Domain Theory in Logical Form Revisited: A 20 Year Retrospective

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The Big Picture

A Basic Intuition

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- The **axiomatic view**: we axiomatize $P \models \phi$ as a logical theory of which properties P satisfies. Thinking of properties as "observations", we can then seek to construct the meaning of P out of the properties it satisfies:

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How can we reconcile these views, and indeed bring them into exact correspondence? An elegant and robust mathematical framework for these ideas is provided by **Stone Duality**.

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$$U_b = \{x \mid b \in x\}$$
Then $B \cong \mathsf{Clop}(\mathsf{Pt}(B))$.
- The spaces that arise this way (totally disconnected compact Hausdorff spaces) are the **Stone spaces**. For every Stone space S,
$$S \cong \mathsf{Pt}(\mathsf{Clop}(S)).$$

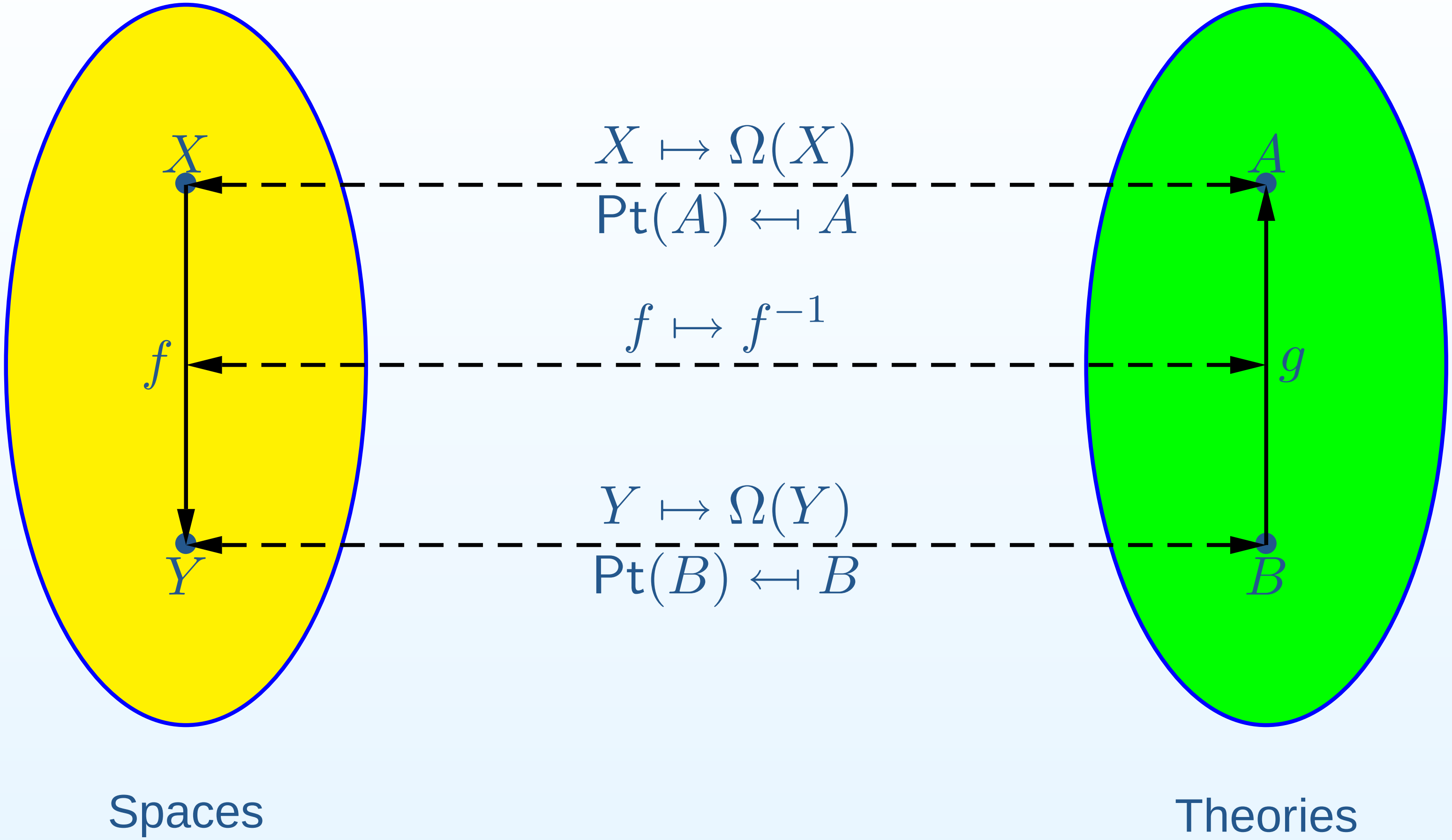
Stone Duality Logically

In logical terms:

