Algebraic and Topological Methods in Non-Classical Logics II

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Index of abstracts

INVITED LECTURES

Subframe logics, nuclei, and pointless topologies

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In this talk I will present a particular instance of fruitful use of algebraic and topological methods in non-classical logics by linking subframe logics with pointless topologies through nuclei, which are unary operations on Heyting algebras satisfying certain identities. The advantages of this approach will be discussed in detail.

Maps and monads for modal frames

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What is the appropriate notion of "morphism" for general modal frames? This talk will give an answer by defining a notion of "modal map" between frames, generalizing the usual notion of bounded morphism/p-morphism, and reducing to it in the case of Kripke frames. The category **Fm** of all frames and modal maps has reflective subcategories **CHFm** of compact Hausdorff frames, **DFm** of descriptive frames, and **UEFm** of ultrafilter enlargements of frames. All three subcategories are equivalent, and are dual to the category of modal algebras and their homomorphisms.

The ultrafilter enlargement of a frame is the free compact Hausdorff frame generated by that frame relative to **Fm**. This free construction has an associated "monad" whose Eilenberg-Moore category is isomorphic to **CHFm**. An equivalence between the Kleisli category of the monad and **UEFm** can defined from a construction that assigns to each frame a unique image-closed frame with the same ultrafilter enlargement (an image-closed frame is one in which the set of alternatives of any point is topologically closed).

These ideas are connected to a certain category shown by S. K. Thomason to be dual to the category of complete and atomic modal algebras and their homomorphisms. Thomason's category turns out to be the full subcategory of the above Kleisli category that is based on the Kripke frames.

Some recent developments in canonicity

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A modal logic is said to be canonical if it is validated by the frame of its canonical model, constructed from maximal consistent sets. Canonicity came to prominence in the mid-1960s and is still perhaps the most popular way of showing Kripke completeness of a modal logic. But there exist non-canonical Kripke complete logics; so a natural question is to characterise the canonical modal logics. Fine, van Benthem, and Goldblatt showed (among other things) that any modal logic sound and complete for some elementary class of frames is canonical, and generalised this to boolean algebras with operators. There are counterexamples to the converse (joint work with Goldblatt and Venema), but currently, the causes of canonicity in them have no general explanation. I will discuss this and related issues, such as the phenomenon of canonical logics with no canonical axiomatisation.

Algebraic Gentzen systems and ordered structures

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Gentzen systems have been used extensively to describe logics and decision procedures. While the standard techniques are largely syntactic, there have been several approaches to providing semantics for Gentzen systems, such as [1,3,4,5,9]. Here we present another algebraic approach to propositional Gentzen systems using quasivarieties of ordered structures. We consider Gentzen rules as quasiidentities over a signature that includes operation symbols for all logical connectives and punctuation symbols (e.g. comma) as well as a binary relation symbol for the sequent separator. Hence sequences of formulas correspond to algebraic terms and sequents are atomic formulas. This allows standard Tarskian semantics to be used for models of the Gentzen systems.

Several varieties of ordered algebras have natural axiomatizations by quasiidentities derived from Gentzen systems of the corresponding logics. In particular, semilattices, lattices, idempotent semirings [6], residuated lattices [7], and several subvarieties and subreducts of these classes of algebras are examined from this point of view. The cut rule corresponds to transitivity of the sequent separator relation, and the axiomatization is particularly helpful if this rule can be eliminated. E.g. for lattices this is simply Whitman's 1941 solution for the word problem in free lattices (already discovered in 1920 by Skolem, see [2]).

Viewing Gentzen systems as axiomatizations of quasivarieties allows them to be easily extended with additional algebraic axioms and provides a simple way of implementing them within existing theorem provers and rewrite systems. We will illustrate this with an implementation of a Gentzen system for Kleene algebras (as well as residuated Kleene algebras with tests) and for generalized BL-algebras (neither of them is known to be cut-free). In each case the automated theorem prover Otter [8] succeeds to prove results from the Gentzen-style quasiequational basis but may not find proofs of the same results from a standard equational basis.

In the latter part of the talk we present some additional results about generalized BL-algebras. In particular we show that all finite ones are commutative, and their congruence lattices are determined by a maximal Heyting subalgebra. It follows that all finite generalized

BL-algebras can be constructed from MV-algebras through a generalization of the ordinal sum construction.

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Amalgamation and interpolation in propositional many-valued logics

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Interpolation is an important property in logic. The book [GM] includes an interesting discussion, together with an almost complete investigation of interpolation in the case of modal and superintuitionistic logics. Roughly speaking, interpolation means that if B can be derived from A, then the relevant information from A which is needed to derive B only contains variables which are common to A and B. There are at least two kinds of interpolation in a logic L: (a) implicative (Craig) interpolation and (b) deductive interpolation. These versions can be formulated as follows:

(a) If $L \vdash A \to B$, then there is C in the common language such that $L \vdash A \to C$ and $L \vdash C \to B$.

(b) If $A \vdash_L B$, then there is C in the common language such that $A \vdash_L C$ and $C \vdash_L CB$.

(Here, $\vdash_L$ denotes the consequence relation in L). Deductive interpolation is connected to amalgamation in the corresponding variety. Such connections are shown in many papers, cf e.g. [CP], [O], [GO], [Ma], [Ma2], [Ma3]. In particular, Galatos and Ono [GO] proved that interpolation in a substructural logic extending FL_e is equivalent to amalgamation in the corresponding variety of (commutative) residuated lattices. This allows for an algebraic treatment of deductive interpolation.

An interesting consequence of (Craig's) interpolation is the Beth definability property, (we will formulate it wrt consequence relation, but a formulation by means of implication is also possible):

Suppose $A(p), A(q) \vdash_L p \leftrightarrow q$ (that is p is implicitly defined by $A(p)$). Then $A(p)$ explicitly defines p, that is, there is a formula D whose variables are in A and do not include p, such that $A(p) \vdash_L p \leftrightarrow D$.

Also the Beth property is related to an algebraic property, namely strong amalgamation. In general, strong amalgamation implies the Beth property but not viceversa. The situation of FL_e, FL_{ec} and FL_{ew} wrt interpolation and amalgamation is rather clear: due to cut-elimination, such logics have interpolation (both Craig and deductive), so the corresponding varieties have the amalgamation property. Problems arise if exchange is not present. The situation of superintuitionistic logics is also well-known: only a finite number of them have interpolation. Moreover, interpolation, amalgamation and strong amalgamation are equivalent for such logics [Ma], [Ma2]. For these logics, the Beth property is weaker than interpolation: all these logics have the Beth property [Kr]. Thus the Beth property does not imply strong amalgamation in this case (and not even interpolation).

In this paper we investigate the above properties (amalgamation, strong amalgamation, deductive interpolation and Beth's property) for the extensions of Hájek's logic BL. Some results are known: for instance MV-algebras have amalgamation [Mu2], and a variety of MV-algebras has the amalgamation property iff it is generated by a chain [DNL2]. The only extension of Łukasiewicz logic having Craig interpolation are classical logic and the inconsistent logic [K]. The only varieties of Gödel algebras having the amalgamation property are the whole variety of Gödel algebras, the variety of Boolean algebras and the variety generated by the three element Gödel algebra [Ma], [Ma2].

Here are our main results:

1. BL algebras, and product algebras have the amalgamation property.
2. There are uncountably many varieties of BL algebras without the amalgamation property.
3. Let Π, **MV** and **G** denote the varieties of product algebras, of MV-algebras and of Gödel algebras. Then the join of two or of three of these varieties has not the amalgamation property except from the join of Π and **G**, (which has the amalgamation property).
4. There are only three varieties of BL-algebras having the strong amalgamation property, and all of them are subvarieties of **G**.
5. There are only three extensions of BL with the Craig interpolation property, and all of them are extensions of Gödel logic.
6. The extensions of BL having the Beth property are precisely the extensions of Gödel logic.

Problems

Do MTL algebras have the amalgamation property?

How many varieties of BL algebras have the amalgamation property? (Conjecture: countably many).

Do commutative GBL-algebras have the amalgamation property? (Conjecture: YES).

Which of the above varieties have the joint embeddability property?

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What can the theory of canonical extensions and dualities tell us about MV-algebras and ℓ-groups?

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The theory of canonical extension has advanced by leaps and bounds during the past decade. It now provides concrete representations, often encoding useful topological dualities, for many varieties of ordered algebras, and through correspondence theory yields powerful relational semantics for a great range of non-classical logics. Both Jónsson and Tarski's seminal work on BAOs and Goldblatt's landmark paper 'Varieties of complex algebras' concentrated on operations (on Boolean algebras and, in the latter case, distributive lattices) which preserve coordinatewise *one* of $\vee$ and $\wedge$. In the unary case this is, of course, just what is required for modal logic. With MV-algebras and ℓ-groups we have *binary* operations which interact well coordinatewise with *both* $\vee$ and $\wedge$. Surely it should be fruitful to study such algebras via canonical extensions. Or maybe not? A variety of MV-algebras is canonical only if it is finitely generated. And N.G. Martínez in the 1990s published no fewer than three alternative dual representations, no one of which seems definitive.

This talk (based on joint work with Mai Gehrke) will present a generalised canonicity theorem applicable, *inter alia*, to MV-algebras and to ℓ-groups, will discuss the interrelationships between the available representations, and will highlight the role played by residuation.

Structural completeness in substructural logics

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A consequence relation C is said to be *structurally complete* if every proper extension of C has some new *theorems* (as opposed to nothing but new rules of inference). In this case, a proper extension of C need not itself be structurally complete. We call C *hereditarily structurally complete* (*HSC*) if every extension of C is structurally complete—including C itself. This turns out equivalent to the demand that every extension of C be an *axiomatic* extension. So when C is HSC, everyone can agree about the meaning of 'a logic over C', whether we conceive of 'logics' as sets of theorems or just as consequence relations.

Partly for this reason, the problem of structural completeness has been studied quite extensively in the context of intermediate and modal logics. It is well known that all significant fragments of classical (propositional) logic are structurally complete, while for intuitionistic logic, the structurally complete fragments are just those that don't contain both implication and disjunction.

In relevance logic, J.K. Slaney and R.K. Meyer proved that the implication-conjunction fragment of the system **R** is structurally complete, and they conjectured the same for the fragment with implication, conjunction and the Ackermann constant t (which behaves like the $\mathbf{1}$ of linear logic). Their proof is quite intricate and it relies on the contraction axiom $(x \to (x \to y)) \to (x \to y)$. Apparently, no results for logics strictly more general than **R** were obtained previously.

This talk will present some new results concerning structural completeness in more general substructural logics. It is well known that *commutative residuated lattices* (CRLs) constitute the equivalent algebraic semantics for positive linear logic (without exponentials). All results to be presented are obtained by algebraic methods. We consider suitable varieties V of *subreducts* of CRLs, i.e., subalgebras of reducts of CRLs, modeling various extensions of fragments of linear logic. To establish that such an extension is HSC, we prove that all subquasivarieties of V are varieties. To disprove structural completeness, we find a proper subquasivariety W of V such that V is the homomorphic closure of W.

The most general system that we consider is the extension $\mathbf{LL}^n$ of linear logic by the axiom $x^n \to x^{n+1}$, where $n > 1$ is arbitrary. It will be proved that *the implication-conjunction fragment of* $\mathbf{LL}^n$ is HSC. Any enlargement of the signature destroys this result. We shall address the question: in which *extensions* of $\mathbf{LL}^n$ can extra connectives like disjunction and fusion be added to the signature without destroying hereditary structural completeness? Some consequences for fragments of certain many-valued logics are obtained in the process.

Setting $n = 1$ in the above result, we get a strictly stronger version of Slaney and Meyer's theorem, viz. *the implication-conjunction fragment of* **R** *is HSC*. We also show that *the full* *negation-free fragment of* **R**–*mingle is HSC.* Here **R**–mingle is formulated without the Ackermann contant t.

Disproving Slaney and Meyer's conjecture, we shall show that *the implication-conjunction-t fragment of* **R** *is not structurally complete.* (This system is also an exact fragment of contractive linear logic.) More generally, our proof shows that in a wide range of substructural logics, the presence of the constant t prevents structural completeness.

The proof breaks down, however, in **R**–mingle. Here, by a different argument, we establish that when **R**–mingle is formulated with t, the full negation-free fragment (including t) *is* HSC.

CONTRIBUTED TALKS

Poset representation of free algebras in some varieties of residuated structures

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Concrete representations of free algebras in varieties constituting the algebraic semantics of a logic, are powerful analysis tools for that logic. A notable example is McNaughton's representation theorem of free n-generated MV algebra (the variety of MV algebras is the algebraic counterpart of Łukasiewicz logic) as the set of continuous piecewise linear functions from $[0,1]^n$ to $[0,1]$, with each of the finitely many pieces having integer coefficients, equipped with pointwise defined operations. McNaughton's theorem allowed Mundici to prove NP-completeness of the satisfiability problem for Łukasiewicz infinite-valued logic.

In some cases free algebras turn out to be even simpler objects. Consider the logic MTL of all left-continuous t-norms and their residua. Its algebraic semantics is constituted by the variety $\mathbf{V}(\mathrm{MTL})$ of MTL-algebras. Among schematic extensions of MTL there are infinite-valued logics $\mathcal{L}$ such that the finitely generated free algebras in the corresponding subvariety $\mathbf{V}(\mathcal{L})$ of $\mathbf{V}(\mathrm{MTL})$ are finite. Here we shall deal with finitely generated free algebras of Gödel and Nilpotent Minimum logics and we represent them combinatorially as algebras of sections over finite posets. As a byproduct, we shall count the number of their elements.

Operations of free algebras in those varieties are understood in order-theoretical terms (they are comparisons between variables or negated variables), while arithmetic plays no significant rôle: this is in sharp contrast for instance with Łukasiewicz logic, where operations are truncated sums and subtractions.

We recall that the (disjoint) union, or *horizontal sum* $A \cup B$ of two disjoint posets A and B is the poset formed by defining $x \leq y$ if and only if either $x, y \in A$ and $x \leq y$ in A, or $x, y \in B$ and $x \leq y$ in B. The linear sum, or *vertical sum* $A \oplus B$ of two disjoint posets A and B is defined by taking the following ordered relation: $x \leq y$ if and only if either $x, y \in A$ and $x \leq y$ in A, or $x, y \in B$ and $x \leq y$ in B, or $x \in A$ and $y \in B$. In the following we shall always assume that posets involved in horizontal and vertical sums are disjoint, by taking isomorphic copies of operands when necessary. We write $\mathbf{1}$ to denote the poset containing only one element. A chain with n elements is isomorphic to $\mathbf{1} \oplus \cdots \oplus \mathbf{1}$, n times, which we denote by $\mathbf{n}$. By nA we denote $A \cup A \cup \cdots \cup A$ n-times. For any poset A we set $A_\perp := \mathbf{1} \oplus A$. Each poset $\langle A, \leq \rangle$ is order isomorphic to a poset $\langle o(A), \leq \rangle$ obtained by replacing each element of A with a copy of $\mathbf{1}$. We call $o(A)$ the *type* of A, since $o(A)$ retains only the order theoretic information about A.

Definition. A finite poset A is *nice* if its type $o(A)$ is described using only operations $\mathbf{1}$, $\cup$ and $\oplus$.

Let A be a finite poset. A *branch* B of A is a chain of maximal length in A, i.e., is a set of elements $b_1 < \ldots < b_u$ with b_1 minimal element in A, b_u maximal element in A and such that if $B \subseteq B' \subseteq A$ and B' is a chain then $B = B'$ or $B' = A$. A *section* over A is a sequence of elements $(p_i)_{i \in I}$ of A such that for each branch B in A there exists exactly one $p_i \in B$. The set of sections over A is denoted by $S(A)$.

Theorem. The free Gödel algebra $Free_n(\mathrm{G})$ over n variables is isomorphic to the Gödel algebra of sections over a nice poset G_n, its type being $o(G_n) = H_n \cup (H_n)_\perp$ with $H_0 = \mathbf{1}$ and

$$H_n = \bigcup_{i=0}^{n-1} \binom{n}{i} (H_i)_\perp \,.$$

Further, $|Free_n(\mathrm{G})| = |S(H_n)|(|S(H_n)| + 1)$, for $|S(H_n)| = \prod_{i=0}^{n-1}(|S(H_i)| + 1)^{\binom{n}{i}}$.

Example. The type of the poset underlying the representation of $Free_2(\mathrm{G})$ as an algebra of sections is $o(G_2) = \mathbf{2} \cup \ \mathbf{3} \cup \ \mathbf{3} \cup \ (\mathbf{2} \cup \ \mathbf{3} \cup \ \mathbf{3})_\perp$.

Theorem. The free NM-algebra over n variables is isomorphic to the NM-algebra of sections over the nice poset NM_n its type being

$$o(NM_n) = \left(\bigcup_{i=0}^{n-1} \binom{n}{i} 2^i (K_i \oplus \mathbf{1} \oplus K_i) \right) \cup 2^n (K_n \oplus K_n) \,,$$

with

$$K_i = \bigcup_{k=0}^{i} k! \left\{ {i \atop k} \right\} (\mathbf{k}_\perp) \,,$$

where $\left\{ {i \atop k} \right\}$ are the Stirling numbers of the second kind. Further,

$$|Free_n(\mathrm{NM})| = |S(K_n)|^{2^{n+1}} \prod_{i=0}^{n-1} (2|S(K_i)| + 1)^{2^i \binom{n}{i}},$$

where $|S(K_i)| = \prod_{k=0}^{i} (k + 1)^{k! \left\{ {i \atop k} \right\}}$.

Example. The type of NM_2 is given by:

$$o(NM_2) = \mathbf{3} \ \cup \ 4(\mathbf{5}) \ \cup \ \ 4((\mathbf{3} \ \cup \ \mathbf{2} \ \cup \ \mathbf{3}) \oplus (\mathbf{3} \ \cup \ \mathbf{2} \ \cup \ \mathbf{3})).$$

Our approach can be extended to other varieties with analogous properties.

