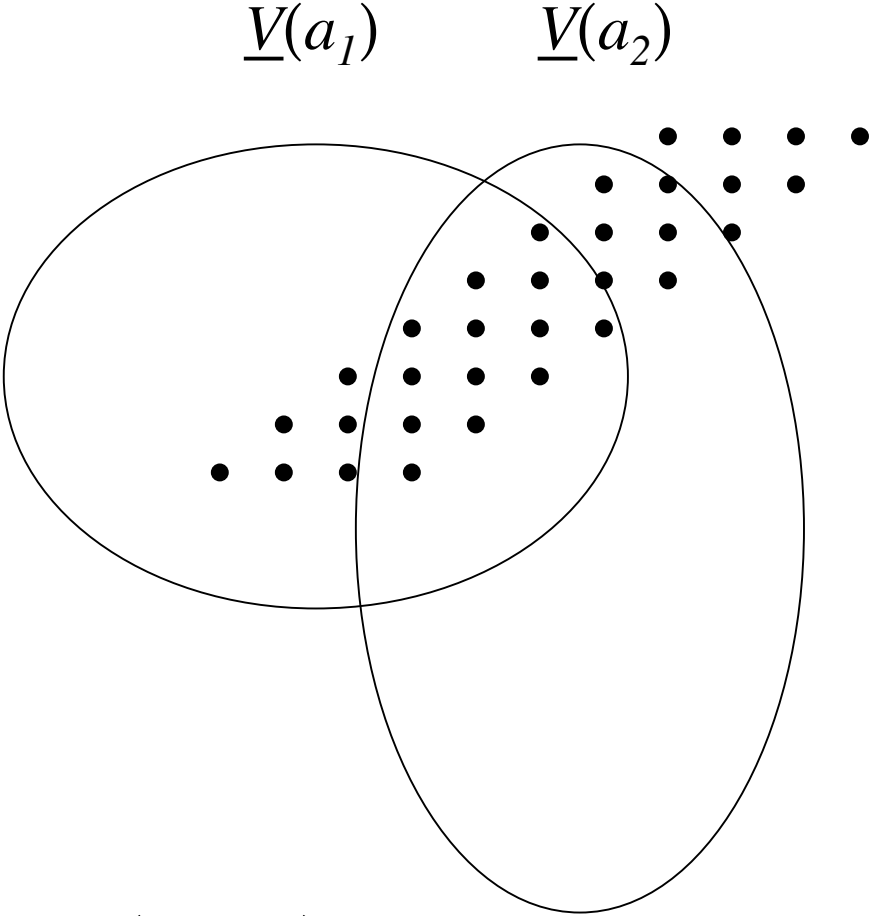
- A second extension of L_{min}
 - Let Σ_{con} be $\{(p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)\}$
 - The **axioms of $L_{\Sigma con}$** are those of L_{min} plus every modal formula like $(a \neq 0) \wedge (a \neq 1) \rightarrow (a \oplus \neg a)$
- **Complexity of sat- $L_{\Sigma con}$** : sat- $L_{\Sigma con}$ is PSPACE-complete

Decidability/complexity

- **Open problems**
 - Find a set Σ of modal formulas such that **sat- L_Σ is EXPTIME-complete**
 - Find a set Σ of modal formulas such that **sat- L_Σ is NEXPTIME-complete**
 - Find a set Σ of modal formulas such that **sat- L_Σ is not decidable**

Topological interpretation

- A **topological model** is a structure of the form $M = \langle X, T, V \rangle$ where
 - $\langle X, T \rangle$ is a topological space
 - V associates a regular closed subset $V(p)$ of $\langle X, T \rangle$ to each Boolean variable p
- \underline{V} associates a regular closed subset $\underline{V}(a)$ of $\langle X, T \rangle$ to each Boolean term a
 - $\underline{V}(p) = V(p)$
 - $\underline{V}(0) = \emptyset$, $\underline{V}(\neg a) = Cl(X \setminus \underline{V}(a))$, $\underline{V}(a_1 \vee a_2) = \underline{V}(a_1) \cup \underline{V}(a_2)$
- Remark that
 - $\underline{V}(1) = X$, $\underline{V}(a_1 \wedge a_2) = Cl(In(\underline{V}(a_1) \cap \underline{V}(a_2)))$



$M \text{ sat } (a_1 \oplus a_2)$

Topological interpretation

- **Satisfiability** of ϕ in $M = \langle X, T, V \rangle$ is defined by:
 - $M \text{ sat } (a_1 \oplus a_2)$ iff $\underline{V}(a_1) \cap \underline{V}(a_2) \neq \emptyset$
 - Not $M \text{ sat } 0$, $M \text{ sat } \neg\phi$ iff not $M \text{ sat } \phi$, $M \text{ sat } (\phi_1 \vee \phi_2)$ iff $M \text{ sat } \phi_1$ or $M \text{ sat } \phi_2$
 - $M \text{ sat } (a_1 = a_2)$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- Remark that
 - $M \text{ sat } 1$, $M \text{ sat } (\phi_1 \wedge \phi_2)$ iff $M \text{ sat } \phi_1$ and $M \text{ sat } \phi_2$