- B is the Lindenbaum algebra of a propositional theory
- Points are models
- The compactness of $\mathsf{Pt}(S)$ subsumes the Compactness Theorem
- The existence of enough points to achieve the above isomorphisms subsumes the Completeness Theorem — *i.e.*

$$\phi \not\leq \psi \implies \exists x \in \mathsf{Pt}(B).\, x \models \phi \ \wedge \ x \not\models \psi.$$

Stone Duality Generalized



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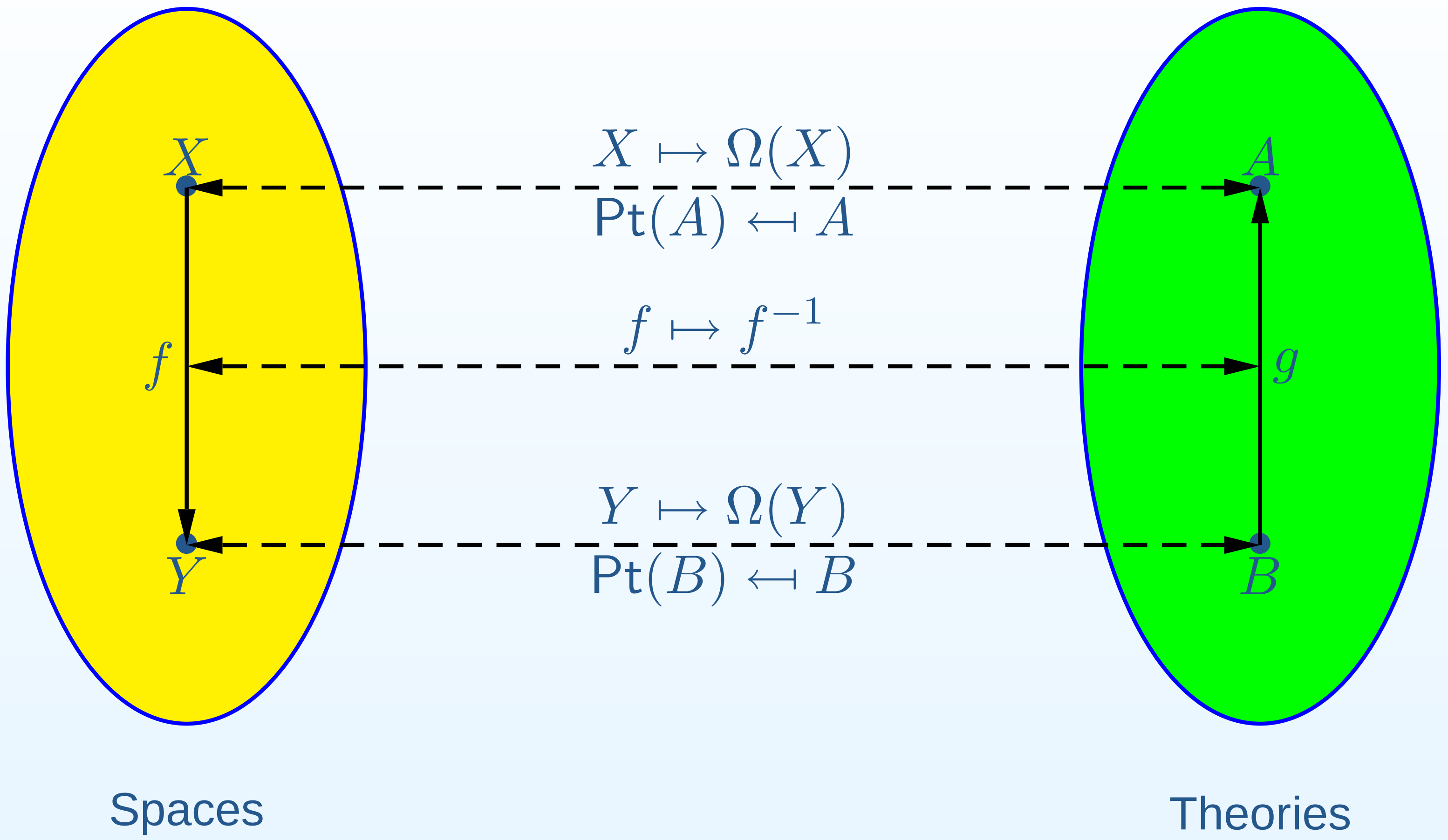

Note that **maps** are part of the picture, as well as spaces. This is important from the CS point of view — it generalizes the duality of **state transformers** and **predicate transformers**.