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On the Löb algebras

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Formal propositional logic, **FPC**, invented by A. Visser in 1981, is the propositional logic of the provability logic or the Gödel-Löb logic. We study the variety of Löb algebras, the algebraic structures associated with (**FPC**). Among other things, we prove a completeness theorem for **FPC** with respect to the variety of all Löb algebras. We show that the variety of Löb algebras has the weak amalgamation property. Some interesting subclasses of the variety of Löb algebras, e.g., linear, faithful and strongly linear Löb algebras are studied.

Deduction theorems in logics without exchange

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For substructural logics that have the Exchange rule, it is known that a local deduction theorem holds, for example, in the case with Weakening,

$$\Gamma, \varphi \vdash \psi \quad \text{iff} \quad \Gamma \vdash \varphi^n \to \psi \quad \text{for some natural number } n.$$

Furthermore, a deduction theorem holds iff the logic satisfies some form of n-potence axiom:

$$\vdash \varphi^n \to \varphi^{n+1},$$

in which case

$$\Gamma, \varphi \vdash \psi \quad \text{iff} \quad \Gamma \vdash \varphi^n \to \psi.$$

When the Exchange rule is omitted the situation is significantly altered. The Full Lambek Calculus **FL** (i.e., IPC without Exchange, Contraction and Weakening) has no local deduction theorem. Thus, the problem arises of characterizing the axiomatic extensions of **FL** that have a deduction or local deduction theorem.

We approach this problem algebraically. It is well-known that the algebraic equivalents of the deduction and local deduction theorem are, respectively, equationally definable principal congruences (EDPC) and the congruence extension property (CEP). The equivalent algebraic semantics for **FL** is the variety $\mathcal{R}$ of residuated lattices. Thus, we seek to characterize the subvarieties of $\mathcal{R}$ that have EDPC or the CEP. We make strong use of the fact that the varieties are ideal determined so that we may consider ideals rather than congruences. A finite basis of ideal terms for $\mathcal{R}$ will be given, as well as a simpler one for the integral subvariety of $\mathcal{R}$. We define the following ideal-version of EDPC:

A class $\mathcal{K}$ has EDPI* if there exists an ideal term $u(x, y)$ in x for $\mathcal{K}$ such that for all $\boldsymbol{A} \in \mathcal{K}$ and $a, b \in A$, $b \in \langle a \rangle^{\boldsymbol{A}}$ iff $b = u(a, b)$.

The main result is the following:

> **Theorem:** *If $\mathcal{V}$ is a variety of residuated lattices generated by a class $\mathcal{K}$, then $\mathcal{V}$ has EDPC if and only if $\mathcal{K}$ has EDPI*.*

As a consequence of this result, it is decidable whether a finite residuated lattice generates a variety with EDPC.

For the CEP, we define a condition PIEP*, similar in nature to EDPI*, for which we obtain the corresponding result:

> **Theorem:** *If $\mathcal{V}$ is a variety of residuated lattices generated by a class $\mathcal{K}$, then $\mathcal{V}$ has the CEP if and only if $\mathcal{K}$ has PIEP*.*

Some examples illustrating these results will be given.

The methods presented here for residuated lattices are applicable more generally to any variety that is either ideal determined or congruence permutable.

Weakly algebraizable Gentzen systems

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A deductive system (Hilbert-style) is an algebraic invariant closure system over the set of formulas of a given propositional language, where invariant means that it is closed under inverse substitutions. Similarly, a Gentzen system can be seen as an algebraic invariant closure system over the set of all sequents, i.e., finite sequences of formulas, of this language. (Substitutions act on sequents componentwise.) The main result of the work is a technique that allows us to adapt the methods, previously developed in the area of algebraic logic for Hilbert deductive systems, to the case of Gentzen systems. Using the properties of the Tarski congruence, a generalization of the Leibniz congruence, we develop an algebraic hierarchy for Gentzen systems that closely parallels the well-known algebraic hierarchy of Hilbert deductive systems. This approach allows us to unify in a single framework several previously known results about algebraizable and equivalential Gentzen systems. We also obtain a characterization of weakly algebraizable Gentzen systems.

The significance of Gentzen systems and related axiomatizations by Gentzen rules is due in large part to the fact that various metatheoretical properties of Hilbert deductive systems can be formulated in terms of Gentzen systems. In particular, it was observed that a number of important non-protoalgebraic deductive systems that have a natural algebraic semantics also have so-called fully adequate Gentzen systems associated with them, the conjunction-disjunction fragment of the classical propositional logic being a paradigmatic example. Using the fact that any fully adequate Gentzen system is weakly algebraizable in our sense, we formulate a general criterion for the existence of a fully adequate Gentzen system, which works both for protoalgebraic and non-protoalgebraic Hilbert deductive systems, and show that many of the known partial results can be explained based on this general criterion. This includes such cases as the existence of fully adequate Gentzen systems for self-extensional logics with conjunction or implication, and the criteria for the existence of a fully adequate Gentzen system for protoalgebraic and weakly algebraizable logics.

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The logic of quantum information flow

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I present a dynamic-epistemic logic for reasoning about quantum information flow. This presentation covers developments arising from recent joint work [2] with Sonja Smets on the *dynamic logic of entanglement* (itself a development of our joint work in [1] on the logic of *single* quantum systems). The setting combines modal, algebraic and topological reasoning about *compound quantum systems*, modeled as finite-dimensional Hilbert spaces.

Quantum Mechanics represents a system composed of n subsystemsas a *tensor product* $\mathcal{H} = \bigotimes_{1\leq i\leq n} H^{(i)}$ of n Hilbert spaces$H^{(i)}$. For the purposes of quantum computtion, it is enough to consider a system composed of n "qubits", i.e. the case when each basic Hilbert space $H^{(i)}$ is *two-dimensional.* In this case, one can choose for each space an orthonormal basis composed of two vectors (the "qubits") $\mid 0\rangle_i$ and $\mid 1\rangle_i$. The possible *experimental (or testable) properties* of a physical system are represented by the *closed linear subspaces* of $\mathcal{H}$. For a set $M \subseteq \mathcal{H}$ of vectors, we put $\overline{M}$ for the closed linear subspace generated by M. The testable properties form a non-distributive lattice $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$ (with set-inclusion as the order), with meet given by intersection (corresponding to *classical conjunction*) and join given by the *quantum join* $W \sqcup W' = \overline{W \cup W'}$ (which is thus different from classical disjunction, and encodes all the possible superpositions of states satisfying either of two properties). The *state* of a physical system is given by an *atom* of the lattice of properties, i.e. a *one-dimensional linear subspace* $s = \overline{x} := \overline{\{x\}}$ (for some non-zero vector $x \in \mathcal{H}$). We put Σ for the set of all possible states of $\mathcal{H}$. The possible *physical actions* that can be performed on the system are represented by*linear maps*on $\mathcal{H}$: e.g. a*successful measurement* (or a *"quantum test" S?*) of a physical property S corresponds to a *projector* P_W in $\mathcal{H}$ (onto the subspace W corresponding to S); the possible evolutions of a physical system in the absence of any measurement are given by (reversible) linear maps U, called *unitary transformations.* An important relation between properties (or states) is *orthogonality* $S \perp P$, which encodes a sense of "necessary failure" of a measurement: if the state has property S, then it is impossible to perform a successful measurement of property P. To represent non-deterministic actions, we close actions under arbitrary unions. Quantum actions are thus closed under arbitrary unions $R \cup R'$ and relational compositions $R; R'$.

For any action R and any set $S \subseteq \Sigma$ of states, we can consider the *weakest precondition* $[R]S = \{t \in \Sigma : \forall s \in \Sigma(tRs \Rightarrow s \in S)$ and the *image* $R(S) = \{t \in \Sigma : \exists s \in S\, sRt\}$. Any quantum action R has an *adjoint* $R^{\dagger}$, which is defined as the Galois dual of R w.r.t. orthogonality.

Let $N = \{1, 2, \ldots, n\}$ be the set of (all indices for) the basic parts (qubits) of our system. We also consider all *possible subsystems*, given by subsets $I \subseteq N$ of indices. A state s is said to be *I-separated* if it has a well-defined I-subsystem, i.e. if is of the form $s = \overline{x_I \otimes y_{N\setminus I}}$, with $x_I \in \bigotimes_{i\in I} H^{(i)}$, $y_{N\setminus I} \in \bigotimes_{j\notin I} H^{(j)}$. In this case, the state $s_I := \overline{y_I}$ is called the *(I-)local component (or local state)* of s. A state is *entangled* if it is not separated. An *(I-)local action* R is one that can be thought of as an action on the I-subsystem only, i.e. there exists a relation R_I on I-local states s.t. $R(\overline{x_I \otimes y_{N\setminus I}}) = \overline{R(x_I \otimes y_{N\setminus I})}$. For any unary linear map F and any two distinct qubits $i \neq j$, there exists a unique $\{i, j\}$-local state $\overline{R}_{ij}$ that is *"entangled according to R"*, i.e.: any measurement of qubit i performed on the system $\overline{R}_{ij}$ with (local) result x_i will collapse the qubit j to a local state $F(x)_j$. This "non-locality" captures the essence of entanglement as *correlation* of spatially separated local measurements. In Quantum Computation, one uses this property to encode "programs" F into entangled states $\overline{F}_{ij}$. See [3,2] for details.

In our setting, we think of any possible subsystem $I \subseteq N$ as a (potential) "agent", and we think of some of the quantum actions as being "done" by agents. We encode the fact that some information$P$ is *potentially available* to the local subsystem I by defining a notion of *knowledge*. We say that I *knows* P, and write $K_I P$, if the fact that the system satisfies P can be inferred only by looking at the I-subsystem: $s \in K_I P$ iff for all I-local programs π we have $\pi(s) \in P$. The underlying accessibility relation is *reflexive and transitive*, so the knowledge modality will satisfy the $S4$ axioms (truthfulness and positive introspection). The restriction of the accessibility relation to I-separated states is also *symmetric*, so that *knowledge is negatively introspective (i.e. satisfies the modal axiom* 5*) in all (*I*)-separated states.* For any action R, we consider an (I-)local action R_I: the intuitive meaning is that "agent I performs action R". We simply put: sR_It iff t is R is I-local, t is I-separated and sRt.

For a given set C of propositional constants (among which are $\mid 0\rangle$ and $\mid 1\rangle$), a given set of unitary operations $U \in \mathcal{U}$, and an set of propositional variables $p, q \ldots$, the (PDL-like) syntax of our quantum dynamic-epistemic logic consists of propositional *formulas* and of *programs*, defined by mutual induction:

$$\begin{array}{llllllllllll} \varphi ::= p & \mid & c & \mid & \neg\varphi & \mid & \varphi \wedge \varphi & \mid & [\pi]\varphi & \mid & K_I\varphi \\ \pi ::= \varphi? & \mid & U & \mid & \pi^\dagger & \mid & \pi \cup \pi & \mid & \pi ; \pi & \mid & \pi_I \end{array}$$

with the obvious semantics. I present a sound proof system for this logic, including an *Entanglement Axiom*, that captures the fundamental property of entangled states $\overline{R}_{ij}$ mentioned above. In this system, one can prove many non-trivial properties of quantum information flow in compound systems, in particular interesting theorems about the interplay between *knowledge, quantum measurements and entanglement* (including a formal correctness proof for the famous "Quantum Teleportation" protocol.

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The variety generated by the Rieger-Nishimura lattice

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We investigate the variety **RN** generated by the Rieger-Nishimura lattice. It is well known that the Rieger-Nishimura lattice is the one-generated free Heyting algebra. The equation theory of **RN** is the greatest 1-conservative extension of the equation theory of the variety of all Heyting algebras. In other words, the logic of **RN** is the greatest 1-conservative extension of intuitionistic propositional calculus **IPC**.

The variety **RN** was first studied by Kuznetsov and Gerciu [3] and Gerciu [1], and independently by Kracht [2]. The papers cited above claim interesting results but contain sketchy proofs, some of which have serious flaws. We provide a systematic study of **RN** and its subvarieties.

We describe the finitely generated and finite subdirectly irreducible **RN**-algebras using sums of Heyting algebras. We give a complete proof of Gerciu's theorem that every subvariety of **RN** is generated by its finite members, and construct a supervariety of **RN** which is not generated by its finite members. A criterion for a subvariety of **RN** to be locally finite is established. A finite axiomatization of **RN** using Jankov formulas and subframe formulas is provided, and it is shown that every subvariety of **RN** is finitely axiomatizable. It follows that the lattice of subvarieties of **RN** is countable. Finally, we prove that the equation theory of every subvariety of **RN** is coNP-complete.

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Residuation on weakly Heyting algebras

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This contribution is an attempt to combine modal algebras with the residuation law (and so, with the substructural world). The implication that we consider in the residuation law is the strict implication of modal algebras. This explains why we use the framework of weakly Heyting algebras, and not the one of modal algebras. Anyway, we stress that both frameworks are really close [1].

The variety of weakly Heyting algebras, or WH-algebras, was introduced in [2]. *A weak Heyting algebra* is an algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\rightarrow$ is a binary operation satisfying the equations:

1. $(x \rightarrow y) \wedge (x \rightarrow z) \approx x \rightarrow (y \wedge z)$,
2. $(x \rightarrow z) \wedge (y \rightarrow z) \approx (x \vee y) \rightarrow z$,
3. $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$,
4. $x \rightarrow x \approx 1$.

If we consider a modal algebra and define $\rightarrow$ as the box of the Boolean implication $\supset$ then what we obtain is a weakly Heyting algebra. And from the Priestley-style duality developed in [2] it is clear that every weakly Heyting algebra is embeddable into one that is obtained from a modal algebra. That is, the variety of weakly Heyting algebras corresponds to the strict implication reduct (also with $\wedge, \vee, 0, 1$) of the modal algebras (see [3] for the logical counterpart).

In the talk we will start giving a purely algebraic proof of the previous fact (cf. [4, pp. 128–130]). Then, we will consider two new varieties in the language enlarged with $\star$. The variety of *residuated weakly Heyting algebras*, or RWH-algebras, is the one obtained by adding:

(5) $x \star (x \rightarrow y) \leq y$,

(6) $x \leq y \rightarrow (y \star x)$,

(7) $x \star (y \wedge z) \leq x \star y$.

The members of this variety are exactly the weakly Heyting algebras such that *the law of residuation* holds, i.e., $a \leq b \to c$ iff $b \star a \leq c$ for every $a, b, c \in A$. And we also introduce the variety of *Boolean residuated weakly Heyting algebras*, or BRWH-algebras, obtained from all the previous equations by adding:

(8) $(x \wedge y) \star z \approx x \wedge (y \star z)$.

Then, it is possible to see that **RWH** $\neq$ **BRWH** while both **RWH** and **BRWH** are conservative expansions of **WH**. This is an easy consequence from the finite embeddability property of these varieties. The law of residuation determines univocally the operation $\star$ but it does not always exist, e.g., it is possible to give a complete WH-algebra where it is not possible to define $\star$. One of the laws that $\star$ satisfies over the BRWH-algebras is the monotonicity in both components. However, all the following equations are not valid: $x \star y \approx y \star x$, $x \star (y \star z) \approx (x \star y) \star z$, $1 \star 1 \approx 1$, $x \star y \leq y$ and $x \wedge y \leq x \star y$.

Finally, I would like to point out that every BRWH-algebra is embeddable into a RWH-algebra that admits Boolean implication, i.e., there is a certain binary operation $\supset$ under which the lattice becomes a Boolean algebra. This justifies the name of the variety. Thus it is easy to see that the equational logic associated with **BRWH** corresponds to the local consequence defined by Kripke models where

$$\mathcal{M}, w \Vdash \varphi \star \psi \quad \text{iff} \quad \mathcal{M}, w \Vdash \varphi \text{ and exists } u \in R^{-1}[w] \text{ such that } \mathcal{M}, u \Vdash \psi.$$

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On partially ordered algebras of relations with the domino operations

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It is known that algebras of binary relations can be considered as the modal algebras of two-dimensional modal logics [1]. We shall concentrate our attention on the operation of relation product (dyadic operator [1]) $\circ$, and the unary domino operations ∇_1, ∇_2 (see [1, 2]) which are defined as follows:

$$\nabla_1(\rho) = \{(x, y) : (\exists z)(z, x) \in \rho\}, \quad \nabla_2(\rho) = \{(x, y) : (\exists z)(y, z) \in \rho\}.$$

For any set Ω of operations on binary relations, let $R\{\Omega, \subset\}$ be the class of partially ordered by the set-theoretical inclusion $\subset$ algebras whose elements are binary relations and whose operations are members of Ω, and let $Var\{\Omega, \subset\}$ be a variety generated by $R\{\Omega \subset\}$. The following theorems give a basis of identities for the varieties $Var\{\circ, \nabla_1, \subset\}$ and $Var\{\circ, \nabla_2, \subset\}$.

Theorem 1. *An partially ordered algebra* $(A, \cdot, ^\star, \leq)$ *of the type* $(2, 1)$ *belongs to the variety* $Var\{\circ, \nabla_1, \subset\}$ *if and only if the following identities hold:*

$(xy)z = x(yz)$, $x^{\star\star\star} = x^{\star\star}$, $(x^\star)^2 = x^\star$, $x^\star y^\star = x^{\star\star} y^\star$, $x^\star y^{\star\star} = y^\star x^{\star\star}$,
$xy^\star = xx^\star y^\star$, $(xy^\star)^\star = y^\star x^\star$, $(xy^\star)^\star = (yx^\star)^\star$, $(x^{\star\star} y)^\star = x^{\star\star} y^\star$,
$x^{\star\star}(xy^\star)^\star = (xy^\star)^\star$, $x^\star yz^\star = x^\star(yz^\star)^\star$, $(xy^\star z)^\star = z^\star(xy^\star)^\star$,
$x \leq x^{\star\star}$, $(xy)^\star \leq x^{\star\star}$, $(xy)^\star \leq y^\star$, $x^\star y^\star \leq x^\star$.