Topological interpretation

- A first topological extension of L_{min}
 - Let $\Sigma_{ref,sym}$ be $\{(p \neq 0) \rightarrow (p \oplus p), (p \oplus q) \rightarrow (q \oplus p)\}$
 - The **axioms of $L_{\Sigma_{ref,sym}}$** are those of L_{min} plus every modal formula like $(a \neq 0) \rightarrow (a \oplus a), (a \oplus b) \rightarrow (b \oplus a)$
- **Completeness of $L_{\Sigma_{ref,sym}}$** : For all modal formulas ϕ , ϕ is provable from the axioms of $L_{\Sigma_{ref,sym}}$ iff ϕ is valid in the class of all topological models iff ϕ is valid in the class of all frames $F = \langle W, R \rangle$ such that $\forall w \in W (w R w)$ and $\forall w, w' \in W (w R w' \rightarrow w' R w)$
- **Complexity of sat- $L_{\Sigma_{ref,sym}}$** : sat- $L_{\Sigma_{ref,sym}}$ is NP-complete

Topological interpretation

- A second topological extension of L_{min}
 - Let $\Sigma_{ref,sym,con}$ be $\{(p \neq 0) \rightarrow (p \oplus p), (p \oplus q) \rightarrow (q \oplus p), (p \neq 0) \wedge (p \neq 1) \rightarrow (p \oplus \neg p)\}$
 - The **axioms of $L_{\Sigma_{ref,sym,con}}$** are those of L_{min} plus every modal formula like $(a \neq 0) \rightarrow (a \oplus a), (a \oplus b) \rightarrow (b \oplus a), (a \neq 0) \wedge (a \neq 1) \rightarrow (a \oplus \neg a)$
- **Completeness of $L_{\Sigma_{ref,sym,con}}$** : For all modal formulas ϕ , ϕ is provable from the axioms of $L_{\Sigma_{ref,sym,con}}$ iff ϕ is valid in the class of all connected topological models iff ϕ is valid in the class of all frames $F = \langle W, R \rangle$ such that $\forall w \in W (w R w), \forall w, w' \in W (w R w' \rightarrow w' R w)$ and R is **connected**
- **Complexity of sat- $L_{\Sigma_{ref,sym,con}}$** : sat- $L_{\Sigma_{ref,sym,con}}$ is PSPACE-complete

Conclusion

- We have considered logics based on a language that contains the operators \oplus and $=$
 - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
 - $\phi ::= (a_1 \oplus a_2) \mid 0 \mid \neg \phi \mid (\phi_1 \vee \phi_2) \mid (a_1 = a_2)$
 - $M \text{ sat } (a_1 \oplus a_2)$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
 - $M \text{ sat } (a_1 = a_2)$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
- $(a_1 \oplus a_2)$ is equivalent to $\langle U \rangle (a_1 \wedge \langle R \rangle a_2)$ and $(a_1 = a_2)$ is equivalent to $[U](a_1 \leftrightarrow a_2)$

Conclusion

- We might also consider logics based on the more general language that contains the \otimes as well
 - $a ::= p \mid 0 \mid \neg a \mid (a_1 \vee a_2)$
 - $\phi ::= (a_1 \oplus a_2) \mid (\mathbf{a}_1 \otimes \mathbf{a}_2) \mid 0 \mid \neg \phi \mid (\phi_1 \vee \phi_2) \mid (a_1 = a_2)$
 - $M \text{ sat } (a_1 \oplus a_2)$ iff there exists $w \in \underline{V}(a_1)$ such that for some $w' \in R(w)$, $w' \in \underline{V}(a_2)$
 - $M \text{ sat } (a_1 = a_2)$ iff $\underline{V}(a_1) = \underline{V}(a_2)$
 - $M \text{ sat } (\mathbf{a}_1 \otimes \mathbf{a}_2)$ iff there exists $w \in \underline{V}(a_1)$ such that for each $w' \in R(w)$, $w' \in \underline{V}(a_2)$
- $(a_1 \oplus a_2)$ is equivalent to $\langle U \rangle (a_1 \wedge \langle R \rangle a_2)$, $(a_1 = a_2)$ is equivalent to $[U](a_1 \leftrightarrow a_2)$ and $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ is equivalent to $\langle U \rangle (a_1 \wedge [R]a_2)$