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- A generalized Dynamic Logic covering all the constructs of denotational semantics — higher order functions, recursive types, non-determinism, etc.
- Using the logical form to unpack the structure of complex, recursively defined semantic domains.

Domain Theory in Logical Form

Denotational Metalanguage

Syntax of Types:

$$\sigma \quad ::= \quad \sigma \times \tau \mid \sigma \rightarrow \tau \mid \sigma \oplus \tau \mid \sigma_\perp \mid P\sigma \mid X \mid \mathsf{rec}\, X.\, \sigma \mid \ldots$$

Typed terms $t : \sigma$.

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The programme:

- We assign, in a syntax-directed (*i.e.* compositional) fashion, a propositional theory $\mathcal{L}(\sigma)$ to each type σ, such that (the Lindenbaum algebra of) $\mathcal{L}(\sigma)$ is the **Stone dual** of $\mathcal{D}(\sigma)$, the domain associated (in conventional denotational semantics) to σ.

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- We axiomatize the meaning of terms $t : \sigma \to \tau$ as
  - Endogenous version: $\phi\{t\}\psi$
  - Exogenous version: $\phi \leq [t]\psi$

  with the intended meaning

$$[\![\phi]\!] \subseteq [\![t]\!]^{-1}([\![\psi]\!]).$$

Domain as Theories

This requires giving constructions on theories which yield the Stone duals of denotational constructions such as function spaces and powerdomains:

$$\begin{aligned}\mathcal{L}(\sigma \to \tau) &= \mathcal{L}(\sigma) \to_{\mathcal{L}} \mathcal{L}(\tau) \\ \mathcal{L}(P\sigma) &= P_{\mathcal{L}}(\mathcal{L}(\sigma))\end{aligned}$$

Recursive types are handled by inductive definitions of the logics — in effect, allowing arbitrary nesting of (generalized) modalities. Cf. current work on "coalgebraic logic".

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In this way, we can **read off presentations of complex semantic domains as propositional theories**, hence unpacking their structure.

Some Details

Stone Duality For Domains

"Domains" here are ω-algebraic cpo's. Can be viewed as topological spaces under the Scott topology.

We are interested in categories of such domains which are **cartesian closed**, and closed under various constructions, e.g. **powerdomains**:

- Scott domains
- SFP

Stone duality for these categories can be seen as restrictions for the Stone duality between distributive lattices and "coherent" or "spectral" spaces — those for which the compact-open sets generate the topology.

Some Trade Secrets

The lattices of compact-opens for domains are **coprime-generated**, *i.e.* every element is a finite join of coprimes, where a is **coprime** if:

$$a \leq \bigvee_{i \in I} b_i \;\; \Rightarrow \;\; \exists i. \, a \leq b_i.$$

(These correspond to “2/3 SFP domains”. SFP or Scott domains require additional axioms, which in the case of SFP are **non-elementary**.)

Coprimes correspond to the basic opens $\uparrow(a)$, a a compact element of D, in the Scott topology on D.

Our axiomatization involves a predicate $\mathbf{C}(a)$ for coprimeness. The key point in proving completeness (and effectiveness) of the axiomatization is to show that there are **effective coprime normal forms**.

Philosophical Aside

How should we interpret the basic concepts of Domain Theory:

$$d \sqsubseteq e$$

and “partial elements”, e.g. in $\mathbb{N}_\bot$, Σ^∞?

Two views:

- **Ontological**: Partial elements are possible states of the computation system, independently of any observer: necessary extensions to our universe of discourse.
- **Epistemic**: We (implicitly) assume an observer; (compact) partial elements are **observable properties**.