Theorem 2. *An partially ordered algebra $(A, \cdot, {}^\star, \leq)$ of the type $(2,1)$ belongs to the variety $Var\{\circ, \nabla_2, \subset\}$ if and only if the following identities hold:*
$(xy)z = x(yz)$, $x^{\star\star\star} = x^{\star\star}$, $(x^\star)^2 = x^\star$, $x^\star y^\star = x^\star y^{\star\star}$, $x^{\star\star} y^\star = y^{\star\star} x^\star$,
$x^\star y = x^\star y^\star y$, $(x^\star y)^\star = y^\star x^\star$, $(x^\star y)^\star = (y^\star x)^\star$, $(x^{\star\star} y)^\star = x^\star y^{\star\star}$,
$(x^\star y)^\star x^{\star\star} = (x^\star y)^\star$, $x^\star yz^\star = (x^\star y)^\star z^\star$, $(xy^\star z)^\star = (y^\star z)^\star x^\star$,
$x \leq x^{\star\star}$, $(xy)^\star \leq y^{\star\star}$, $(xy)^\star \leq x^\star$, $x^\star y^\star \leq y^\star$.

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Geometry of Robinson consistency in Łukasiewicz logic

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For all unexplained notions about MV-algebras and Łukasiewicz (always propositional in the present talk) logic: we refer to [1]. For X an arbitrary set of variables, L_X denotes the set of formulas ψ whose variables are in X. Any such ψ is said to be an *L_X-formula.* The definition is the same for boolean logic and for many-valued logic. A proper subset Θ of L_X is called a *theory* (or, an L_X-theory if necessary) if

(i) Θ contains all L_X-tautologies of Łukasiewicz infinite-valued propositional logic, and

(ii) Θ is closed under modus ponens.

Theories are in one-one correspondence with ideals of free MV-algebras. An L_X-theory Θ is said to be *prime* (also called "complete" in Hájek's monograph [2]) if for any L_X-formulas φ and ψ either $\varphi \to \psi$ or $\psi \to \varphi$ belongs to Θ. Prime theories are in one-one correspondence with prime ideals of free MV-algebras. Every prime theory Θ has a unique *maximally consistent* completion Θ'. In other words, $L_X \supseteq \Theta' \supseteq \Theta$ and there is no theory $\Theta'' \subseteq L_X$ properly extending Θ'. By contrast with boolean logic Θ' generally does not coincide with Θ. Maximally consistent theories are in one-one correspondence with maximal ideals of free MV-algebras. The *Robinson consistency property* for boolean, as well for Łukasiewicz logic, can be stated as follows:

> Suppose Θ is a prime L_X-theory, and Ψ is a prime L_Y-theory. Let $L_Z = L_X \cap L_Y$ and $L_W = L_X \cup L_Y$. If $\Theta \cap L_Z = \Psi \cap L_Z$ then there is a prime L_W-theory Φ such that $\Theta = \Phi \cap L_X$ and $\Psi = \Phi \cap L_Y$.

We give a proof of the Robinson consistency property for Łukasiewicz propositional logic. As a corollary we obtain a new proof of the amalgamation property for MV-algebras. For the proof of our main results we make no use of lattice-ordered groups and the Γ functor. Rather, we make use of geometric tools naturally arising from the rich theory of MV-algebras, such as McNaughton's representation of free MV-algebras via $[0,1]$-valued piecewise linear functions, unimodular triangulations of the n-cube, and the classification of spectral spaces of free MV-algebras via bases in euclidean space.

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The logic of pregroups and the Lambek calculus

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A pregroup is a p.o. monoid such that, for every element a, there exist elements a^l and a^r, called the left adjoint and the right adjoint, respectively, of a which satisfy the conditions: $a^l a \leq 1 \leq aa^l$, $aa^r \leq 1 \leq a^r a$. The logic of pregroups, introduced under the name compact bilinear logic by Lambek (1999), can be obtained from the multiplicative fragment of noncommutative linear logic of Abrusci (also called bilinear logic by Lambek) by identifying $\otimes$ with $\oplus$ and 0 with 1.

A direct formalization of the above algebraic conditions leads to a rewriting system whose term algebra is a quasi-pregroup, and a natural quotient-structure of this term algebra is a free pregroup. Lambek (1999) proves a normalization theorem (often called the Lambek Switching Lemma) which states that any derivation of Δ from Γ can be transformed into a normal form in which all contraction rules precede all expansion rules. In Buszkowski (2003) it has been shown that this theorem is equivalent to the cut-elimination theorem for an appropriate two-side sequent system; the same paper also provides a one-side sequent system, admitting cut elimination. It follows from these results that the complexity of this logic is PTIME.

In the present paper we consider a one-side sequent system for this logic enriched with non-logical rules, corresponding to Lambek's induced steps. We prove the cut-elimination theorem for this system and the equivalence of this theorem and the Switching Lemma. As a consequence, we show that the logic of pregroups is faithfully interpretable (in a sense) in the logic of residuated monoids (the Lambek calculus). Interestingly, both logics radically differ in several fundamental aspects. For instance: (1) the former is PTIME, but the latter is NP-complete, (2) the latter possesses the finite model property, but the former does not, (3) the latter is complete with respect to powerset monoids over monoids, but the former is not, and similarly for monoids of all binary relations on a set. (Since the logic of pregroups can be presented as an axiomatic extension of the Lambek calculus, then we consider only models satisfying the additional axioms.)

We show that these facts follow from basic properties of pregroups, established in Buszkowski (2001, 2002); the results on Lambek calculus are due to different authors, e.g. Andréka and Mikulás, Buszkowski, Došen, Ono, Okada and Terui, Pentus, van Benthem and others. We also discuss certain problems concerning lattice ordered pregroups.

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Lattice representation of MV-algebras

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It is well known that every MV-algebra is a bounded distributive lattice with respect to the induced order. We will characterize distributive lattices which can be MV-algebras with respect to the certain operations. In particular, we will show that a finite distributive lattice is an MV-algebra if and only if it is a direct product of chains and we derive MV-operations in such a lattice.

Deduction theorems in weakly implicative logics

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In this paper we study the different variants of deduction theorem in the context of weakly implicative logics. We use the achieved results to give alternative characterizations of a special subclass of weakly implicative logics, the so-called weakly implicative fuzzy logics. Because of the lack of space we present the starting points, basic definitions, and *some examples of our results* only.

Weakly implicative logics The class of *weakly implicative logics* (introduced in [1]) extends the well-known class of Rasiowa's implicative logics (see [2]). A logic (understood as a consequence relation $\vdash$) is weakly implicative iff it contains a (definable) connective $\to$ that satisfies the following conditions:

$$\begin{array}{rcl} & \vdash & \varphi \to \varphi \\ \varphi, \varphi \to \psi & \vdash & \psi \\ \varphi \to \psi, \psi \to \chi & \vdash & \varphi \to \chi \\ \varphi \to \psi, \psi \to \varphi & \vdash & c(\dots, \varphi, \dots) \to c(\dots, \psi, \dots) \quad \text{for all connectives } c \end{array}$$

Observe that for the implication we do not assume any structural rule (exchange, weakening, contraction). By *axiomatic system* we understand the set of axioms and deduction rules, closed under arbitrary substitution.

Definition 1 Let $\mathcal{AS}$ be an axiomatic system. An $\mathcal{AS}$*-proof* of the formula φ in theory T is a founded tree labelled by formulas; the root is labelled by φ and leaves by either axioms or elements of T; and if a node is labelled by ψ and its preceding nodes are labelled by $\psi_1, \psi_2, \dots$ then $<\{\psi_1, \psi_2, \dots\}, \psi> \in \mathcal{AS}$. We say that $\mathcal{AS}$ is a presentation of $\vdash$ if $T \vdash \varphi$ iff φ has an $\mathcal{AS}$-proof in theory T.

Definition 2 Let m be a natural number and φ and ψ formulas. We define the formula $\varphi^m \to \psi$ inductively as: $\varphi^0 \to \psi = \psi$ and $\varphi^{i+1} \to \psi = \varphi \to (\varphi^i \to \psi)$.

Deduction theorems In this section we restrict ourselves to finitary logics only. There are many variants of deduction theorem. The common one is the *Local deduction theorem* (LDT). The logic $\vdash$ has LDT if

$$T, \varphi \vdash \psi \text{ iff there is natural } n \text{, such that } T \vdash \varphi^n \to \psi$$

Another variant of deduction theorem says more about the n. The logic $\vdash$ has the *n-Implicative deduction theorem* (nIDT) if $\vdash$ has a presentation $\mathcal{AS}$ such that $T, \varphi_1, \dots \varphi_n \vdash \psi$ iff $T \vdash (\varphi_1^{k_1} \to (\varphi_2^{k_2} \to (\dots (\varphi_n^{k_n} \to \psi) \dots)$, where k_i, is the number of occurrences of the formula φ_i in some $\mathcal{AS}$-proof of ψ.

Theorem 1 Let $n > 2$. The logic $\vdash$ has nIDT iff $\vdash$ has presentation $\mathcal{AS}$, where Modus Ponens is the only deduction rule and the implicative fragment of $\vdash$ extends BCI.

As a consequence we get that 3IDT entails nIDT for all n. However, the question whether Theorem 1 works for 1IDT and 2IDT seems to be open. Of course, nIDT entails LDT. Although, the nIDT looks a little complicated it is a non-trivial specification of LDT—we know that there is a logic strictly weaker than BCI with LDT (thus without 3IDT!).

This was just an example of finer analysis of a notion of *deduction theorem* inside the class of weakly implicative logics. We can also present several other variants, mainly resulting from enhancing the expressive power of our logic by adding some new connective. For example with unary connective $\triangle$ we can formulate an "S4-like" deduction theorem, etc.

Weakly implicative fuzzy logics Weakly implicative logics can be characterized as those which are complete w.r.t. a class of *ordered matrices* (in which the set D of designated values is upper), if the ordering of the elements of the matrix is defined as

$$x \leq y \ \equiv_{\text{df}} \ x \to y \in D$$

By (weakly implicative) *fuzzy* logics we call such weakly implicative logics that are complete w.r.t. *linearly* ordered matrices. If we restrict to the finitary logics, the main (syntactical) equivalent characterization of this class of logics is the so-called *Prelinearity property*:

$$T,\ \varphi \to \psi \vdash \chi \text{ and } T,\ \psi \to \varphi \vdash \chi \text{ entails } T \vdash \chi.$$

The presence of some form of deduction theorem allows us to formulate the more "direct" characterization of (some subclass of) weakly implicative fuzzy logics. For example we can prove:

Theorem 2 Let $\vdash$ be a finitary logic with LDT. Then $\vdash$ is weakly implicative fuzzy logic iff $(\varphi \to \psi)^i \to \chi, (\psi \to \varphi)^j \to \chi \vdash \chi$ for each i and j.

Theorem 3 Let $\vdash$ be a finitary logic with 3IDT fulfilling $\varphi \vdash \psi \to \varphi$. Then $\vdash$ is weakly implicative fuzzy logic iff $\vdash ((\varphi \to \psi)^i \to \chi) \to (((\psi \to \varphi)^i \to \chi) \to \chi)$ for each i.

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General theory of the commutator for deductive systems

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Let $(\boldsymbol{S}, \vdash)$ be finitary k-dimensional deductive system. Let $\underline{x}_1 = \langle x_1^1, \ldots, x_k^1 \rangle$, $\ldots$, $\underline{x}_m = \langle x_1^m, \ldots, x_k^m \rangle$, and $\underline{y}_1 = \langle y_1^1, \ldots, y_k^1 \rangle$, $\ldots$, $\underline{x}_n = \langle x_1^n, \ldots, x_k^n \rangle$ be strings of disjoint variables. Each string has length k. A k-formula $\alpha(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u})$ is called a *commutator formula for* $\vdash$ *in the variables* $\underline{x}_1, \ldots, \underline{x}_m$ *and* $\underline{y}_1, \ldots, \underline{y}_n$ if the following condition is satisfied:

(1) $\quad \underline{x}_1, \ldots, \underline{x}_m \vdash \underline{\alpha}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u})$ and $\underline{y}_1, \ldots, \underline{y}_n \vdash \underline{\alpha}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, \underline{u})$.

Let $M = (\boldsymbol{A}, \underline{D})$ be a model for $\vdash$. (Note that $D \subseteq A^k$.) $Fi_{\vdash}(M)$ denotes the class of all $\vdash$-filters on A which include the $\vdash$-filter D. For $\underline{E}, \underline{F} \in Fi_{\vdash}(M)$ let

$[\underline{E}, \underline{F}]_M :=$ the least $\vdash$-filter in $Fi_{\vdash}(M)$ which includes the set
$\{\underline{\delta}(\underline{a}_1, \ldots, \underline{a}_m, \underline{b}_1, \ldots, \underline{b}_n, e_1, \ldots, e_r) : \underline{\delta}(\underline{x}_1, \ldots, \underline{x}_m, \underline{y}_1, \ldots, \underline{y}_n, u_1, \ldots, u_r)$
is a commutator k-formula, $\underline{a}_1, \ldots, \underline{a}_m \in \underline{E}, \underline{b}_1, \ldots, \underline{b}_n \in \underline{F}, e_1, \ldots, e_r \in A\}$.

$[\underline{E}, \underline{F}]_M$ is called the *commutator of the filters* $\underline{E}$ *and* $\underline{F}$ *over* M (relative to the system $\vdash$). The following observation is immediate:

Theorem 1. *For any* $\underline{F}, \underline{G} \in Fi_{\vdash}(M)$*, where* $M = (\boldsymbol{A}, \underline{D})$*, the following conditions hold:*

(i) $\underline{D} \subseteq [\underline{F}, \underline{G}]_M$;

(ii) $[\underline{F}, \underline{G}]_M \subseteq \underline{F} \cap \underline{G}$;

(iii) $[\underline{F}, \underline{G}]_M = [\underline{G}, \underline{F}]_M$;

(iv) *The commutator is monotone in both arguments, i.e., if* $\underline{F}_1, \underline{F}_2$ *and* $\underline{G}_1, \underline{G}_2$ *are filters on* M*,* $\underline{F}_1 \subseteq \underline{F}_2$*, and* $\underline{G}_1 \subseteq \underline{G}_2$*, then* $[\underline{F}_1, \underline{G}]_M \subseteq [\underline{F}_2, \underline{G}]_M$ *and* $[\underline{F}, \underline{G}_1]_M \subseteq [\underline{F}, \underline{G}_2]_M$.

The purpose of this talk is to present in a uniform way the commutator theory for k-deductive system of arbitrary positive dimension k. We are interested in the logical perspective of the research - an emphasis is put on an analysis of the interconnections holding between the commutator and logic. This research thus qualifies as belonging to "abstract algebraic logic", an area of universal algebra that explores to a large extent the methods provided by the general theory of deductive systems.

The focus of the talk is on the following two issues:

(1) the discussion of various simplifications of the definition of the commutator. In this context several notions of centralizator for deductive systems is investigated.

(2) the discussion of the additivity and correspondence properties of the commutator.

But the theory is mainly centered about special cases of the general definition, viz. 1-dimensional deductive systems and 2-dimensional ones.

As to (1), the talk deals with the issue of equivalence of different concepts of a centralizator. Since the commutator of two deductive filters is equal to the intersection of appropriate deductive filters that are centralizators of the two filters, much space is devoted to the discussion of various forms of the ternary relation that two filters are centralized relative a third filter. In the 1-dimensional case, it is proved that for a wide variety of protoalgebraic logics, viz. weakly regularly algebraizable systems, the general notion of a centralizator of deductive filters is equivalent to the centralizator defined in terms of binary commutator formulas (with parameters). The theory outlined here much extends some other approaches as e.g. that promoted by Gumm and Ursini [1984].

In the 2-dimensional case, the commutator for equational logics is mainly investigated. In the context of the centralizator theory for equational logics, the focus of the talk is on the idea of applying a general notion of an implication viewed as a set of quaternary equations having jointly the property of detachment relative to a given equational system. This idea was outlined in the author's monograph [2001] and applied to various concrete problems in the theory of quasivarieties of algebras.

We underlie similarities between the two cases—they are basically handled by "isomorphic" methods. Freese and McKenzie [1987] and Kearnes and McKenzie [1992] laid the foundations of the commutator for equational systems. Our contribution to the theory consists in an attempt to disentangle various intricate (often syntactic) characterizations of the commutator and to render them in a more transparent logical form provided by the conceptual framework of the contemporary Abstract Algebraic Logic.

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Galois connections and (quasi-)uniform structures

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In [3], J. Picado introduced a way of composing Galois connections (contravariantly), which endows the set $Gal(A, A)$ of Galois endomaps of a partially ordered set A with a structure of quantale whenever A is a frame. Recently, M. Erné and J. Picado proved that for any complete lattice A, $Gal(A, A)$ with this composition forms a quantale if and only if A is pseudocomplemented [1].

Having this quantalic structure as starting point, in this talk we show how it may provide a unified description of uniform and quasi-uniform structures both in the classical and the non-classical (i.e., pointfree) contexts [2].

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Free algebras over a poset

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Let **V** be a variety of algebras such that they have an underlying ordered structure definible by means of certain equations. In this note, a construction of the free **V**-algebras over a poset I, whenever **V** is generated by an algebra C, is obtained.

Some results about diagonal–free two–dimensional cylindric algebras

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The variety $\mathbf{Df}_2$ of all diagonal-free two–dimensional cylindric algebras has been widely studied by different authors, but little research has focused on those problems inherent to finite algebras.

In this note, the finite $\mathbf{Df}_2$-algebras are characterized by means of certain partitions of their set of atoms. By adjusting this characterization for finite subdirectly irreducible $\mathbf{Df}_2$-algebras, a formula for computing the number of subdirectly irreducible $\mathbf{Df}_2$-algebras with a finite number of atoms is given.

Besides, it is shown that every finite $\mathbf{Df}_2$-algebra can be expressed as a direct product of simple algebras, determining the simple algebras that form this direct product as well as the number of each simple algebra which takes place in it.

