Algebraic Analysis of Visser's Propositional Logic

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Introduction

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$$\vdash_{IPL} A \quad \text{iff} \quad \vdash_{S_4} A^t$$

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Language

- The language of BPC contains:
  1. a countably infinite set of individual variables,
  2. logical constance $\top$ and $\bot$, and the logical connectives $\wedge$, $\vee$ and $\rightarrow$.
- The axioms and rules of BPC are in sequent notation.
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Axiomatization

- $A \Rightarrow A$,
- $A \Rightarrow \top$,
- $\bot \Rightarrow A$
- $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$,
- $\frac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}$,
- $\frac{A \Rightarrow B \quad A \Rightarrow C}{A \Rightarrow B \wedge C}$ (double line),
- $\frac{A \Rightarrow B \quad C \Rightarrow B}{A \vee C \Rightarrow B}$ (double line),
- $\frac{A \wedge B \Rightarrow C}{A \Rightarrow B \rightarrow C}$,
- $(A \rightarrow B) \wedge (B \rightarrow C) \Rightarrow A \rightarrow C$,
- $(A \rightarrow B) \wedge (A \rightarrow C) \Rightarrow A \rightarrow B \wedge C$,
- $(A \rightarrow B) \wedge (C \rightarrow B) \Rightarrow A \vee C \rightarrow B$.

Formal Propositional logic, FPC

FPC, is the extension of *BPC* by the Löb's axiom schema or equivalently by all substitution instances of Löb's rule:

$$(\top \to A) \to A \Rightarrow \top \to A, \qquad \frac{A \wedge (\top \to B) \Rightarrow B}{A \Rightarrow B}.$$

Some definitions an notations

- For every formula A, we denote $\top \to A$ by $\top A$.
- A theory Γ over BPC is a set of sequents and rules closed under derivability.
- A theory Γ is structural if $\Gamma \vdash A \Rightarrow B$ implies $\sigma(\Gamma) \vdash \sigma(A) \Rightarrow \sigma(B)$ for all substitution σ, where $\sigma(\Gamma) = \{\sigma(A) \Rightarrow \sigma(B) \ : \ A \Rightarrow B \in \Gamma\}$.
- An intermediate logic is a consistent structural sequent theory. The theories BPC, $IPC = BPC + \top \to A \Rightarrow A$, $CPC = IPC + \Rightarrow (A \vee \neg A)$, and FPC are all intermediate logics.

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- *A Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and R is a transitive relation on W. And a *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ such that $\mathfrak{F}$ is a Kripke frame and V is a valuation; that is, a map $V : PROP \to \mathcal{P}(W)$, satisfying the condition:

$$w \in V(p) \text{ and } wRv \text{ implies } v \in V(p),$$

- Let $\mathbf{K} = \langle K, \prec \rangle$ be an irreflexive Kripke frame. We say the height of $\mathbf{K}$, $h(\mathbf{K})$, is $n \in \omega$, if n is the largest number of elements $\alpha_0, \alpha_1, ..., \alpha_{n-1} \in K$ such that $\alpha_0 \prec \alpha_1 \prec ... \prec \alpha_{n-1}$. Otherwise $h(\mathbf{K}) = \infty$.

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logics of finite hight

F_n- logics

For every $n \in \omega$, we consider intermediate logics,
$F_n = BPC + \Rightarrow \Box^n \bot$, where $\Box^0 \bot = \bot$ and $\Box^n \bot = \top \to \Box^{n-1} \bot$.
we have $F_1 \supset F_2 \supset \cdots$.

Theorem

E_n is strongly complete with respect to the class of all irreflexive models $\mathcal{K}$ with $h(\mathbf{K}) \leq n$.

Corollary

$E_\omega = FPC$. where
$E_\omega = (\bigcap_{n=1}^{\infty} E_n) = \{A \Rightarrow B : E_n \vdash A \Rightarrow B$, for all $1 \leq n < \omega\}$.

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Basic algebra

A *Basic algebra* $\mathfrak{B} = \langle |\mathfrak{B}|, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is a structure with constants 0 and 1, and binary functions $\wedge$, $\vee$, and $\rightarrow$, such that

- with respect to 0, 1, $\wedge$, and $\vee$ we have a distributive lattice with top and bottom; and
- for $\rightarrow$ we have the additional identities and quasi-identities

1. $a \rightarrow b \wedge c = (a \rightarrow b) \wedge (a \rightarrow c)$;
2. $b \vee c \rightarrow a = (b \rightarrow a) \wedge (c \rightarrow a)$;
3. $a \rightarrow a = 1$;
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Models

An *algebraic model* of *BPC* consists of a pair $\underline{\mathcal{B}} = \langle \mathcal{B}, I \rangle$ with $\mathcal{B}$ a Basic algebra and I a map from the set of all propositional variables of the language of *BPC* to B. The map I can be uniquely extended to all formulas. A sequent $\phi \Rightarrow \psi$ is satisfied by a model , $\models \phi \Rightarrow \psi$, if $I(\phi) \leq I(\psi)$. A sequent $\phi \Rightarrow \psi$ is valid in a Basic algebra $\mathcal{B}$, if it is satisfied in for all interpretations I. Let T be a theory and s a sequent. s is a logical consequence of T, written: $T \models s$, if $\models T$ implies $\models s$, for all algebraic models .

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Algebraic Completeness and soundness

Theorem

For all theories T and sequents s, $T \vdash s$ iff $T \models s$.

subdirectly irreducible Ba

Proposition

Let F be a filter of basic algebra $\mathfrak{A}$. Then the binary relation $\theta(F) = \sim$ on A defined by

$$a \sim b \text{ iff } \exists f \in F \text{ such that } a \wedge f = b \wedge f.$$

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Corollary

Let $\mathfrak{A}$ be a non-trivial basic algebra, then the following conditions are equivalent.