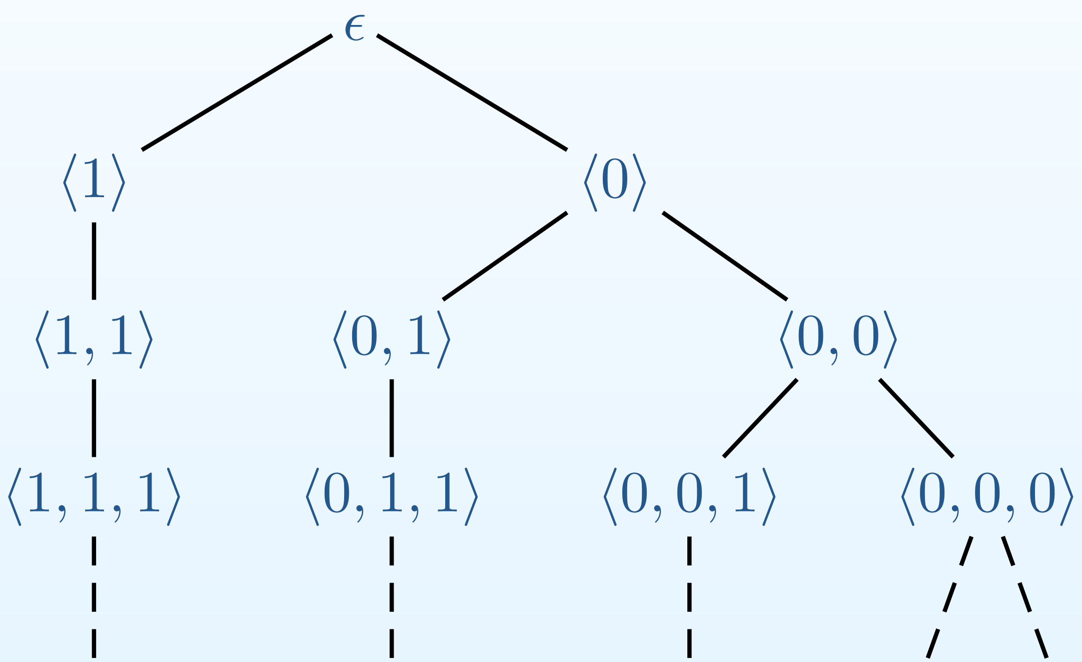
In fact, both readings are useful. The particular feature of domains which allows this creative ambiguity between points and properties to be used so freely is that **basic points and basic properties (or observations) are essentially the same things**. E.g. finite streams.

Domain Constructions In Logical Form

The Plotkin Powerdomain

If D is a domain, we want to make a domain $P(D)$ of subsets of D, to represent **non-deterministic computation over** D.

Which sets? the **finitely generable** ones.



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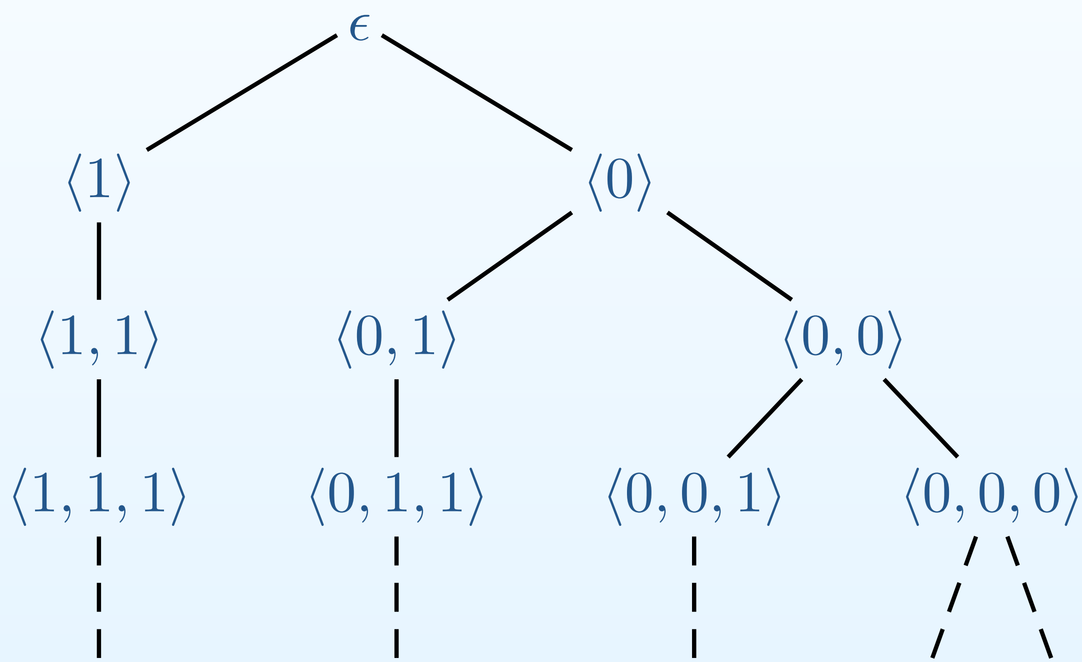