By applying all results stated above, a formula for computing the number of $\mathbf{Df}_2$-algebra's structures that can be defined over a finite Boolean algebra is obtained.

Finally, the study of the lattice $\Lambda(\mathrm{Df}_2)$, started by N. Bezhanishvili in [1] is completed by giving a full description of it. The tools used to do so are the aforementioned characterization of the finite subdirectly irreducible Df_2-algebras, some results established by N. Bezhanishvili as well as the well-known results of B. Jónsson ([5]) and B. Davey ([2]).

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Extending the monoidal t-norm based logic with an independent involutive negation

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The Monoidal T-norm based Logic **MTL** was introduced by Esteva and Godo in [4], and was shown in [7], by Jenei and Montagna, to be standard complete w.r.t. **MTL**-algebras over the real unit interval, i.e. algebras defined by left-continuous t-norms and their residua. In this logic a negation is definable from the implication and the truth constant $\overline{0}$, so that $\neg\varphi$ stands for $\varphi \to \overline{0}$. This negation behaves quite differently depending on the chosen left-continuous t-norm and in general is not an involution. Such an operator can be forced to be involutive by adding the axiom $\neg\neg\varphi \to \varphi$ to **MTL**. The system so obtained was called in [4] **IMTL** (Involutive Monoidal T-norm based Logic). However, in such a logic the involution does depend on the t-norm, so that **IMTL** singles out only those left-continuous t-norms which yield an involutive negation. Clearly, operators like Gödel and Product t-norms are ruled out. This motivated then the interest in studying a logic of left-continuous t-norms with an independent involutive negation.

Our approach is somehow related to the one carried out in [5] in which the logics $\mathbf{G}_\sim$, $\mathbf{SBL}_\sim$ (obtained by the introduction of an involutive negation not dependent on the t-norm in Gödel Logic (**G**) and in the Strict Basic Logic (**SBL**)) and their related predicate calculi were investigated. Indeed we introduce in **MTL** the operator Δ [1], which resulted to be very useful for basic (but fundamental) results. Moreover, we also add to **MTL** a unary connective $\sim$ and the following axioms which capture the behavior of involutive negations:

(~ 1) $\sim \overline{0}$,

(~ 2) $\sim\sim \varphi \equiv \varphi$,

(~ 3) $\Delta(\varphi \to \psi) \to (\sim \psi \to \sim \varphi)$.

Following such ideas, then, we introduce in our work the logic $\mathbf{MTL}_\sim$, the variety of $\mathbf{MTL}_\sim$-algebras (its algebraic structures) and we provide algebraic and standard completeness results. In other words we prove that $\mathbf{MTL}_\sim$ is sound and complete with respect to the class of all linearly ordered $\mathbf{MTL}_\sim$-algebras and that $\mathbf{MTL}_\sim$ is sound and complete with respect to the standard $\mathbf{MTL}_\sim$-algebra, that is $\mathbf{MTL}_\sim$-algebras having as a domain the real unit interval $[0, 1]$.

The logic obtained is interesting, since by defining a new connective

$$\varphi \underline{\vee} \psi \equiv \sim (\sim \varphi \& \sim \psi),$$

it allows to represent by means of left-continuos t-norms and involutions all dual t-conorms. This is not possible in any other residuated fuzzy logic. Moreover, notice that we can also define S-implications as follows:

$$\varphi \rightsquigarrow \psi \equiv \sim \varphi \underline{\vee} \psi.$$

This suggests that the work carried out in [2] might be recovered under our framework.

We also introduce the predicate calculus $\mathbf{MTL}\forall_\sim$ obtained, as usual, by enlarging the propositional language with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$ together with the two classical quantifiers $\forall$ and $\exists$ and axioms on quantifiers capturing their usual behaviours.

In this context the involutive negation allows to define a quantifier by the other one, namely $\mathbf{MTL}\forall_\sim$ proves

$$(\exists x)\varphi(x) \equiv\sim (\forall x)(\sim \varphi(x)),$$

and thus the calculus axiomatization can be simplified.
Also for $\mathbf{MTL}\forall_\sim$ we provide algebraic and standard completeness.

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Pointwise discontinuous functions from a modal point of view

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The subject explored in our work has a lengthy historical development which seems to converge in the 1905 memoir *"Leçons sur les functions discontinues"* by René-Louis Baire. Let f be a function defined on a topological space X and with values lying in the set $\mathbf{R}$ of real numbers (with Euclidean topology). Recall that the function f is *continuous at the point* x if $f^{-1}(U)$ contains an open neighbourhood of x for every open interval U containing $f(x)$. A point $x \in X$ at which the function f is not continuous is called a *discontinuity* of f. The set of the discountinuities of f is denoted by D_f. The condition for the continuiity of f is clearly $D_f = \emptyset$. If D_f is a nowhere dense set the function f is said to be *pointwise discontinuous*; in this case f has a point of continuity in any open set. The function f is called *almost continuous* if f is *hereditarily pointwise discontinuous*, i.e. $f|A$ is pointwise discontinuous for every closed subset $A \subseteq X$. The *characteristic function* of a subset A of X is the function χ_A defined by $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$, otherwise. Baire has shown that almost continuous characteristic functions are exactly those characteristic functions that are the pointwise limits of sequences of continuous functions. One more bit of notation. Let $\mathcal{A}_X$ denote the class of all characteristic functions, $\mathcal{PD}_X = \{f \in \mathcal{A}_X \mid f \textit{ is pointwise discontinuous}\}$ and $\mathcal{AC}_X = \{f \in \mathcal{A}_X \mid f \textit{ is almost continuous}\}$.

Recall that the standard topological semantics of the modal system **S4** is based on the notion of a topological model, that is a pair (X, ν) with X a topological space and $\nu : X \to \{0, 1\}$ a valuation. We wish to impose certain topological restrictions on valuations. We deal here with a sharpening of topological semantics, namely topological models with valuations that are "nearly continuous", that is belong either to $\mathcal{AC}_X$ or to $\mathcal{PD}_X$. It is not hard to verify that the modal logic of the class of topological models (X, ν) with continuous valuations is the trivial logic, i.e. $\mathbf{S4} + p \leftrightarrow \Box p$. However, we hope that some justification of our amended semantics is containing in the following observations.

Recall that the modal logic **S4.Grz** is the system that results when the axiom $\Box(\Box(p \to \Box p) \to \Box p) \to p$ is added to the Lewis system **S4**. Note that **S4.Grz** is the largest modal system in which Intuitionistic propositional logic can be embedded by the Gödel modal translation. The system **S4.1** (first defined by McKinsey) is the modal system obtained by adding $\Box \diamond p \to \diamond \Box p$ to **S4** as a new axiom.

Observation 1. (a) **S4.Grz** is the modal logic of the class of topological models (X, ν) such that $\nu \in \mathcal{AC}_X$; (b) **S4.1** is the modal logic of the class of topological models (X, ν) such that $\nu \in \mathcal{PD}_X$.

Observation 2. (a) **S4.Grz** is the modal logic of the Euclidean models $(\mathbf{R}, \nu)$ with almost continuous valuations; (b) **S4.1** is the modal logic of the Euclidean models $(\mathbf{R}, \nu)$ with pointwise discontinuous valuations;

This observation shows the system **S4.Grz** has a certain "Euclidean completeness property": every formula p which is true in Euclidean space $\mathbf{R}$ for every almost continuous valuation ν is provable in **S4.Grz**. Hence by contraposition, we see also that if a formula is not provable in **S4.Grz**, then we can be sure of finding an almost continuous counter-example for it in the space $\mathbf{R}$.

Equivalence of consequence relations: an order-theoretic and categorical perspective, II

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The focus of our study is the class of modules for which the Blok-Pigozzi result in [BP] can be generalized. More specifically, we wish to consider the class of modules for which the abstractions of the two conditions for algebraizability, discussed in Constantine Tsinakis's abstract (see page 83), coincide. Throughout this abstract, we fix an order complete partially ordered monoid $\mathbf{A}$ and the full subcategory ${}_{\mathbf{A}}\mathcal{C}$ of ${}_{\mathbf{A}}\mathcal{M}$ whose objects are complete lattices.

Given two $\mathbf{A}$-modules $\mathbf{P}$ and $\mathbf{Q}$, we seek conditions that will guarantee that two consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively, are equivalent, i.e. the modules of their theories are isomorphic, if and only if there exist morphisms (translators) between the original modules that induce this isomorphism. The latter condition can be described in more detail as follows. Let $\vdash_\gamma$ and $\vdash_\delta$ be two consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively, and let γ and δ be the closure operators on $\mathbf{P}$ and $\mathbf{Q}$ that correspond to $\vdash_\gamma$ and $\vdash_\delta$. Then, for every isomorphism f between the modules of theories $\mathbf{P}_\gamma = \mathbf{Th}_{\vdash_\gamma}$ and $\mathbf{Q}_\delta = \mathbf{Th}_{\vdash_\delta}$, there exist translators $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ such that $\delta\tau = f\gamma$ and $\gamma\rho = f^{-1}\delta$.

We observe that the objects of ${}_{\mathbf{A}}\mathcal{C}$ for which the generalization of the Blok-Pigozzi result holds are precisely the *projective* objects of this category. We further prove that if $\mathbf{P}$ is an $\mathbf{A}$-module, $\{v_i \mid i \in I\}$ is a subset of P and $\{u_i \mid i \in I\}$ is a subset of A, such that the conditions

(R1) $\bigvee_{i \in I}(x/v_i) \star v_i = x$, for all $x \in P$, (R2) $[(a \star v_i)/v_i]u_i = au_i$, for all $a \in A$, $x \in P$.

(R3) $u_i \star v_i = v_i$.

(R4) $[(\bigvee a_i \star v_i)/v_n]u_n = [(a_n \star v_n)/v_n]u_n$ for all $n \in I$.

are satisfied, then **P** is projective. Here we denote by $\star : \mathbf{A} \times \mathbf{P} \to \mathbf{P}$ the residuated action of **A** on **P** and by \ and / its residuals.

Additionally, we prove that the $\wp(\mathbf{\Sigma})$-modules $\wp(\mathbf{Fm})$ of formulas and $\wp(\mathbf{Eq})$ of equations satisfy the preceding conditions – actually the set I can be chosen to be a singleton – and are therefore projective. Moreover each of these modules is cyclic; i.e. it is generated by a single element. It is important to mention that each projective cyclic **A**-module is isomorphic to a submodule of **A** that is generated by an idempotent element.

Let *Seq* be a set of *sequents* (intuitionistic, classical, non-associative, see [GO]), multi-sequents or hypersequents. Unless all elements in *Seq* have bounded length, the $\wp(\mathbf{\Sigma})$-module $\wp(\mathbf{Seq})$ is not cyclic, but we prove that it is projective. This result can be proved by verifying conditions (R1) - (R4) above or by noting that $\wp(\mathbf{Seq})$ is a co-product of cyclic projective modules.

Rebagliato and Verdú [RV] give a definition of the equivalence of two consequence relations on (associative) sequents. The results in [BJ99] do not cover the case of sequents, but it follows from our results that the isomorphism of the modules of theories is equivalent to the definition of Rebagliato and Verdú.

Our main result guarantees that under natural additional assumptions the desired translators τ and ρ are finitary; i.e. they send compact elements to compact elements. In the case of powersets they simply send finite subsets to finite sets.

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Generalized Kripke frames

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In recent joint work with J. Michael Dunn and Alessandra Palmigiano we give a uniform account of relational completeness for the implication-fusion fragment of various substructural logics. These results are obtained by applying canonical extension together with a discrete duality and Sahlqvist-like correspondence theory for certain complete (non-distributive) lattices. The approach in that work is purely algebraic, but the outcome is a two-sorted type of relational structures including what we think might best be thought of as worlds and information quanta, respectively. The talk will be a preliminary account of these structures.

Local MV-algebras and quasi-constant functions

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In this work we characterize local MV-algebras as algebras of *quasi-constant* functions. Combining such result with the representation theorem of MV-algebras, we get that every MV-algebra is embeddable in an algebra of quasi-constant functions over a suitable quotient of the spectrum.

Definition. An MV-algebra A is *local* if it satisfies one of the following equivalent conditions:
i) for any $a \in A$, either $ord(a) < \infty$ or $ord(a^*) < \infty$,
ii) the set $\{a \in A : ord(a) = \infty\}$ is a proper ideal of A
iii) A has one and only one maximal ideal.
For any MV-algebra A, the *radical* of A (denoted by $Rad(A)$) is the intersection of all maximal ideals of A. Note that $Rad(A) = \{a \in A \mid ord(a) = \infty\}$. An MV-algebra A is *perfect* if $A = Rad(A) \cup Rad^*(A)$. Every perfect algebra is local. The most important example of perfect MV-algebra, is Chang's algebra [2] $C = \{nc : n \in \omega\} \cup \{1 - nc : n \in \omega\}$ where $c^* = 1 - c$ and $ord(c) = \infty$ and $ord(1 - c) < \infty$. We have $Rad(C) = \{nc : n \in \omega\}$.
Any linearly ordered MV-algebra is local. Let us describe an example of local MV-algebra that will result as a kind of prototypical local MV-algebra.
Let X be an arbitrary non empty set, A an MV-algebra and $\mathbf{K}(A^X)$ the subset of the MV-algebra A^X defined as follows:

$$\mathbf{K}(A^X) = \{f \in A^X \mid f(X) \subseteq [a]_{Rad(A)} \quad \text{for some} \quad a \in A\}.$$

$\mathbf{K}(A^X)$ is the MV-algebra of *quasi constant* functions from X to A.

Proposition $\mathbf{K}(A^X)$ is a local MV-algebra.

Let $SpecA$ be the set of all prime ideals of A. Note that for any MV-algebra A we have

$$Rad(\prod_{P \in SpecA} (A/P)) = \prod_{P \in SpecA} (Rad(A/P))$$

and

$$(\prod_{P \in SpecA} (A/P))/Rad(\prod_{P \in SpecA} (A/P)) \cong \prod_{P \in SpecA} ((A/P)/Rad(A/P)).$$

Let A be a local MV-algebra. Then each $(A/P)/Rad(A/P)$ is simple (i.e. its only ideal is $\{0\}$). Further, for every $x \in A$ and for every $P, Q \in SpecA$, up to isomorphism, it holds:

$$\{(x/P)/Rad(A/P)\}_{P \in SpecA} = \{r\}, \quad \text{for some} \quad r \in [0, 1].$$

Proposition A is local if and only if it can be embedded in $\mathbf{K}(\prod_{P \in Spec(A)} A/P)$.

By applying Di Nola representation theorem we have:

Theorem Every local MV-algebra can be embedded into an MV-algebra of quasi constant functions on an ultrapower of $[0, 1]$.

Consider now any MV-algebra A and let Σ be a binary relation defined on $Spec(A)$ as follows: P, $Q \in \Sigma$ iff for every $a \in A$,

$$\frac{(a/P)}{Rad(A/P)} = \frac{(a/Q)}{Rad(A/Q)}.$$

It is easy to see that Σ is an equivalence on $Spec(A)$. Then we denote the quotient set by $\Sigma\,(Spec(A))$ and the equivalence class of a prime ideal P by $\Sigma(P)$.
Hence the canonical embedding: $A \hookrightarrow \prod_{P \in Spec(A)} A/P$ given by Chang representation theorem, can be written as follows:

$$A \hookrightarrow \prod_{\Sigma(P)\in\Sigma(Spec(A))} \prod_{Q\in\Sigma(P)} A/Q$$

and it can be easily checked that each $\prod_{Q\in\Sigma(P)} A/Q$ is a local MV-algebra.
In [3] the authors shown that every MV-algebra can be represented as an algebra of global sections over a compact sheaf (defined over $Max(A)$), whose stalks are local. We can modify such result, obtaining the following

Theorem Every non trivial MV-algebra A can be embedded in the MV-algebra of global sections (whose stalks are algebras of radical constant functions) of an MV-sheaf space $\mathcal{E} = (E, X, p)$ with $X = Spec(A)$.

Note that the $\prod_{Q\in\Sigma(P)} A/Q$ are the largest local stalks that we can find in such a kind of representation. Further, we have a sort of refinement of Di Nola representation of MV-algebras as algebras of functions taking values in an ultrapower of $[0, 1]^*$. We showed that we can organize prime ideals in such a way to obtain *piecewise radical constant functions*.

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On an infinite-valued Łukasiewicz logic that preserves degrees of truth

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Infinite-valued Łukasiewicz logic L_∞ is a very well-known propositional logic studied for instance in [6], [4] and [1]. Among its properties, we point out the more relevant for the motivation of the results in this paper:

1. L_∞ can be defined from a class of matrices constituted by Wajsberg algebras (also called MV-algebras) where the set of designated values is an arbitrary implicative filter (recall that implicative filters in Wajsberg algebras are the lattice filters that are closed under the fusion operation).

2. From the algebraic point of view, L_∞ is protoalgebraic and algebraizable. Protoalgebraicity follows from the facts that the identity law is a theorem of L_∞ and that Modus Ponens is an inference rule of L_∞.
3. L_∞ is not selfextensional, i.e., interderivability in the infinite valued Łukasiewicz logic is *not* a congruence relation on the algebra of formulas.
4. L_∞ does not satisfy the *Deduction-Detachment Theorem*, nor the *Graded Deduction Theorem*. L_∞ does satisfy, though, the *Local Deduction-Detachment Theorem* (see [3]).

In this paper we will study a new logic, determined also by Wajsberg algebras, but focusing on the order relation instead of the implication. This new logic, denoted as $L_\infty^{\leq}$, is an example of a "logic that preserves degrees of truth" and in parallel with the properties of L_∞, we will have the following:

1. $L_\infty^{\leq}$ can be defined from a class of matrices constituted by Wajsberg algebras where the set of designated values is an arbitrary lattice filter.
2. From the algebraic point of view, $L_\infty^{\leq}$ is *not protoalgebraic* and hence it is obviously *not algebraizable*. Since $L_\infty^{\leq}$ still has the identity law as a theorem, non protoalgebraicity means that no rule like Modus Ponens is admissible as an inference rule of $L_\infty^{\leq}$, for any binary term conceivable as implication.
3. $L_\infty^{\leq}$ *is selfextensional*, i.e., in this caseinterderivability is a congruence relation on the algebra of formulas.
4. $L_\infty^{\leq}$ does not satisfy the *Deduction-Detachment Theorem*, nor the *Local Deduction-Detachment Theorem*, but it *does satisfy* the *Graded Deduction Theorem*.