1. *$\mathfrak{A}$ is subdirectly irreducible,*
2. *$\mathfrak{A}$ contains a least prime filter(least with respect to the inclusion relation),*
3. *$\mathfrak{A}$ has a second greatest element.*

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The only simple algebras in the variety of Basic algebras are two elements Basic algebras.

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Embedding

L_1-algebra

A basic algebra $\mathfrak{A}$ is called an L_1-algebra iff $\Box 0 = 1$. $\mathbf{2}^{\mathbf{I}}$ is the zero-generated L_1-algebra.

Proposition

Let $\mathfrak{A}$ be an L_1- algebra with $a, b \in A$ such that $a \not\leq b$. Then there is a homomorphism h of $\mathfrak{A}$ onto $\mathbf{2}^{\mathbf{I}}$ so that $h(a) \not\leq h(b)$.

Theorem(**Embedding**)

Let $\mathfrak{A}$ be an L_1-algebra. Then there is an index set I such that $\mathfrak{A}$ can be embedded into $\prod_{i \in I} \mathbf{2}^{\mathbf{I}}$.

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Minimal varieties

Theorem

The minimal varieties of basic algebras are the class of all Boolean algebras and the class of all $\mathbf{L_1}$*-algebras.*

Proof.

Let $\mathcal{V}$ be a non-trivial subvariety of the variety of all basic algebras. Then $\mathcal{V}$ has a simple algebra $\mathfrak{B}$. So $\mathfrak{B}$ is either $\mathbf{2}$ or $\mathbf{2^I}$. Therefore $\mathcal{V}$ contains either $V(\mathbf{2})$ or $V(\mathbf{2^I})$, i.e., $\mathcal{V}$ contains either the class of all Boolean algebras or the class of $\mathbf{L_1}$-algebras. □

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Maximal intermediate logics

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CPC *and* $\mathbf{F}_1$ *are the only maximal intermediate logics among the intermediate logics ordered by* $\subseteq$*. Each intermediate logic is contained in* **CPC** *or* $\mathbf{F}_1$*.*

Proof.

Let T be an intermediate logic and put $\mathcal{V}(T) = \{\mathfrak{A} \mid \mathfrak{A} \models T\}$. Then it contain either the class of all Boolean algebras or the class of all L_1-algebras. Hence T contained in **CPC** or $\mathbf{F}_1$. Note that **CPC** + $\mathbf{F}_1$ is inconsistent. □

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Definitions

Löb algebra

A basic algebra $\mathfrak{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$ is called a *Löb algebra*, *La*, iff for all $x \in A$, $\Box x \rightarrow x = \Box x$, where $\Box x = 1 \rightarrow x$.

L_n-algebra

A basic algebra $\mathfrak{A}$ is called an L_n-algebra, for $n \in N$, iff $\Box^n 0 = 1$.

Remark

$\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots \subseteq \mathcal{L}_n \subseteq \ldots \subseteq \mathcal{L} \subseteq \mathcal{B}$.

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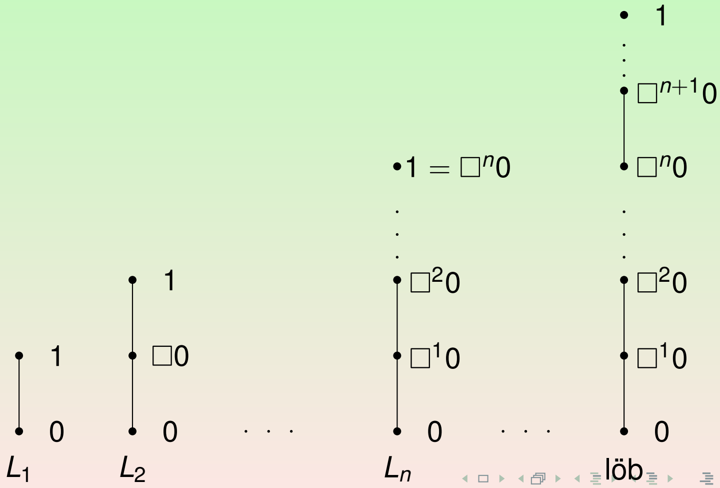
Zero-generated $\mathbf{L_n}$-algebras



Some facts

Proposition

Every L_n-algebra, for $n \in N$, is a Löb algebra.

Proof.

For every $m \geq 0$, we have $\Box^{m+1} a \to a = \Box a \to a$.
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Example

Consider the set of natural numbers with Sup ω (i.e., the ordinal $\omega + 1$) and define $n \wedge m := min(n, m)$, $n \vee m := max(n, m)$,

$$n \to m = \begin{cases} m + 1, & \text{if } n > m, \\ \omega, & \text{if } n \leq m, \end{cases} \text{ for } n, m \in \omega + 1.$$

$\perp := 0$ and $\top := \omega$. The ordering is the natural one. This Löb algebra is not a $\mathbf{L}_n$-algebra for any $n \in N$.

Some facts

Proposition

Every Löb algebra with no infinite chain of elements is a L_n-algebra, for some $n \in N$. In particular every finite Löb algebra is a L_n-algebra, for some $n \in N$.

Amalgamation

proposition

Every proper filter of an $\mathbf{L_n}$-algebra is irreflexive.

proposition

The length of every chain of prime filters with respect to $\prec$ in an L_n-algebra is at most n.

Corollary

The Stone algebra of an L_n-algebra is an $\mathbf{E}_n$-algebra.

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The variety $\mathcal{L}_n$, for $n \in N$, has the amalgamation property.

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Thank You!