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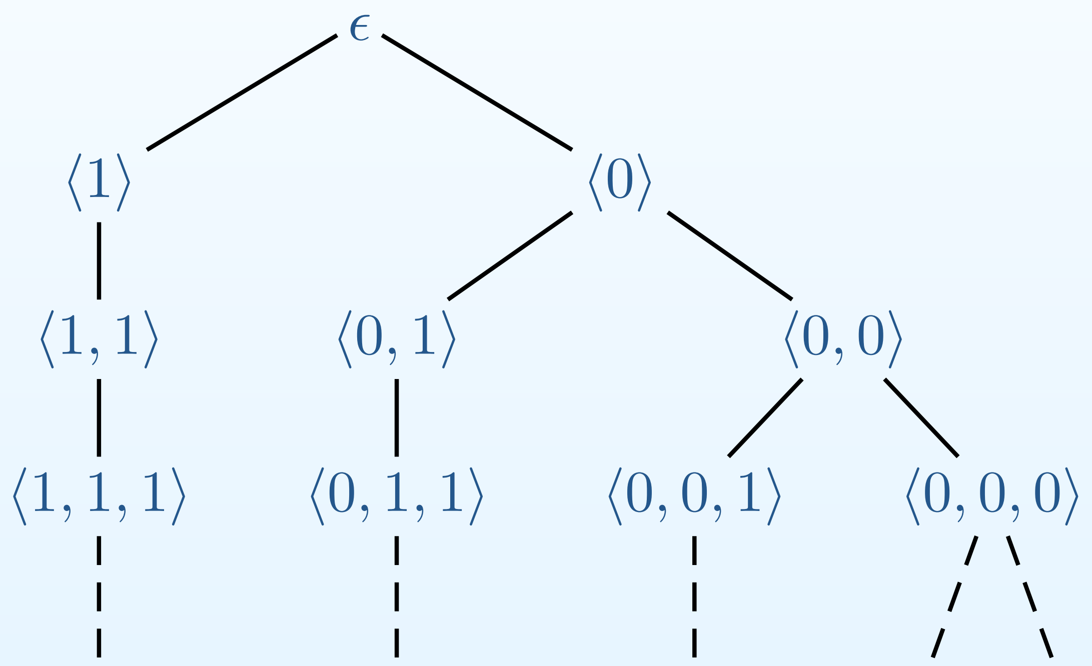

The set generated is $0^*1^\omega \cup 0^\omega$. Can we generate 0^*1^ω?

How to order sets

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Note that:

- Each node labelled with b in X_n has one or more successors in X_{n+1}, each labelled with some b' such that $b \sqsubseteq b'$.
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Abstracting from this situation, we have sets X and Y such that:

- $\forall x \in X. \exists y \in Y. x \sqsubseteq y$
- $\forall y \in Y. \exists x \in X. x \sqsubseteq y$

We write this as $X \sqsubseteq_{\mathrm{EM}} Y$: the **Egli-Milner order**.

Axiomatizing The Plotkin Powerdomain

(i) The generators:

$$G(P(A)) \equiv \{\Box a : a \in |A|\} \cup \{\Diamond a : a \in |A|\}$$

(ii) Axioms:

$$\begin{array}{ll}
(\Box - \wedge) & \Box \bigwedge_{i \in I} a_i = \bigwedge_{i \in I} \Box a_i \\
(\Diamond - \vee) & \Diamond \bigvee_{i \in I} a_i = \bigvee_{i \in I} \Diamond a_i \\
(\Box - \vee) & \Box (a \vee b) \leq \Box a \vee \Diamond b \\
(\Diamond - \wedge) & \Box a \wedge \Diamond b \leq \Diamond (a \wedge b) \\
(\Box - 0) & \Box 0 = 0
\end{array}$$

(iii) Rules:

$$(\Box - \leq) \quad \frac{a \leq b}{\Box a \leq \Box b}$$

$$(\mathbf{C} - \Box - \Diamond) \quad \frac{\{\mathbf{C}_A(a_i)\}_{i \in I} \quad (I \neq \varnothing)}{\mathbf{C}(\Box \bigvee_{i \in I} a_i \wedge \bigwedge_{i \in I} \Diamond a_i)}$$

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- There is a tight link between **bisimulation** and the Egli-Milner ordering, or the Vietoris construction, first identified in this setting, in my paper: "A Domain Equation for Bisimulation".
- Note that this Vietoris construction gives "one-level" or "flat" modalities. To get the usual iterated modalities we must combine this with a **recursive domain equation** — yielding a finer analysis.

Axiomatizing Function Spaces

(i) The generators:

$$G(A \to B) \;\equiv\; \{(a \to b) : a \in |A|, b \in |B|\}.$$

(ii) Relations:

$$(\to -\wedge) \qquad (a \to \bigwedge_{i\in I} b_i) = \bigwedge_{i\in I}(a \to b_i)$$

$$(\to - \vee -L) \qquad (\bigvee_{i\in I} a_i \to b) = \bigwedge_{i\in I}(a_i \to b)$$

$$(\to - \vee -R) \qquad \frac{\mathbf{C}_A(a)}{(a \to \bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a \to b_i)}$$

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Note the key use of coprimeness in $(\to - \vee -R)$.

Note also the resemblance to **intersection types** and **filter models** — again on one level, combining with recursive types to yield a finer analysis.

Coprimeness

The coprimeness axiom:

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Horrific — but **effective**.