The general theory of abstract algebraic logic (see for instance [2]) guarantees that every selfextensional logic with conjunction $\mathcal{S}$ has a Gentzen system $\mathcal{G}_{\mathcal{S}}$ that is both fully adequate for it and algebraizable, having the same algebraic counterpart as the logic $\mathcal{S}$. For an arbitrary $\mathcal{S}$ the Gentzen system $\mathcal{G}_{\mathcal{S}}$ is defined non-constructively, but in our case we find a finite presentation (i.e., a sequent calculus) of the Gentzen system $\mathcal{G}_{L_\infty^{\leq}}$, we study its properties and models, and we determine several relationships between the Gentzen system and the logics L_∞ and $L_\infty^{\leq}$. The sequent calculus given here is a modification of the one introduced in [5].

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Categorical abstract algebraic logic: the isomorphism theorem

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In [1] a characterization of algebraizability of a sentential logic $\mathcal{S}$ is obtained in terms of the existence of an isomorphism between the lattice of theories of $\mathcal{S}$ and the lattice of theories of the equational consequence of its equivalent algebraic semantics $\mathbf{K}$, commuting with substitutions. In [2] we find a generalization of this result in Teorem V.3.5, stating that two deductive systems are equivalent if, and only if, there exists an isomorphism between their lattices of theories, commuting with substitutions. In turn, in [5] a new generalization of this result is exhibited for Gentzen systems: Theorem 2.19.

Each of these systems is a generalization of the previous one: sentential logics and equational consequences of a class of algebras are particular cases of k-deductive systems, which are particular cases of Gentzen systems. But there are other kinds of deductive systems with a similar Isomorphism Theorem that are not, nor extend, Gentzen systems.

A generalization of all these systems are π-institutions, which were introduced for the first time by Fiadeiro and Sernadas in [3], inspired by the work on institutions of Goguen and Burstall in [4]. Institutions cover all theses deductive systems and also formalize others multi-sorted ones which came form computing science. π-institutions in turn focus attention in the syntax instead of in semantics, as do institutions. For both, a categorical context organizing the information is required.

In [6] Voutsadakis proved the following result:

Theorem (Voutsadakis). If $\mathcal{I}$ and $\mathcal{I}'$ are two term π-institutions, then they are deductively equivalent if, and only if, there exists an adjoint equivalence $\langle F, G, \eta, \varepsilon \rangle : \mathbf{Th}\mathcal{I} \rightharpoonup \mathbf{Th}\mathcal{I}'$ that commutes with substitutions.

This result is a generalization of Theorem V.3.5 of [2]. However, in spite of its abstraction level, it is not a generalization of Theorem 2.19 of [5], since not all Gentzen systems can be exhibited (in fact, only those that are k-systems can) as term π-institutions. This shows that this condition is probably too strong. But it cannot be just removed.

I will provide two π-institutions which are not deductively equivalent but with isomorphic categories of theories (through an isomorphism commuting with substitutions). The objective of eliminating absolutely the conditions over the π-institutions in the characterization theorem of deductive equivalence is then vane. However, I will offer a way of extending the result which will cover Gentzen systems among others. To do this, the notion of Grothendieck construction of a **Cat**-valued functor will be used. We obtain then the following extended version of the Isomorphism Theorem:

Theorem. If $\mathcal{I}$ and $\mathcal{I}'$ are two multi-term π-institutions, then they are deductively equivalent if, and only if, there exists an adjoint equivalence $\langle F, G, \eta, \varepsilon \rangle : \mathbf{Th}\mathcal{I} \rightharpoonup \mathbf{Th}\mathcal{I}'$ that commutes with substitutions.

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On product logic with truth-constants

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In the context of fuzzy logical systems, introducing truth-constants in the language is an elegant means to be able to explicitely reasoning with partial degrees of truth. This goes back to Pavelka [5] who built a propositional many-valued logical system over Łukasiewicz logic by adding into the language a truth constant $\overline{r}$ for each real $r \in [0, 1]$, together with a number of additional axioms. Although the resulting logic is not strong complete (like Łukasiewicz logic), Pavelka proved that his logic, we will called it PL, is complete in a weaker sense. Namely, by defining the truth degree of a formula φ in a theory T as

$$\| \varphi \|_T = \inf\{e(\varphi) \mid e \text{ evaluation model of } T\}$$

and the degree of provability of φ in T as

$$| \varphi |_T = \sup\{r \mid T \vdash_{PL} \overline{r} \to \varphi\},$$

Pavelka proved that these degrees coincide. This kind of completeness, is usually known as Pavelka-style completeness, and strongly relies in the continuity of Łukasiewicz truth functions. Novák extended Pavelka approach to Łukasiewicz first order logic.

Later, Hájek [3] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results, indeed it is enough to extend the language only by a countable number of truth-constants, one per each *rational* in $[0, 1]$, and by two additional axiom schemata, called book-keeping axioms:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow s}$$

where $*$ and $\Rightarrow$ are Łukasiewicz t-norm and its residuum respectively. He denoted this new system Rational Pavelka Logic, RPL for short. Moreover he proved that RPL is strong complete for finite theories.

Similar *rational* extensions for other popular fuzzy logics can be obviously defined, but remark that Pavelka-style completeness cannot be obtained since Łukasiewicz is the only fuzzy logic with continuous truth-functions in the real unit interval [0, 1]. Among different works in this direction we may cite [3] where an extension of G_Δ (the extension of Gödel logic with Baaz's Delta operator) with a finite number of rational truth-constants, and [1] where the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel logic with an involutive negation) and to Π (Product logic) and $\Pi_\sim$ (Product logic with an involutive negation), but in these cases is necessary to add an infinitary rule to obtain the Pavelka-style completeness. More recently, in [2] the authors consider the extension of Gödel, Nilpotent minimum, and some Weak Nilpotent Minimum logics with rational truth-constants. Weak standard completeness is shown for those logics.

In this talk we will consider expansions of another popular fuzzy logic, the Product fuzzy logic $\mathbf{\Pi}$ [3,4], with any countable subsets of truth-constants closed by the product logic truth-functions, and we will prove weak standard completeness for them.

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On the algebraization of valuation semantics

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In 1989, W. Blok and D. Pigozzi proposed a precise mathematical definition of the notion of *algebraizable logic* [1], which generalizes the traditional Lindenbaum-Tarski method. Nevertheless, many interesting logics fall out of the scope of this approach. It is the case of the so-called *non-truth-functional logics*, and in particular of the paraconsistent systems of da Costa [4]. The major problem with these logics is the lack of congruence for some connective(s), the key ingredient in the algebraization process. Our goal is to generalize the Blok-Pigozzi approach by dropping the assumption that formulas should be homomorphically evaluated over algebras of truth-values of the same type. As an example we shall consider da Costa's system $\mathcal{C}_1$, whose non-algebraizability has been studied in [7,6].

As in the Blok-Pigozzi case, we shall focus on logic systems $\mathcal{L} = \langle L, \vdash \rangle$ which are tarskian and structural, in the sense that L is freely generated by a signature Σ from a set of propositional variables and $\vdash$ is invariant under substitutions. As observed by J. Czelakowski and R. Jansana in [3], an equivalent characterization of the algebraizability of $\mathcal{L}$ can be obtained in terms of the existence of a mutual interpretability between $\mathcal{L}$ and unsorted equational logic. Hence $\mathcal{L}$ is represented over the unsorted equational logic $Eqn(\Sigma, K)$, where Σ is precisely the signature of $\mathcal{L}$ and K is a class of Σ-algebras. Each Σ-algebra $A \in K$ can thus be seen as an interpretation for $\mathcal{L}$ along the unique homomorphism $[[\_]]_A : L \to A$. Our generalization proceeds by replacing unsorted equational logic with another suitable base logic. Following the idea in [2], we will work with a two-sorted equational logic with sorts ϕ and τ of formulas and truth values, respectively, plus an operation v from ϕ to τ, that represents the valuation map. Moreover, the two-sorted signature Σ_2 will have the operations of Σ on sort ϕ, but can have an arbitrarily chosen signature Σ' of operations on truth-values. We shall represent $\mathcal{L}$ over the two-sorted equational logic $Eqn(\Sigma_2, K_2)$, where K_2 is a class of Σ_2-algebras. In this case we say that $\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able. Here, each $A \in K_2$ can be seen as an interpretation for $\mathcal{L}$ along the valuation map v_A over the set of truth-values A_τ. The crucial observation is that v does not have to be an homomorphism, as advocated in valuation semantics [5]. Of course, we can recover the Blok-Pigozzi case by choosing $\Sigma' = \Sigma$ and K_2 as the class of all algebras whose ϕ-fragment is L and whose τ-fragment is in K, and such that v satisfies the homomorphism conditions $v(c(y_1, \ldots, y_n)) = c(v(y_1), \ldots, v(y_n))$ for every n-ary constructor $c \in \Sigma$.

Under this new approach, we show that $\mathcal{C}_1$ is algebraizable using the class K_2 of two-sorted algebras whose truth-values form a Boolean algebra, in such a way that the valuation map v fulfills the homomorphism conditions for every connective, except for the paraconsistent negation. Every formula $\varphi \in L$ is translated to the τ-equation $v(\varphi) = \top$, and every τ-equation $t_1 = t_2$ is translated to a formula $t_1^* \equiv t_2^*$. Here, t^* is obtained by taking advantage of the usual representation of classical negation in $\mathcal{C}_1$.

One of the most important tools of the Blok-Pigozzi approach is the *Leibniz operator* Ω that maps each theory of $\mathcal{L}$ to the largest congruence on L that is compatible with the theory. In our case, since the valuation map v is not necessarily an homomorphism, we may end up having contexts δ (formulas in one variable) such that $v(\varphi) = v(\psi)$ does not imply $v(\delta(\varphi)) = v(\delta(\psi))$. If the previous implication holds we call $\delta(\_)$ a *congruent context*. An equivalence relation $\sim$ on L is then called a *semi-congruence* if $\varphi \sim \psi$ implies $\delta(\varphi) \sim \delta(\psi)$ for every congruent context $\delta(\_)$. In analogy, we can now also define an operator "Ω'' that maps each theory of $\mathcal{L}$ to the largest semi-congruence on L compatible with the theory. Using "Ω'' we can generalize the notion of protoalgebraization. We say that $\mathcal{L}$ is proto-$Eqn(\Sigma_2, K_2)$-able if $\langle \varphi, \psi \rangle \in$ "$\Omega''(\Gamma)$ implies $\Gamma \cup \{\varphi\} \dashv\vdash \Gamma \cup \{\psi\}$.

We can now prove that the general characterization properties of the Blok-Pigozzi approach with respect to the Leibniz operator carry over to our more general setting.

Theorem *$\mathcal{L}$ is proto-$Eqn(\Sigma_2, K_2)$-able iff "Ω'' is monotone.*

When a logic is $Eqn(\Sigma_2, K_2)$-able we can relate, in a strong sense, the operator "Ω'' with the corresponding representation map $\theta : \mathcal{L} \to Eqn(\Sigma_2, K_2)$. In fact, for any theory Γ of $\mathcal{L}$, we can show that "$\Omega''(\Gamma) = \{\langle \varphi, \psi \rangle : \theta(\Gamma) \models_{Eqn(\Sigma_2, K_2)} v(\varphi) = v(\psi)\}$.

Theorem *$\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able iff "Ω'' is injective and sup-preserving.*

Since sup-preservation implies monotonicity, we also have that every $Eqn(\Sigma_2, K_2)$-able logic is proto-$Eqn(\Sigma_2, K_2)$-able.

In this work we generalized the Blok-Pigozzi theory of algebraization of logics and got some similar results. Still there are many open problems. In particular, there are two very important open questions. Given a logic $\mathcal{L}$ is it always possible to find a truth-values signature Σ' such that $\mathcal{L}$ is $Eqn(\Sigma_2, K_2)$-able? If we fix a truth-values signature Σ' and the valuation axioms, is there a proto-$Eqn(\Sigma_2, K_2)$-able logic that is not $Eqn(\Sigma_2, K_2)$-able? Nevertheless this work leaves good perspectives for further generalizations, namely by choosing other interesting base logics to replace the role played by unsorted equation logic in the Blok-Pigozzi case. We have already some preliminary work on abstracting the relevant properties of unsorted equational logic that are essential to the algebraization process.

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Co-products of Heyting algebras

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A *Heyting algebra* $(H, \wedge, \vee, \to, 0, 1)$ is a bounded distributive lattice $(H, \wedge, \vee, 0, 1)$ with an additional binary operation $\to: H \times H \to H$ satisfying

$$x \leq a \to b \text{ if and only if } a \wedge x \leq b.$$

It is well known that the class $\mathbb{HA}$ of Heyting algebras is equationally definable, hence forms a variety.

Suppose $\mathbb{K}$ is a class of Heyting algebras and $A, B \in \mathbb{K}$. The $\mathbb{K}$-*coproduct* of A and B is a Heyting algebra $A \otimes B \in \mathbb{K}$ with Heyting algebra homomorphisms $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ satisfying the following universal property: For every Heyting algebra $H \in \mathbb{K}$ with Heyting algebra homomorphisms $f : A \to H$ and $g : B \to H$, there exists a unique Heyting algebra homomorphism $h : A \otimes B \to H$ such that $h \circ i_A = f$ and $h \circ i_B = g$. It follows that if $A \otimes B$ exists, then it is unique up to Heyting algebra isomorphism. If we replace in the definition of $\mathbb{K}$-*coproduct* of A and B the homomorphism $i_A : A \to A \otimes B$ and $i_B : B \to A \otimes B$ to be injective, then we have the definition of $\mathbb{K}$-*free product*. We call $A \otimes A$ the $\mathbb{K}$-*copower* of A. Since $\otimes$ is associative, we can also define the *n-th* $\mathbb{K}$-*copower* of A as $A \otimes \cdots \otimes A$ (n-times).

Proposition 1 If $\mathbb{V} \subseteq \mathbb{HA}$ is a variety of Heyting algebras and $A, B \in \mathbb{V}$, then the $\mathbb{V}$-coproduct of A and B exists.

Proposition 2 In the variety $\mathbb{HA}$ a co-product coincides with a free product.

Let (X, R) be a partially ordered set (for short, a *poset*) and $Q \subseteq X$. Then we say that Q is an *upper cone* (or simply *a cone*) if, whenever $x \in Q$ and $R(x, y)$, it follows that $y \in Q$. When (X, R) is a poset, we sometimes represent the one as $(X, \leq)$. We say that x *covers* y if $y \leq x$ and $y \neq x$ and there is no z such that $z \neq y$, $z \neq x$ and $y \leq z \leq x$. Say that $Y \subseteq X$ *totally covers* an element $x \in X$, in the notation $x \prec Y$ or $Y \succ x$, if Y coincides with the set of all elements which cover x. If Y is a singleton then we say that the element totally covers $x \in X$.

Let (X, R) be a poset and $x \in X$. A *chain out* of x is a linearly ordered subset (i.e. for every y, z from the subset either yRz or zRy) of X with the least element x; a *depth of* x, in the natation $d(x)$, denotes the supremum cardinality of chains out of x.

Let $(Y_1, \leq_1)$, $(Y_2, \leq_2)$ be finite posets. Let the elements of $Y_1 \times Y_2$ be colors of our desired poset (X, R) (i.e. any element $x \in X$ has a color $Col(x) \in Y_1 \times Y_2$), some upper cones of which will form a Heyting algebra corresponding to the co-product of H_1 and H_2, which are Heyting algebras of all upper cones of $(Y_1, \leq_1)$ and $(Y_2, \leq_2)$, respectively. Let us construct (X, R) by levels (i.e. by elements of fixed depth) in the following way. The set $maxX$ of maximal elements of X is $maxY_1 \times maxY_2$. For every $x \in maxX$ $Col(x) = x \in Y_1 \times Y_2$. $maxX$ is the set of elements of X having depth 1, i.e. $maxX = \{x \in X : d(x) = 1\}$, which we denote by X_1 and (X_1, R_1) is a poset, where $R_1(x, y) \Leftrightarrow x = y$. For any element $(x, y) \in X_1$ there exists an element $U(x_1, y_1)$ with $Col(U(x_1, y_1)) = (x_1, y_1)$ such that $U(x_1, y_1) \prec (x, y)$ iff $[(x_1 \prec_1 x \ \& \ y_1 \prec_2 y) \vee (x = x_1 \ \& \ y_1 \prec_2 y) \vee (x_1 \prec_1 x \ \& \ y = y_1)]$. For anti-chain $U \subseteq X_1$ there exists an element $U(x, y)$ with $Col(U(x, y)) = (x, y)$ such that $U(x, y) \prec U$ iff $[(x \prec_1 \pi_1(U) \ \& \ y \prec_2 \pi_2(U)) \vee (x = \pi_1(u)$ for every $u \in U \ \& \ y \prec_2 \pi_2(U)) \vee (x \prec_1 \pi_1(U) \ \& \ y = \pi_2(u)$ for every $u \in U)]$. Here and further $\pi_i(U) = \{\pi_i(u) : u \in U\}$, $i = 1, 2$. By these elements we have constructed the set of all elements of depth 2. We denote through X_2 the set of all elements having a depth less or equal 2 and (X_2, R_2) is a poset, where R_2 is an order relation obtained by the construction.