Examples

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- A Domain Equation for Bisimulation

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  Connections with ideas from filter models and intersection types.

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  Connections with ideas from filter models and intersection types.
- The Finitary Non-Well Founded Sets

$$S = V(S)$$

  The Stone Space of the free modal algebra! Carries an interesting set-theory, in which the universe is a set.

Example: The Finitary Non-Well Founded Sets

The Finitary Non-Well-Founded Sets

A quiz for modal logicians:

What is the Stone space of the free modal algebra?

Which kind of set theory does it provide a model for?

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We study an example: the space $\mathbb{F}$ of **finitary non-well-founded sets**. By finitary we mean:

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We can describe $\mathbb{F}$ as **the Stone space of the free modal algebra (on no generators)**. Here we take a modal algebra to be a Boolean algebra B equipped with a unary operator $\Diamond$ satisfying the axioms

$$\text{(MA)} \quad \Diamond(a \vee b) = \Diamond a \vee \Diamond b \qquad\qquad \Diamond 0 = 0.$$

This is the algebraic variety corresponding to the minimal normal modal logic $\mathbf{K}$. The Boolean algebra is equipped with a constant 0, so the free algebra over no generators can be non-trivial. We shall show that it is indeed non-trivial!

First description of $\mathbb{F}$

We firstly describe $\mathbb{F}$, *qua* topological space, as the solution of a domain equation in $\mathbf{Stone}$. We use the **Vietoris construction** $\mathcal{P}_V$. Given a Stone space S, $\mathcal{P}_V(S)$ is the set of all compact (which since S is compact Hausdorff, is equivalent to closed) subsets of S, with topology generated by

$$\Box U = \{C \mid C \subseteq U\} \qquad (1)$$
$$\Diamond U = \{C \mid C \cap U \neq \varnothing\} \qquad (2)$$

where U ranges over the open sets of S. We can read $\Box U$ as the set of all C such that C **must** satisfy U, and $\Diamond U$ as the set of C such that C **may** satisfy U. The allusion to modal logic notation is thus deliberate, and we shall shortly see a connection to standard modal notions.

Interlude: Domain Equations

We consider domain equations $X \cong F(X)$ for endofunctors $F : \mathcal{C} \longrightarrow \mathcal{C}$. We want **extremal solutions** of such an equation: either an **initial algebra** $\alpha : FA \rightarrow A$, or a **final coalgebra** $\beta : A \rightarrow FA$. (The Lambek lemma then guarantees that the arrow is an isomorphism). These concepts generalize the lattice-theoretic notions of least and greatest fixpoint. In most cases of interest, initial algebras can be constructed as colimits:

$$\lim_{\rightarrow}(\mathbf{0} \rightarrow F\mathbf{0} \rightarrow F^2\mathbf{0} \rightarrow \cdots)$$

generalizing the construction of the least fixpoint as $\bigvee_k F^k \bot$, while final coalgebras can be constructed as limits:

$$\lim_{\leftarrow}(\mathbf{1} \leftarrow F\mathbf{1} \leftarrow F^2\mathbf{1} \leftarrow \cdots)$$

generalizing the construction of the greatest fixpoint as $\bigwedge_k F^k \top$. (In the domain theoretic case, the **limit-colimit coincidence** means that the two constructions coincide, and we obtain **both** an initial algebra **and** a final coalgebra.) For the finitary case we are considering, the functors will be ω-continuous in the appropriate sense, and the limit or colimit can be taken with respect to the ω-chain of finite iterations.

First description

We define $\mathbb{F}$ as the **final coalgebra of the Vietoris functor on $\mathbf{Stone}$**. Since $\mathcal{P}_V$ is cocontinuous, $\mathbb{F}$ is constructed as the limit of the ω^{op}-chain

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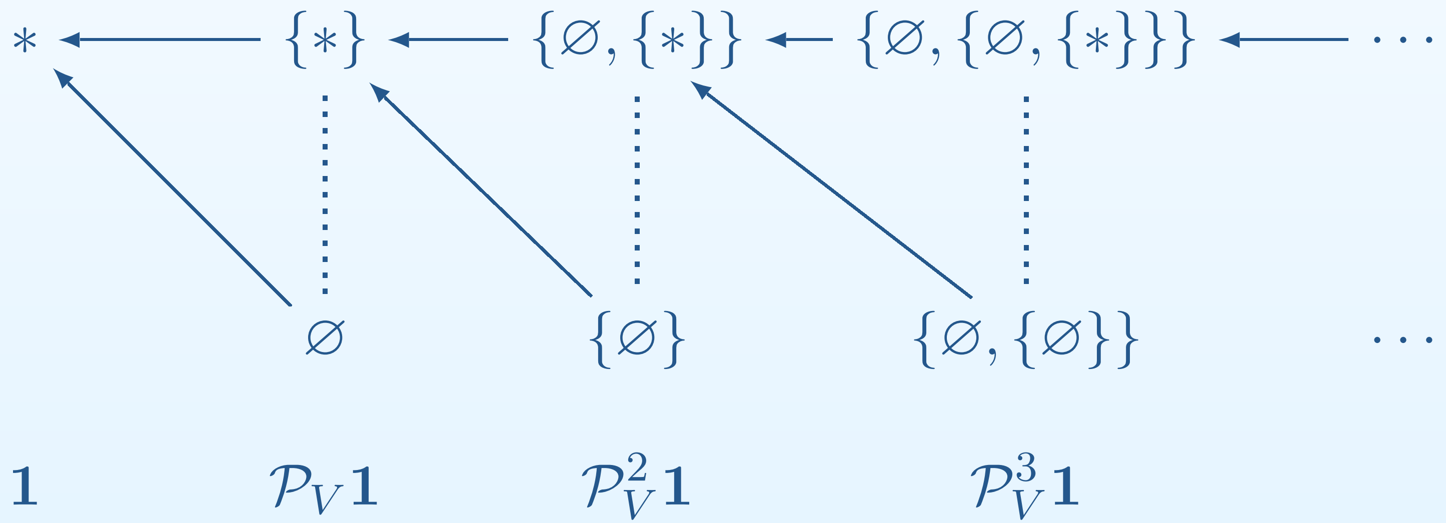
We give a picture of the first few terms of the construction:

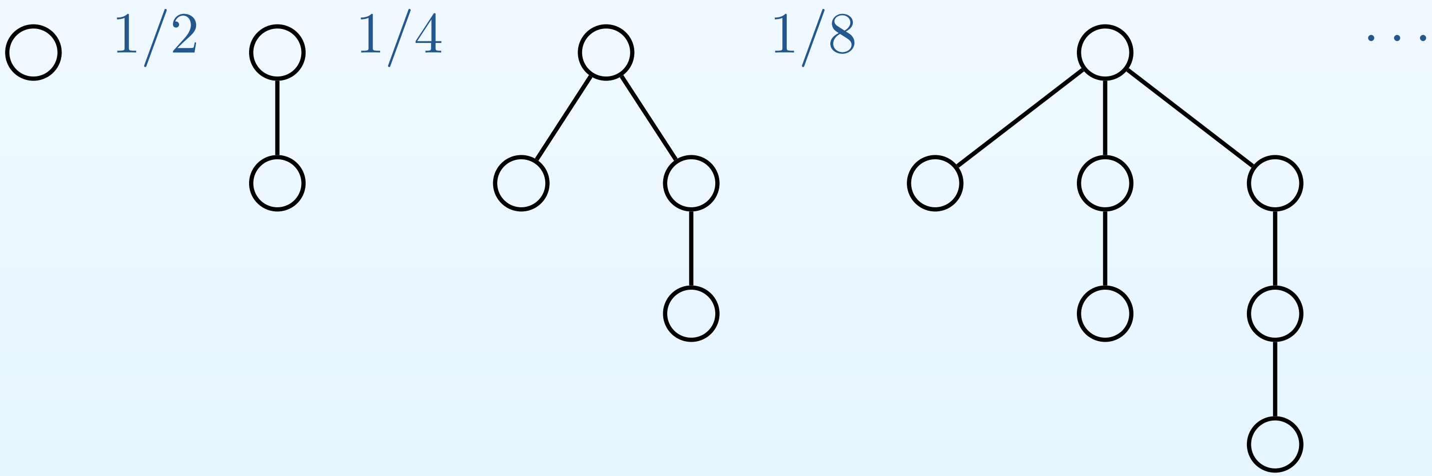


$\mathbb{F}$ As An Ultrametric Completion

$\mathbb{F}$ can equivalently be described as the **ultrametric completion of the hereditarily finite sets**.

$$d(S,T) \quad = \quad \begin{cases} 0, & S \sim T \\ 2^{-k}, \ \text{least } k \text{ such that } S \not\sim_k T & \text{otherwise} \end{cases}$$

Example of Cauchy Sequence:



The corresponding sequence of sets is

$$\varnothing, \quad \{\varnothing\}, \quad \{\varnothing, \{\varnothing\}\}, \quad \{\varnothing, \{\varnothing\}, \quad \{\{\varnothing\}\}\}, \ldots$$

Vietoris Logically

As we have seen, the Vietoris construction can be described **logically** as an operation on **theories**. For the coherent case, $V(L)$, for a distributive lattice L, is the distributive lattice generated by $\Box a$, $\Diamond a$, ($a \in L$), subject to the axioms:

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \Diamond(a \vee b) = \Diamond a \vee \Diamond b \tag{3}$$

$$\Box 1 = 1 \qquad \Diamond 0 = 0 \tag{4}$$

$$\Box(a \vee b) \leq \Box a \vee \Diamond b \qquad \Diamond(a \wedge b) \geq \Diamond a \wedge \Box b. \tag{5}$$

In the boolean case, where we have a classical negation, $\Box$ and $\Diamond$ are inter-definable (e.g. $\Box a = \neg \Diamond \neg a$), and the axiomatization simplifies to (MA).