Let us suppose that a poset (X_k, R_k) is constructed for $k \geq 2$. Let us construct (X_{k+1}, R_{k+1}) in the following way. For any element $u \in X_k$, with $Col(u) = (x, y)$, there exists an element $u(x_1, y_1)$ with $Col(u(x_1, y_1)) = (x_1, y_1)$ such that $u(x_1, y_1) \prec u$ iff $[(x_1 \prec_1 x \ \& \ y_1 \prec_2 y) \vee (x = x_1 \ \& \ y_1 \prec_2 y) \vee (x_1 \prec_1 x \ \& \ y = y_1)]$. For anti-chain $U \subseteq X_k$, such that $U \cap (X_k - X_{k-1}) \neq \emptyset$, there exists an element $U(x, y)$ with $Col(U(x, y)) = (x, y)$ such that $U(x, y) \prec U$ iff $[(x \prec_1 \pi_1(U) \ \& \ y \prec_2 \pi_2(U)) \vee (x = \pi_1(u)$ for every $u \in U \ \& \ y \prec_2 \pi_2(U)) \vee (x \prec_1 \pi_1(U) \ \& \ y = \pi_2(u)$ for every $u \in U)]$. By these elements we have constructed the set of all elements of depth $k+1$. We denote through X_{k+1} the set of all elements having a depth less or equal $k+1$ and (X_{k+1}, R_{k+1}) is a poset, where R_{k+1} is an order relation obtained by the construction.

It is clear that $X_k \subset X_{k+1}$, $R_k \subset R_{k+1}$. Let $(X, R) = \bigcup_{k=1}^{\infty}(X_k, R_k)$. Let V_p be the set of all elements a of X such that $Col(a) = (p, y)$, $y \in Y_2$, and V_q be the set of all elements b of X such that $Col(b) = (x, q)$, $x \in Y_1$. According to the construction of (X, R) the sets V_p and V_q, for $(p, q) \in Y_1 \times Y_2$, are upper cones. Let H be a Heyting algebra generated by $\{V_p : p \in Y_1\} \cup \{V_q : q \in Y_2\}$.

Theorem 3 The Heyting algebra H is a $\mathbb{HA}$-coproduct of the Heyting algebras H_1 and H_2, i.e. $H = H_1 \otimes H_2$.

Theorem 4 For a variety $\mathbb{V}$ of Heyting algebras the following conditions are equivalent:
1) $\mathbb{V}$ is locally finite.
2) The $\mathbb{V}$-coproduct of any two finite $\mathbb{V}$-algebras is finite.
3) Finite $\mathbb{V}$-copowers of finite $\mathbb{V}$-algebras are finite.
4) Either $\mathbb{V}$ is the variety of Boolean algebras or finite $\mathbb{V}$-copowers of $\mathbf{3} \in \mathbb{V}$ are finite, where $\mathbf{3}$ is three-element Heyting algebra.

For a constant more

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1. Let s be an increasing map from a poset X *onto* a poset Y, satisfying $s \leq_X = \leq_Y s$ (juxtaposition denotes composition of relations). Generate from this datum the (sketch of) diagram obtained by adding the relations $r = \leq_Y s$ and $d = \overset{-1}{\leq_Y} s$ from Y to X.

Such a diagram in the category of relations between posets enters in a duality * with an analogous diagram in the category of maps between complete, completely distributive, algebraic and "atomic" Heyting algebras ("atomic" means that for every x, y such that $x \not\leq y$, there is a non-null join irreducible α such that $\alpha \leq x$ and $\alpha \not\leq y$), in which $s^\star, r^\star, d^\star$ are, respectively, a complete monomorphism of such H.a's from $K = Y^\star$ into $H = X^\star$, the graph of a complete existential quantifier on H and the graph of a complete universal quantifier on H; these last two determine on H a complete "atomic" Heyting algebra.

2. Let the first diagram be taken in the category of relations between Priestley spaces dual of H.a's in the Priestley duality, and be so that, in addition, s is continuous, and the images under r (resp. d) of clopen increasing (resp. clopen decreasing) sets are clopen increasing (resp. clopen decreasing).

Then such a diagram enters in an extended Priestley-Cignoli-Lafalce-Petrovich duality * with an analogous diagram in the category of maps between H.a's, in which s^*, r^*, d^* are, respectively, a monomorphism of H.a's from $K = Y^*$ into $H = X^*$, the graph of an existential quantifier on H and the graph of an universal quantifier on H; these last two determine on H a monadic H.a.

3. Leaving tacit the intermediary application of a "forgetting topology" covariant functor, the covariant functor ** is a genuine *perfect completion* functor, extending that of Hansoul, and which apply to H.a's, homomorphisms, $\vee$-hemimorphisms and $\wedge$-hemimorphisms between H.a's, so that it can be extended to *monadic* H.a's and to *monadic* homomorphisms between monadic H.a's (this can be proven using representations by suitable diagrams).

4. The graph of a *constant* of a monadic H.a - say $\mathcal{M}$ - represented as before, including a monomorphism of H.a's j from K into H, is the graph of an homomorphism of H.a's γ from H onto K satisfying $\gamma j = \mathrm{Id}$.

Hence $\gamma^{**} j^{**} = \mathrm{Id}$ holds, so that the graph of γ^{**} is the graph of a constant of $\mathcal{M}^{**}$, of which the intersection with $H \times K$ is the graph of γ.

Hence $\mathcal{M}^{**}$ admits of at least as many constants as $\mathcal{M}$. However, as some monadic H.a's admit of closed elements which forbid the existence of any constant, there is no theorem generalizing Halmos' theorem of existence of rich extensions of monadic boolean algebras to monadic H.a's.

5. In place of the theorem so refuted, we propose the following:

From any monadic H.a - say $\mathcal{M}$ - given as before, including in particular a monomorphism of H.a's j from a H.a K into a H.a H, one can build a monadic H.a $\mathcal{M}^+$ extending $\mathcal{M}$ so that

(a) every constant γ of $\mathcal{M}$ is induced on $\mathcal{M}$ by a constant γ^+ of $\mathcal{M}^+$;

(b) $\mathcal{M}^+$ admits of a constant γ_0 *more*, *i.e.* which *differs* of all the preceding ones, and which is such that $k \leq a \leq k'$ holds *iff* $k \leq \gamma_0(a) \leq k'$ holds, for every $a \in H$ and every $k, k' \in K$.

Starting from j, the construction begins by passing to the epimorphism $\mathrm{Id} \times j^*$ from $H^* \times H^*$ onto $H^* \times K^*$.

Then, replace the space $H^* \times H^*$ by its closed subspace S to which pertain the $(x, y) \in H^* \times H^*$ satisfying the condition $j^*(x) = j^*(y)$. The diagonal Δ of $H^* \times H^*$ is contained in S and closed, so that Δ is homeomorphic to H^*.

Replace simultaneously the epimorphism $\mathrm{Id} \times j^*$ by its yet epimorphic restriction s from S onto $s[S] = s[\Delta]$; indeed, Δ is a system of representatives of the classes modulo the equivalence $\sim$ defined by the condition $s((x, y)) = s((x', y'))$ or, said otherwise, $\Delta = S/\sim$ holds.

Now, the dual of such a system generates the filter kernel of a constant, sometimes as an "ideal element", if it is closed increasing, in the duality *, and properly just if it is increasing, in the duality *.

In general, Δ isn't increasing; it can just be said that it possesses a greatest non void increasing part N. The preceding operations associated, to each constant γ of $\mathcal{M}$, a closed increasing system of representatives of the classes modulo $\sim$ in S, of which the intersection with Δ is contained in N. By passing to the complete atomic H.a given as aforesaid, including the monomorphism $\bar{s}^\star$, where $\bar{s}$ is an extension of s which will be described in the next paragraph, this system, which remains untouched, produces the generator of the filter kernel of the constant γ^+.

In order to create the dual of a generator for the filter kernel of γ_0, the poset S will then been extended in a set $\bar{S}$ by *grafting* to S a "copy" of $\Delta - N$, the complement of N relatively to Δ. In the same time, s will be extended in an epimorphism $\bar{s}$ so that $\bar{s}(\bar{x}) = s((x,x))$ for every $x \in H^*$, where $\bar{x}$ is the "copy" of (x,x), extending the definition of $\bar{x}$ as equal to (x,x) on N. The order of S will also be extended in an order on $\bar{S}$, which is well defined by the conventions that $\bar{x}$ covers (x,x) on $\Delta - N$ and that $(x,x) < (y,y)$ implies $\bar{x} < \bar{y}$ on Δ. So $\bar{\Delta}$ becomes an increasing system of representatives of the classes modulo the extended equivalence on $\bar{S}$ defined by the condition $\bar{s}(x) = \bar{s}(y)$.

Then, the last step remaining to do is to form $\mathcal{M}^+$ by introducing $\bar{r} = \leq \bar{s}$ and $\bar{d} = \overset{-1}{\leq} \bar{s}$ and by passing then from $\bar{s}, \bar{r}, \bar{d}$ to $\bar{s}^\star, \bar{r}^\star, \bar{d}^\star$ respectively.

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Weakly standard BCC-algebras

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The notion of a BCK-algebra was introduced in 60's by Imai and Iséki as an algebraic formulation of Meredith's BCK-implicational calculus. Left-distributive BCK-algebras, known as Hilbert algebras, form an algebraic counterpart of the logical connective implication in intuitionistic logic. When solving the problem whether the class of all BCK-algebras forms a variety, Komori introduced the class of BCC-algebras and proved that this class is not a variety. The axioms of BCC-algebras allow us to define a natural order relation on a base set. It is well known that there is no restriction to the corresponding posets in that sense that one can define on every poset a structure of a BCC-algebra. This holds even for Hilbert algebras and the corresponding structures are called order-algebras. Order algebras satisfy a very strong property that every subset containing the distinguished element 1 (considerd as a logical value "true") form a subalgebra. A natural problem to describe all BCC-algebras in which every 3-element subset containing 1 is a subalgebra was solved by the author in 2002, the resulting algebras are here called standard BCC-algebras. The aim of my talk is to present a new construction of BCC-algebras from posets requiring a weaker condition on its subalgebras. Resulting structures are called weakly-standard BCC-algebras.

Priestley duality for distributive semilattices

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Stone duality for Boolean algebras (1936) was easily extended to bounded distributive lattices (1937) and to bounded distributive semilattices (see for instance GrŠtzer, 1960). In 1970, H.A. Priestley developed a new duality for bounded distributive lattices, giving a very convenient alternative to Stone duality and also reducing to Stone duality in the Boolean case. Priestley duality is important in itself due to its numerous applications, and also because it opened the way to the most fruitful theory of natural dualities. So it is quite natural to try to extend Priestley duality to the semilattice setting. Our purpose here is twofold.

Priestley topology on the set of prime ideals of a bounded distributive lattice is nothing else than the patch topology associated to the Stone topology. We first show that it is not possible to obtain a duality for distributive semilattices by considering the patch topology associated to their Stone duals.

Then we device a substitute duality, which reduces to Priestley duality when distributive lattices are concerned. To this end, we need to consider not only the set X_1 of all prime ideals on a bounded distributive semilattice S (ideals whose complement is lower directed) but also the set X of all weak prime ideals (I is weak prime if for all $x_1 \dots x_n \notin I$ and $i \in I$ there is $x \leq x_1, , \dots, x_n$, such that $x \nleq i$). It can be proved that the object mapping

$$S \mapsto (X, X_1)$$

where X is endowed with the analog of Priestley topology and order (X is indeed a Priestley space in which X_1 is dense) can be lifted to a dual equivalence for the category of bounded distributive semilattices. Conditions are given under which X is the (order-) Stone-Cech compactification of X_1.

MacNeille completions of modal algebras

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For a modal algebra (B, f) there are two natural ways to extend the operation f to an operation on the MacNeille completion of the Boolean algebra B. The resulting structures are called the lower and upper MacNeille completions of (B, f). We consider lower and upper MacNeille completions for various varieties of modal algebras. In particular, we characterize the varieties of closure algebras and diagonalizable algebras that are closed under lower and/or upper MacNeille completions. We also show that there is a variety of modal algebras that is closed under neither lower nor upper MacNeille completions, but with the axiom of choice one can show that each member of this variety has a MacNeille completion that belongs to the variety, and that this this result implies the Boolean ultrafilter theorem.

Archimedean completeness and subvarieties of ΠMTL-algebras

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The variety $\mathcal{V}$ of ΠMTL-algebras was introduced by Hájek as an algebraic counterpart of Product Monoidal T-norm Based Logic (ΠMTL for short). Each algebra belonging to this variety is in fact bounded commutative integral residuated lattice $(L, *, \to, \wedge, \vee, 0, 1)$ which satisfies prelinearity $(p \to q) \vee (q \to p) = 1$ and the cancellative law, i.e., $x * z = y * z$ implies $x = y$ for $z \neq 0$. It was shown in [1] that $\mathcal{V}$ is generated by its totally ordered members (ΠMTL-chains). A ΠMTL-chain is called Archimedean if for any $0 < x < y < 1$ there is an n such that $y^n \leq x$.

In [2] Hájek posed an interesting question whether Product Monoidal T-norm Based Logic satisfies so-called Archimedean completeness, i.e. a formula φ is provable iff φ is a tautology over each Archimedean ΠMTL-chain. This problem is equivalent to the question whether the class of Archimedean ΠMTL-chains generates the variety of ΠMTL-algebras. As a partial answer to this problem, he introduced the following quasi-identity Q:

$$(p \to q) \to q = 1 \Rightarrow p \vee q \vee \neg q = 1$$

and proved that the quasi-variety given by the identities defining $\mathcal{V}$ and Q ($\mathcal{V}$+Q for short) contains all Archimedean ΠMTL-chains but is strictly smaller than $\mathcal{V}$.

In this talk we give a negative answer to the previous question and show that $\mathcal{V}$ is not generated by the Archimedean ΠMTL-chains. We in fact prove that there is an identity A_1:

$$((p \to q) \to q)^2 \leq p \vee q \vee \neg q$$

defining a subvariety of $\mathcal{V}$ containing all Archimedean ΠMTL-chains. Moreover, we even show that the subvariety $\mathcal{V}$+A_1 is equal to the quasi-variety $\mathcal{V}$+Q.

Then we generalize the identity A_1 and get the sequence of identities A_n for a natural number n:

$$\bigwedge_{i=1}^{n} ((p_{i-1} \to p_i) \to p_i)^2 \leq p_0 \vee \bigvee_{i=1}^{n} (p_i \vee \neg p_i).$$

The varieties $\mathcal{V}$+A_n form a strictly increasing chain whose limit is $\mathcal{V}$. Thus we have found infinitely many varieties between the variety of product algebras and the variety of ΠMTL-algebras $\mathcal{V}$.

Finally, we give also some characterization of algebras belonging to the particular $\mathcal{V}$+A_n. For example, an algebra L is a member of $\mathcal{V}$+A_1 iff L is a product algebra or L is subdirectly irreducible and L/θ is a product algebra, where θ is the monolith (i.e., the minimal nontrivial congruence).

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On BL and weak-BL (MTL) algebras and related algebras

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We continue the investigations made in [5–8], based on [4].

Let $\mathcal{A} = (A, \vee, \wedge, \rightarrow, 0, 1)$ be a BCK(P) lattice (i.e. a reversed left BCK algebra with condition (P) (product) which is a lattice) (BCK(P) lattices are categorically isomorphic to residuated lattices [4]), where:
(P) for all x, y, $x \odot y \overset{notation}{=} \min\{z \mid x \leq y \rightarrow z\}$.

Let us consider in $\mathcal{A}$ the two conditions, divisibility and pre-linearity, from the definition of a BL algebra [3]:
(div) $x \wedge y = x \odot (x \rightarrow y)$,
(prel) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

A Hájek(P) algebra (BL algebra) is a BCK(P) lattice (residuated lattice) satisfying (div) and (prel) conditions and a weak-Hájek(P) algebra (weak-BL algebra [2] = MTL algebra [1]) is a BCK(P) lattice (residuated lattice) satisfying only (prel) condition.

Any linearly ordered BCK(P) lattice (residuated lattice) satisfies (prel) condition, hence is a weak-Hájek(P) algebra (weak-BL algebra = MTL algebra).

We make the following decompositions:
(prel) $\Leftrightarrow$ $(C_\vee)$+$(C_\rightarrow)$ $\Leftrightarrow$ $(C_\wedge)$ + $(C_\diamondsuit)$, (div) $\Leftrightarrow$ $(C_\rightarrow)$+ $C_\wedge)$+ (C_X).

Consequently, a Hájek(P) algebra (BL algebra) is a weak-Hájek(P) algebra (weak-BL algebra = MTL algebra) satisfying (C_X) condition.

If we define the negation by: $x^- = x \rightarrow 0$, for all x, and we introduce the (DN) (double negation) condition, the (WNM) (weak nilpotent minimum) condition and the (R6) condition [11]:
(DN) $(x^-)^- = x$, for all x,
(WNM) $(x \odot y)^- \vee [(x \wedge y) \rightarrow (x \odot y)] = 1$, for all x, y,
(R6) $(x \rightarrow y) \vee [(x \rightarrow y) \rightarrow (x^- \vee y)] = 1$, for all x, y,
we have that: a Wajsberg algebra (MV algebra) is a Hájek(P) algebra (BL algebra) satisfying (DN) condition, a weak-R_0 algebra [11] (IMTL algebra [1]) is a weak-Hájek(P) algebra (weak-BL algebra = MTL algebra) satisfying (DN) condition, an R_0- algebra is a weak-R_0 algebra satisfying (R6) condition and an NM algebra is an IMTL algebra satisfying (WNM) condition. Note that R_0 algebras and NM algebras are categorically isomorphic [10].

Let us finally introduce the (R0) condition:
(R0) $(x \rightarrow y^-) \vee [(x \rightarrow y^-) \rightarrow (x^- \vee y^-)] = 1$, for all x, y.

We prove that in an weak-R_0 algebra (IMTL algebra) we have:
$(R6) \Longleftrightarrow (WNM) \Longleftrightarrow (R0)$,

[1] Dedicated to the memory of my dear father, Angelică, teacher of Latin, Greek and mathematics, left for eternity this month. (28 March 2005)

while in a weak-Hájek(P) algebra (weak-BL algebra= MTL algebra), we have:
$(R6) \Longrightarrow (WNM) \Longrightarrow (R0)$.