Free Modal Algebra As A Fixpoint

The construction $\mathsf{MA}(B)$ lifts to a functor on $\mathbf{Bool}$, the category of Boolean algebras. We can iterate this construction to get the initial solution of $\mathbb{B} = \mathsf{MA}(\mathbb{B})$ in $\mathbf{Bool}$:

$$\lim_{\rightarrow}(\mathbf{2} \hookrightarrow \mathsf{MA}(\mathbf{2}) \hookrightarrow \mathsf{MA}^2(\mathbf{2}) \hookrightarrow \cdots)$$

Concretely this is the Lindenbaum algebra of the propositional theory which is inductively generated by these iterates. This is the standard modal system $\mathbf{K}$—but with no propositional atoms. Thus another role for domain equations is revealed: systematizing the inductive definition of the formulas and inference rules of a logic.

To see how hereditarily finite sets can be completely characterized by modal formulas (the "master formula" of the set), we define:

$$\mathcal{F}(\varnothing) = \Box 0 \qquad (= \neg\Diamond 1) \tag{6}$$

$$\mathcal{F}(\{x_1, \ldots, x_n\}) = \Box \bigvee_{i=1}^{n} \mathcal{F}(x_i) \wedge \bigwedge_{i=1}^{n} \Diamond \mathcal{F}(x_i). \tag{7}$$

The Characterization

The link between $\mathbb{B}$ and $\mathbb{F}$ is given by **Stone duality**:

Proposition 1 $\mathbb{B}$ *is the Stone dual of* $\mathbb{F}$.

Again, this is an instance of very general results.

Relation To Domains

We define a domain D as the solution (both initial algebra and final coalgebra) of the equation

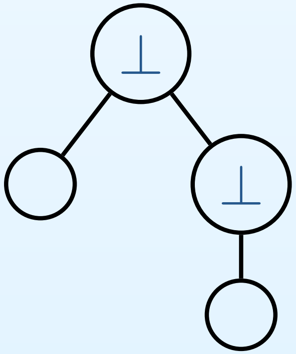
$$D \;=\; \mathcal{P}^0_P(D) = \; \mathbf{1}_\perp \oplus \mathcal{P}_P(D). \tag{8}$$

Here $\mathcal{P}_P(\cdot)$ is the Plotkin powerdomain.

Proposition 2 $\mathbb{F} \cong \mathsf{Max}(D)$, *where* D *is the solution of the domain equation (8).*

We note that D has "partial sets".

Example 3



$\leadsto \quad \{\perp, \varnothing, \{\perp, \varnothing\}\}$

Set Theory In $\mathbb{F}$

We now consider $\mathbb{F}$ as a set-theoretic universe $(\mathbb{F}, \in, =)$. Since we have

$$\mathbb{F} \cong \mathcal{P}_V(\mathbb{F}) \qquad \mathsf{unfold} : \mathbb{F} \xrightarrow{\cong} \mathcal{P}_V(\mathbb{F})$$

we can define

$$S \in T \quad \equiv \quad S \in \mathsf{unfold}(T).$$

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This set theory, and generalizations to "κ-finitary" universes, has been studied by Forti and Honsell.

In Conclusion

Precursors and Successors

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Some Precursors (among many):

- Scott information systems, intersection types and filter models, Martin-Löf, Plotkin, Kozen, . . .
- Mike Smyth, *Powerdomains and Predicate Transformers: A Topological View*, in ICALP 1983. Smyth's slogan: open sets "are" c.e. sets. My slogan: open sets are **finitely observable properties**.

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Some Successors (among many):

- Thomas Jensen, Strictness Analysis in Logical Form
- Marcelo Bonsangue et al.
- Achim Jung and Drew Moshier
- Michael Huth, Marta Kwiatkowska et al.
- Coalgebraic Logic (e.g. Moss, Jacobs, Kurz, Venema et al.)
- Work on Quantales (Resende et al.)
- Work on Constructive Mathematics taking the observational point of view (e.g. Spitters)

References

These papers are all available from my web pages
`http://web.comlab.ox.ac.uk/oucl/work/samson.abramsky/`

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