We give examples of weak-Hájek(P) algebra (MTL algebra) and Hájek(P) algebra (BL algebra) satisfying (R6) condition, and hence satisfying (WNM) and (R0). We give examples of weak-Hájek(P) algebra (MTL algebra) and Hájek(P) algebra (BL algebra) satisfying (WNM) condition, and hence satisfying (R0) condition, but not satisfying (R6). We give examples of weak-Hájek(P) algebra (MTL algebra) and Hájek(P) algebra (BL algebra) satisfying (R0) condition and not satisfying (WNM) and (R6).

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Geometry of associativity - theory and application

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Commutativity of (binary) operations, that is the interchangeability of their arguments is easily seen from the graph of the operations. The meaning of commutativity is just the invariance of the graph with respect to a reflection to the plane defined by $x = y$. Similar geometrical description for associativity is not known. That is, associativity of binary operations can not be seen simply by "looking at" their graphs. Investigation of associativity is one of the major problems in algebra. In our opinion the reason of the difficulty of investigation of associativity is that we are able to "see things" in three dimensions only. In three dimensions the graph of an operation is defined as follows: There are two independent variables x and y, and the value $x*y$ is taken in the third axle. The meaning of associativity together with commutativity is that we can freely interchange the operands of the operation, that is, any two operands are interchangeable. We have seen above that interchangeability is just the invariance of the graph with respect to a reflection to a plane. Consider now the graph of an associative and commutative operation in four dimensions: There are three independent variables x, y, and z, and the value $x*y*z$ is taken in the fourth axle. It follows from the previous arguments that associativity and commutativity together are equivalent to the invariance of the four-dimensional graph with respect to three reflections to the "spaces" $x = y$, $x = z$, and $y = z$, respectively. That is, if we were able to "see things" in four dimension, then associativity together with commutativity were easily seen from the graph of the operation "for the first sight".

Similar geometrical description of associativity is not known as of today.

We have reported on a surprising geometrical property of a special class of associative functions in [9]. Namely, if we, in addition to commutativity and associativity, assume that the "border line" in between the 0 and the positive part of the graph is the function $y = 1 - x$, then the corresponding graphs are rotation-invariant with respect to a rotation with 120 degree. Moreover, vertical sections of graphs of such operations show as well a kind of symmetry.

The mentioned geometrical property does not characterize associativity. That is, there exist rotation-invariant functions which are not associative. The question suggests itself: —Does there exist a geometrical characterization which does not assume the "border line" property, and which do characterize associativity ?

In this talk we shall give a geometric characterization of commutative residuated semigroups (in particular, left-continuous t-norms) based on the notion of rotation-invariance and the notion of nuclei of quantale structures (see [11]). As a consequence, associativity can be "seen" even from the three-dimensional graph. This geometrical understanding of associativity has already led to:

1. New results in the field of residuated lattices (rotation-construction and rotation-annihilation construction [3], for example) and in the corresponding logics (see for instance [1] for a rotation-invariance based new axiomatization for IMTL), and

2. An elegant solution of a long-standing open problem of C. Alsina, M. J. Frank and B. Schweizer concerning the convex combination of t-norms [9]. Namely, at the end of the talk we shall present an answer to the question whether the convex combination of two left-continuous t-norms can ever be a t-norm.

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Critical elements in the lattice of varieties of Fan algebras

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It is known that the Lewis modal system $S4$ contains only five critical (alias, *pretabular*) normal extensions (Esakia, Meskhi). The modal system $S4 + p \to \Box(\Diamond p \to p)$ is one of them. We study the modal system $Fan = K4 + p \to \Box(\Diamond p \to p)$ and the lattice of its normal extensions in the setting of varieties of corresponding modal algebras. We present full description of this lattice and prove that there are exactly eight critical normal extensions of the modal system Fan. First we investigate a special proper extension of Fan; namely the modal system $SM = K4 + \Box(\Diamond p \to p)$. When $\Diamond$ is interpreted as the topological *limit* operation, it is known, that a formula ϕ is provable in the system SM iff ϕ is valid in every submaximal topological space, moreover, SM is the logic of submaximal topological spaces (D. Gabelaia, PhD thesis). Recently, submaximal spaces have been investigated by A. Arhangelskii and P. Collins.

The methods we are using in this paper are of algebraic nature, so instead of dealing with modal systems we will consider varieties of corresponding modal algebras.

McKinsey and Tarski in their early work introduced the definition of Derivative algebras - boolean algebras with additional unary operator d, called derivative operator, satisfying $d0 = 0,\ d(a \vee b) = da \vee db,\ dda \leq da$.

Definition. A derivative algebra (B, d) is called a *submaximal algebra* (respectively *Fan algebra*) if the operator d satisfies the additional identity: $d(da - a) = 0$ (respectively $a \wedge d(da - a) = 0$). The variety of all *submaximal algebras* (*Fan algebras*) is denoted by **SM** (respectively by **Fan**).

We show, that the descriptive frame associated with an arbitrary derivative algebra of the variety **SM** has depth no more than two, hence variety **SM** is locally finite. It follows that any subvariety of **SM** is generated by its finite subdirectly irreducible members. Using the fact that finite subdirectly irreducible algebras correspond to finite rooted Kripke frames, we can reduce the study of the lattice of subvarieties of **SM** to the study of finite rooted SM-frames.

Call a Kripke frame (X, R) *weak poset* and R - weak partial order iff the reflexive closure of the relation R is a partial order.

The relation R of a descriptive frame (X, R) associated with any submaximal algebra is a weak poset.

A rooted weak poset (X, R) with root point x_0 is called fan if $(X - \{x_0\}, R) \neq \emptyset$ is an anti-chain.

Lemma. A Kripke frame corresponding to a finite subdirectly irreducible submaximal algebra is either a finite fan with an irreflexive root point or the one point Kripke frame.

It follows from the above lemma that the set of all finite subdirectly irreducible submaximal algebras is completely determined by finite fans with irreflexive root point. With each finite subdirectly irreducible submaximal algebra we associate a certain invariant - an ordered pair of natural numbers. Using this fact, we prove that:

Theorem. The lattice of subvarieties of the variety **SM** is countable.

Theorem. The varieties V_i, where $i = 1, 2, 3, 4$, are critical subvarieties of **SM**. There are no more critical subvarieties in **SM**.
$V_1 = (SM) + dd1 = 0;$
$V_2 = (SM) + d1 = 1;$
$V_3 = (SM) + dd1 = d1 + da - ((d - d1) \vee a) = 0 + C(a - b - d1) \wedge C(b - a - d1) = 0;$
$V_4 = (SM) + dd1 = d1 + d(da - db) \wedge d(db - da) = 0 + da - ((d - d1) \vee a) = 0;$
Where $Ca = da \vee a$.

The variety **SM** is a proper subvariety of **Fan** variety. In fact, we show, that **Fan** is the smallest variety containing **SM**$\cup$**T**, where **T** is a variety generated by all algebras corresponding to finite fans with the reflexive root.

We show, that like **SM**, the variety **T** contains exactly four critical subvarieties and consequently we have the following theorem.

Theorem. The lattice of subvarieties of the variety **Fan** is countable. There are eight critical systems in Fan .
The four remaining varieties are:
$V_5 = (Fan) + a \leq da;$
$V_6 = (Fan) + d(da - a) = da - a + d - d1 = d1;$
$V_7 = (Fan) + d(da - a) = da - a + d(da - db) \wedge d(db - da) = 0 + da - ((d - d1) \vee a) = 0;$
$V_8 = (Fan) + d(da - a) = da - a + da - ((d - d1) \vee a) = 0 + C(a - b - d1) \wedge C(b - a - d1) = 0;.$

Halldén completeness and pseudo-relevance property of substructural logics

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We will give algebraic characterizations of Halldén completeness (HC), pseudo-relevance property (PRP) and principle of variable separation (PVS) for commutative substructural logics. Though our characterization goes essentially along the same line as Lemmon (1966), Wroński (1976) and Maksimova (1995), some of algebraic conditions that are equivalent in the case of modal or intermediate logics are shown to diverge by the lack of contraction rule or weakening rule, and sometimes a certain modification of definitions of these properties becomes necessary. This is a joint work with H. Ono.

1. Halldén Completeness

Let $\mathcal{L}$ be a substructural logic over $\mathbf{FL_e}$ (intuitionistic linear logic without exponentials) and $\vdash_{\mathcal{L}}$ be the deducibility relation determined by the logic $\mathcal{L}$. For each FL_e-algebra $\mathbf{A}$, $L(\mathbf{A})$ denote the set of all formulas which are valid in $\mathbf{A}$. Then, the following is a generalization of results by Lemmon (1966) and Wroński (1976).

Theorem 1. The following conditions are equivalent for any substructural logic $\mathcal{L}$ over $\mathbf{FL_{ew}}$ (i.e. $\mathbf{FL_e}$ with weakening rule).

(HC) $\mathcal{L}$ is Hallden complete, i.e, for every formula ϕ and ψ which have no variables in common, $\vdash_{\mathcal{L}} \phi \vee \psi$ implies $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$,

(MI) $\mathcal{L}$ is meet irreducible in the lattice of all substructural logics over $\mathbf{FL_{ew}}$, i.e, $\mathcal{L}$ cannot be represented as the intersection of two incomparable logics,

(WC) $\mathcal{L} = L(\mathbf{A})$ for some well-connected FL_{ew}-algebra $\mathbf{A}$, i.e, FL_{ew}-algebra such that $x \vee y = 1$ implies $x = 1$ or $y = 1$.

A similar result holds also for substructural logics over $\mathbf{FL_e}$. To do so, we need to modify definitions of Halldén completeness and well-connected algebras as follows.

(HC') for every formulas ϕ and ψ which have no variables in common, $\vdash_{\mathcal{L}} (\phi \wedge 1) \vee (\psi \wedge 1)$ implies $\vdash_{\mathcal{L}} \phi$ or $\vdash_{\mathcal{L}} \psi$,

(WC') $\mathcal{L} = L(\mathbf{A})$ for some FL_e-algebra $\mathbf{A}$ satisfying the following: for all $x, y \in A^- = \{a \in A \mid a \leq 1\}$, $x \vee y = 1$ implies $x = 1$ or $y = 1$.

Obviously, each of them is equivalent to the original one, when weakening rule holds in a given logic $\mathcal{L}$. For, weakening rule implies that 1 is equal to the greatest element. Moreover, whenever the axiom of n-potency, i.e, $\alpha^n \to \alpha^{n+1}$, holds in $\mathcal{L}$ (over $\mathbf{FL_e}$), the following condition is also equivalent to the Halldén completeness.

(SI) $\mathcal{L} = L(\mathbf{A})$ for some subdirectly irreducible FL_e-algebra $\mathbf{A}$.

2. Pseudo-Relevance property and Principle of Variable Separation

A logic $\mathcal{L}$ has the PRP if for all formulas ϕ and ψ without common variables the condition $\phi \vdash_{\mathcal{L}} \psi$ implies $\phi \vdash_{\mathcal{L}} \bot$ or $\vdash_{\mathcal{L}} \psi$. It is easy to see that for logics over $\mathbf{FL_{ew}}$, the PRP follows from the deductive interpolation property (DIP). But this is not always the case for logics over $\mathbf{FL_e}$. Maksimova's result (1995) on PRP can be extended to substructural logics over $\mathbf{FL_e}$, as shown below.

Theorem 2. For any substructural logic $\mathcal{L}$ over $\mathbf{FL_e}$, $\mathcal{L}$ has PRP if and only if every two subdirectly irreducible FL_e-algebras of $V(\mathcal{L})$ are jointly embeddable into a suitable algebra in $V(\mathcal{L})$.

Komori's result (1978) can be extended to any logic over $\mathbf{FL_{ew}}$ for which Glivenko's theorem holds.

Theorem 3. PRP holds always for any logic $\mathcal{L}$ which includes $\mathbf{FL_{ew}} + \neg(\alpha \wedge \neg\alpha)$.

Let us consider also PVS. A logic $\mathcal{L}$ has the PVS if for every formulas $\phi_1, \phi_2, \psi_1, \psi_2$, where $\{\phi_1, \phi_2\}$ and $\{\psi_1, \psi_2\}$ have no variables in common, the condition $\phi_1, \psi_1 \vdash_{\mathcal{L}} \phi_2 \vee \psi_2$ implies $\phi_1 \vdash_{\mathcal{L}} \phi_2$ or $\psi_1 \vdash_{\mathcal{L}} \psi_2$. Clearly, both Halldén completeness and PRP are special cases of PVS.

Theorem 4. Let $\mathcal{L}$ be a logic over $\mathbf{FL_{ew}}$. Then the following are equivalent.

1. PVS holds in $\mathcal{L}$,
2. for all subdirectly irreducible FL_{ew}-algebras $\mathbf{A}, \mathbf{B} \in V(\mathcal{L})$ there exist a well-connected (or even a subdirectly irreducible) algebra $\mathbf{C}$ in $V(\mathcal{L})$ and monomorphisms α from $\mathbf{A}$ into $\mathbf{C}$ and β from $\mathbf{B}$ into $\mathbf{C}$.

Since PVS implies HC, the above condition 2 implies **(WC)**. We can give a direct proof of this by using ultraproduct construction. Also for substructural logics over $\mathbf{FL_e}$, a similar result to Theorem 4 holds by modifying definitions of PVS and well-connected algebras in the same way as **(HC')** and **(WC')**. Moreover, we can give an algebraic characterization of PVS in the original form for them as shown below.

Theorem 5. Let $\mathcal{L}$ be a logic over $\mathbf{FL_e}$. Then the following are equivalent.

1. PVS holds in $\mathcal{L}$,

2. for all subdirectly irreducible FL_e-algebras $\mathbf{A}, \mathbf{B}$ in $V(\mathcal{L})$ and for all monoliths $x \in \mathbf{A}, y \in \mathbf{B}$, there exist a suitable (or even a subdirectly irreducible) algebra $\mathbf{C} \in V(\mathcal{L})$ and monomorphisms α from $\mathbf{A}$ into $\mathbf{C}$ and β from $\mathbf{B}$ into $\mathbf{C}$ such that $\alpha(x) \vee \beta(y) \ngeq 1$.

A new generalization of Sahlqvist theorem

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Sahlqvist theorem is known since the seventies and was generalized by many authors; comprehensive bibliography can be found in [5]. We propose a new generalization of this theorem covering some interesting cases which are probably beyond other versions of Sahlqvist theorem

Consider the propositional modal language with unary modalities $\Box_1, \ldots, \Box_n$ and a countable set of propositional variables. The variables are organized in a two-dimensional array $p_1^0, p_2^0, p_3^0, \ldots, p_1^1, p_2^1, p_3^1, \ldots, p_1^2, p_2^2, p_3^2, \ldots$ etc.

Definition 1. A $\Box$-*modality* is an arbitrary (maybe, empty) sequence $\Box_{i_1} \ldots \Box_{i_r}$.

Definition 2. A *regular formula of type 0* has a form Δp_i^0, where Δ is an arbitrary $\Box$-modality. A *regular formula of type k* is a formula $\Delta_1(POS(\bar{p}^1, \ldots, \bar{p}^{k-1}) \to \Delta_2 p_i^k)$, where Δ_1, Δ_2 are arbitrary $\Box$-modalities, $POS(\bar{p}^1, \ldots, \bar{p}^{k-1})$ is a positive modal formula, containing only variables of the form $p_j^l, l < k$.

Definition 3. A *weak Sahlqvist formula of type k* is a formula $GSA \to POS$, where GSA is built from regular formulas of the type $\leq k$ using only $\wedge, \vee, \Diamond_i$, and POS is an arbitrary positive formula.

Theorem 1. Every weak Sahlqvist formula has a computable first-order equivalent.

Theorem 2. Every weak Sahlqvist formula is d-persistent, and hence, canonical.

Weak Sahlqvist formulae arise in some natural logics.

Example 1. The formula cub_1 is theorem of $\mathbf{K}^3$ ([2], 397):

$$cub_1 = [\Diamond_1(\Box_2 p_{12} \wedge \Box_3 p_{13}) \wedge \Diamond_2(\Box_1 p_{21} \wedge \Box_3 p_{23}) \wedge \Diamond_3(\Box_1 p_{31} \wedge \Box_2 p_{32}) \wedge$$
$$\Box_1\Box_2(p_{12} \wedge p_{21} \to \Box_3 q_3) \wedge \Box_1\Box_3(p_{13} \wedge p_{31} \to \Box_2 q_2) \wedge \Box_2\Box_3(p_{23} \wedge p_{32} \to \Box_1 q_1)\ ]$$
$$\to \Diamond_1\Diamond_2\Diamond_3(q_1 \wedge q_2 \wedge q_3).$$

Definition 4. (cf. [4], Section 2.3) Let $F = (W, R)$ be a frame for unimodal language. A δ-*square* of F is a frame $F_\delta^2 = (W \times W, R_1, R_2, \Delta)$, where R_1 and R_2 are binary predicates, Δ is unary predicate, such that $(x_1, x_2)R_1(y_1, y_2)$ iff $x_1 R y_1$ and $x_2 = y_2$; $(x_1, x_2)R_2(y_1, y_2)$ iff $x_1 = y_1$ and $x_2 R y_2$; $(x_1, x_2) \in \Delta$ iff $x_1 = x_2$.

Definition 5. The logic $\mathbf{K}_\delta^2$ is the set of all formulae in the modal language with two 1-modalities $\Box_1$, $\Box_2$ and one 0-modality δ, which are valid in all δ-squares.

Example 2. The following weak Sahlqvist formula is in $\mathbf{K}_\delta^2$:
$\Diamond_2(\Diamond_1(r \wedge \Diamond_1(s \wedge \Diamond_1(\delta \wedge q))) \wedge \Diamond_1(p \wedge \Diamond_1\delta)) \wedge \Box_1(\Diamond_2 p \to \Box_1 v) \to$
$\to \Diamond_1(\Diamond_2 r \wedge \Diamond_1(\Diamond_2 s \wedge \Diamond_1(v \wedge \Diamond_2 q)))$;

Example 3. A simple example of a weak Sahlqvist formula is

$$p \wedge \Box(\Diamond p \to \Box r) \to \Diamond(r \wedge \Diamond p).$$

Remarks on the proofs.

The proofs of Theorem 1, 2 are rather standard. By syntactic reasons it is sufficient to consider only the formulae without disjunctions in the antecedent. Then by induction on the type of $\varphi = GSA \to POS$, we construct the smallest valuation θ making GSA true, and substitute the corresponding first-order formulae in POS.

To prove the d-persistence note that θ is closed in Stone-Esakia topology, and moreover, it can be approximated by clopen valuations η, which preserve the truth value of GSA in the given world.

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On normal extensions of monadic and $\{\to, \Box\}$ fragment of Grzegorczyk's modal logic

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We examine normal extensions of Grzegorczyk's modal logic over language with one propositional variable and signs of $\{\to, \Box\}$ only.

Syntactically, the Grzegorczyk logic **Grz** is characterize as the extension of **S4** by the axiom

$$(grz)\ \ \Box(\Box(p \to \Box p) \to p) \to p$$

The set of rules consists of modus ponens, substitution and necessitation.

Semantically, **Grz** logic is characterized by the class of finite reflexive, transitive trees. We consider trees with restricted depth. In fact, we investigate the following normal extensions of logics $\mathbf{Grz}^{\leq n} = \mathbf{Grz} \oplus J_n$ where

$$\begin{aligned} J_1 &= \Diamond\Box p_1 \to p_1, \\ J_{n+1} &= \Diamond(\Box p_{n+1} \wedge \sim J_n) \to p_{n+1}. \end{aligned}$$

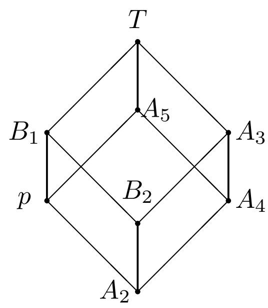
They contain logic **Grz** and the following inclusions hold:

$$\mathbf{Grz} \subset \cdots \subset \mathbf{Grz}^{\leq n} \subset \mathbf{Grz}^{\leq n-1} \subset \cdots \subset \mathbf{Grz}^{\leq 2} \subset \mathbf{Grz}^{\leq 1}.$$

To characterize logics $\mathbf{Grz}^{\leq n}$, we describe the appropriate Tarski-Lindenbaum algebras $\mathbf{Grz}^{\leq n}/_{\equiv}$. The main theorem is as follows:

Theorem 1. *Every algebra $(\mathbf{Grz}^{\leq n}/_{\equiv}, \mathbf{1}, \to, \Box)$ might be extended to a modal algebra consisting of 2^n equivalence classes generated by n atoms.*

As an example, the diagram of the Tarski-Lindenbaum algebra $\mathbf{Grz}^{\leq 3}/_{\equiv}$ is presented:



where

$$
\begin{array}{rcl}
A_1 & = & [p]_{\equiv} \\
A_2 & = & \Box A_1 \\
A_3 & = & A_1 \to A_2 \\
A_4 & = & \Box A_3 \\
A_5 & = & A_3 \to A_4 \\
B_1 & = & A_4 \to A_2 \\
B_2 & = & A_5 \to A_2
\end{array}
$$

Finally, we make an attempt to characterize in a similar way analogous extensions of other modal logics. Unfortunately, in most cases it leads to failure.

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New proof of Boolean amalgamation of orthomodular lattices

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Let G be a graph. We will write $a \perp b$ to denote the fact that $a, b \in G$ are joined by an edge. The *orthocomplement* of any subset S of G, written $S^{\perp}$ is the set of $g \in G$ such that $g \perp s$ holds for all $s \in S$. A subset S of G is *closed* if $S^{\perp\perp} = S$. It is well-known that for any graph G the set of all closed subsets of G is an *ortholattice*, where the meet is intersection, orthocomplementation is the operation just defined, and join is given by $S \vee T = (S^{\perp} \cap T^{\perp})^{\perp}$. Conversely, every ortholattice can be embedded in the algebra of closed subsets of some graph. Employing the exclusion principle characterising *orthomodular* lattices among ortholattices as these that do not contain the six-element ortholattice known as benzene ring, we can define a graph G to be orthomodular if there are no closed $X, Y \subseteq G$ with $X \subset Y$ and $X^{\perp} \cap Y = \emptyset$.

A map $\phi : G \longrightarrow H$, where G and H are graphs, is a *bounded morphism* if it satisfies the following two conditions:

- if $\phi(a) \perp \phi(b)$, then $a \perp b$.
- if $\phi(a) \not\perp u$, then there is a b with $\phi(b) = u$ and $a \not\perp b$.

Let A^G and A^H be the algebras of closed sets of G and H. If ϕ is a surjective bounded morphism, then the inverse map ϕ^{-1} is an embedding of A^H into A^G.

Although all this is folklore, it can be used to define a notion of *amalgam* for graphs, which is just the dual of the usual amalgam of a V-formation of ortho(modular)lattices.

Bruns and Harding proved (cf. Bruns, G., Harding, J., "Amalgamation of Ortholattices" Order, 14:193–209, 1998) on the positive side that (i) ortholattices have the (strong) amalgamation property, and (ii) orthomodular lattices (strongly) amalgamate over a common Boolean subalgebra. On the negative side, they gave a counterexample for amalgamation in orthomodular lattices with the common subalgebra being the Chinese lantern MO3 (with six atoms).

I will present a new proof of their positive results in the setting sketched above. The only novelty I can claim is that the proof is different and perhaps simpler in the case of Boolean amalgamation.

Difference modality in topological spaces

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We consider normal bimodal logics with basic modalities $\Box$ and $[\neq]$; this language is more expressive than the language then the box alone. In topological semantics $\Box A$ is true at point x iff A is true in some neighborhood of x [1], and $[\neq]A$ is true at point x iff A is true at all points y such that $y \neq x$. The universal modality is expressible as follows $[\forall]A \rightleftharpoons [\neq]A \wedge A$. Our basic logic is the fusion of **S4** for $\Box$ and $\mathbf{DL}^-$ [2] for $[\neq]$; this logic is called **S4D**. The logic **S4DEC** is obtained by adding the following extra axioms:

$$(AT_1) \quad [\neq]p \to [\neq]\Box p,$$
$$(DS) \quad [\neq]p \to \Diamond p,$$
$$(AC) \quad [\forall](\Box p \vee \Box\neg p) \to [\forall]p \vee [\forall]\neg p$$
$$(AE_1) \quad [\neq]p \wedge \neg p \wedge \Box(p \to \Box q \vee \Box\neg q) \to \Box(p \to q) \vee \Box(p \to \neg q)$$

Lemma 1 Let X be a topological space. Then

(1) $X| = AT_1$ iff X is a T_1-space;

(2) $X| = DS$ iff X is dense-in-itself.

Definition 2 A topological space is called *locally connected* if every neighborhood of any point contains a connected subneighborhood. A locally connected T_1-space is called *locally 1-connected* if the complement of a point in every its connected open subspace is also connected.

Lemma 3 Let X be a locally connected T_1-space. Then $X| = AE_1$ iff X is locally 1-connected.

Theorem 4 **S4DEC** has the finite model property.

Theorem 5 **S4DEC** is complete with respect to $\mathbb{R}^n$, $n \geq 2$.

The proof is similar to the proof of topological completeness of $\mathbf{S4U} + AC$ with respect to $\mathbb{R}^n$, $n \geq 1$ [3]. But now the construction is more complicated.

Definition 6 Let X be a topological space and let $\mathcal{F} = (W, R, R_D)$ be a finite Kripke frame. A function $f : X \to \mathcal{F}$ is called a *cd-p-morphism* if it is surjective and the following two conditions hold

$$\mathbf{C}f^{-1}(w) = f^{-1}(R^{-1}(w)),$$

$$R_D^{-1}(f^{-1}(w)) = f^{-1}(R_D^{-1}(w)),$$

where R_D in X is the inequality relation.

Lemma 7 If there exists a cd-p-morphism from a space X onto a finite Kripke frame $\mathcal{F}$, then $\mathbf{L}(X) \subseteq \mathbf{L}(\mathcal{F})$.

To prove Theorem 5 we consider finite Kripke frames of special kind (*good frames*) characterizing **S4DEC** and show that any good frame is a cd-p-morphic image of $\mathbb{R}^n$.

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Sectionally residuated semilattices

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Sectionally residuated (semi)lattices are (join-semi)lattices with a largest element such that every principal order filter is a carrier of an integral residuated (semi)lattice. These structures generalize integral residuated lattices and are closely connected with certain pseudo BCK-algebras that were recently introduced as a non-commutative extension of BCK-algebras.

Concrete and abstract logics for coalgebras

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Coalgebras for a functor T on a category X generalise transition systems. Similarly, algebras for a functor L on a category A generalise standard algebras for a signature. If the categories X and A are related by a Stone type duality, one can consider algebras in A as providing a logic for spaces in X. Furthermore, if the functors T and L are dual, one can consider the algebras for the functor L to provide an 'abstract' logic for the coalgebras for the functor T.

Although these logics for T-coalgebras obtained from dualising T enjoy the nice properties of being sound, complete and expressive, they are only 'abstract' logics in the following sense: These logics do not provide a logical calculus or an explicit inductive definition of the set of formulas. We show how to adapt the notion of an algebra being presentable by generators and relations to that of a functor being presentable by operations and equations. We prove that the category of algebras for any such functor is equationally definable in the standard sense. This equational logic then can be turned into a 'concrete' modal logic for T-coalgebras.

Prime ideal theorem for weakly dicomplemented lattices

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To develop a *Boolean Concept Logic* there is a need to have a notion of negation on *formal concepts*. One of the solutions proposed by Rudolf Wille leads to *concept algebras*: these are concept lattices with two unary operations called *weak negation* and *weak opposition*. The motto is that the negation of a formal concept should be a formal concept. Introduced to capture the equational theory of concept algebras, *weakly dicomplemented lattices* are bounded lattices equipped with two unary operations: a *weak complementation* and a *weak opposition*. The *prime ideal theorem* is the corner stone of well-known representation theorem such as topological representation of Boolean algebras by M.H. Stone, of bounded distributive lattices by H.A. Priestley, or of lattices by G. Hartung. For weakly dicomplemented lattices the prime ideal theorem is rather easy to prove. However it seems to be insufficient to get a representation theorem for this class of algebras. In this talk I will present the prime ideal theorem and address the question to what extent it might be useful.

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MVn-algebras with closure operations

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A theory of new algebras having MV-algebra signature enriched with unary operations, which are closure operation type, are introduced. The appropriate logical system with modalities is constructed.

More precisely, a universal algebra $(A, \oplus, \otimes, ^*, \Delta_1, \ldots, \Delta_n, 0, 1)$ $(n \geq 2)$ of type $(2, 2, 1, 1, \ldots, 1, 0, 0)$ is called a C_nMV_m algebra, if $(A, \oplus, \otimes, ^*, 0, 1)$ is MV_m-algebra, and in addition unary operations $\Delta_1, \ldots, \Delta_n$ satisfy the following identities:

E1. $x \leq \Delta_i x, \quad i = 1, \ldots, n.$
E2. $\Delta_i (x \vee y) = \Delta_i x \vee \Delta_i y, \quad i = 1, \ldots, n.$
E3. $\Delta_i \Delta_j x = \Delta_j \Delta_i x, \quad i, j = 1, \ldots, n.$
E4. $\Delta_1 \ldots \Delta_n x = (m-1)x.$
E5. $\nabla_i \Delta_1 \ldots \Delta_{i-1} \Delta_{i+1} \ldots \Delta_n x \vee x = \nabla_i x, \quad i = 1, \ldots, n.$

From these axioms we can deduce equivalent identities for the dual operators $\nabla_i :\ \nabla_i(x) = (\Delta_i x^*)^*$, $i = 1, \ldots, n$.

Example 1. $D_0 = (\{0, 1\}, \oplus, \otimes, ^*, \Delta_1, \ldots, \Delta_n, 1, 0)\,, \quad \Delta_i x = x, \quad i = 1, \ldots, n.$

Example 2.

$$D_i(m) = (\{0, \frac{1}{m-1}, \frac{2}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}, \oplus, \otimes, ^*, \Delta_1, \ldots, \Delta_n, 1, 0),$$

$$\Delta_i x = 2x, \quad \Delta_j x = x, \ j \neq i, \quad i, j = 1, \ldots, n$$

is a C_nMV_m-algebra.

Example 3.

$$P_m = (\{0, \frac{1}{m-1}, \frac{2}{m-1}, \ldots, \frac{m-2}{m-1}, 1\}^n, \oplus, \otimes, ^*, \Delta_1, \ldots, \Delta_n, 1, 0),$$

where $\Delta_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, (m-1)x_i, x_{i+1}, \ldots, x_n)$, is a C_nMV_m-algebra. Moreover,

$$P_m = \prod_{i=1}^{n} D_i(m)$$

Denote the variety of these algebras by $\mathbf{C_nMV_m}$.

Theorem 1. The variety $\mathbf{C_nMV_m}$ contains only $n(d[m-1]-1)+1$ non-isomorphic subdirectly irreducible algebras, where $d[m-1]$ is the number of divisors of $(m-1)$. The subdirectly irreducible C_nMV_m-algebras are $D_i(j)(i = 1, \ldots, n;\ 2 \leq j-1 \in div(m-1))$ and D_0.

Theorem 2. The variety $\mathbf{C_nMV_m}$ is generated by the algebra P_m.

Theorem 3. (Representation Theorem)

i) Any C_nMV_m-algebra A is isomorphic to a subdirect product of D_0 and the algebras $D_i(j)$ $(i = 1, \ldots, n,\ 2 \leq j-1 \in div(m-1))$.

ii) Any finite C_nMV_m-algebra A is isomorphic to a direct product of the algebras D_0, $D_i(j)$ $(i = 1, \ldots, n,\ 2 \leq j-1 \in div(m-1))$.

Now let us define the following sequence:

$$p(2,k) = 2^k, \quad p(i,k) = i^k - \sum_{1<j<i}^{j-1 \in div(i-1)} p(j,k)$$

for every $i > 2$.

Theorem 4. C_nMV_m-algebra

$$F_{C_nMV_m}(k) = D_0^{p(2,k)} \times \prod_{j>2}^{j-1 \in div(i-1)} \prod_{i=1}^{n} D_i(j)^{p(j,k)}$$

is k-generated free C_nMV_m-algebra over the variety $\mathbf{C_nMV_m}$.

Now we define a multi-modal m-valued Łukasiewicz logical system M_nL_m, the language of which consists of: 1) propositional variables $p, q, r, \ldots$ and with indices; 2) connectives: $\rightarrow, \sim, \Delta_1, \ldots, \Delta_n$. Other connectives are defined as follows: $p \vee q = (p \rightarrow q) \rightarrow q$, $p \wedge q = \sim(\sim p \vee \sim q)$, $\nabla_i p = \sim \Delta_i \sim p$, $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$, $p \& q = (p \rightarrow q^*)^*$. Formulas are built in usual way. This logic includes the axioms of m-valued Łukasiewicz logic plus the following axioms:

(A1) $\alpha \rightarrow \Delta_i \alpha, \quad i = 1, \ldots, n$
(A2) $\Delta_i (\alpha \vee \beta) \leftrightarrow \Delta_i \alpha \vee \Delta_i \beta, \quad i = 1, \ldots, n$
(A3) $\Delta_i \Delta_j \alpha \leftrightarrow \Delta_j \Delta_i \alpha, \quad i, j = 1, \ldots, n$
(A4) $(\sim \alpha \rightarrow \alpha) \leftrightarrow \Delta_1 \ldots \Delta_n \alpha, \quad i = 1, \ldots, n$

(A5) $\nabla_i \Delta_1 \ldots \Delta_{i-1} \Delta_{i+1} \ldots \Delta_n x \wedge x \leftrightarrow \nabla_i x, \quad i = 1, \ldots, n$

The inference rules of this logic are: $\alpha, \alpha \to \beta / \beta$ (Modus Ponens) and $\alpha / \nabla_i \alpha, \ i = 1, \ldots, n$.

Theorem 5. A formula α is a theorem of the logic $M_n L_m$ if and only if α is a tautology.

Theorem 6. (Deduction theorem) If Γ is a set of formulas, α and β are any formulas and β is deduced from $\Gamma \cup \{\alpha\}$ in $M_n L_m$, then $\&^{m-1}\alpha \to \beta$ is deduced from Γ, where $\&^{m-1}\alpha$ means $\alpha \& \ldots \& \alpha$ $(m-1)$-times.

On strong completeness

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For a normal propositional modal logic Λ, one can formulate at least three potentially distinct notions of being Kripke complete:

(a) *weakly complete*, i.e., for every non-theorem $\varphi \notin \Lambda$, there exists a frame F for Λ, which refutes φ at some point of F under some valuation;

(b) *strongly locally complete*, i.e., for every set of formulas Γ and every φ which cannot be deduced from Γ and all theorems of Λ by means of MP only, there exists a frame F for Λ, a valuation Val in F and a point $x \in F$ s.t. $x \in Val(\Gamma)$ and $x \notin Val(\varphi)$. Equivalently, every Λ-consistent set of formulas is satisfied at some in some model based on a frame for Λ;

(c) *strongly globally complete*, i.e., for every set of formulas Γ and every φ which is not deducible from Γ and all theorems of Λ by means of MP and Necessitation, there exists a frame F for Λ, a valuation Val in F and a point $x \in F$ s.t. $Val(\Gamma) = F$ and $x \notin Val(\varphi)$.

It is known that (a) does not imply (b) and, a fortiori, does not imply (c) (the Löb logic being a prominent example). Nevertheless, it has been actually shown by Wolter that (b) is equivalent to (c). Moreover, it is known that these two conditions in turn are equivalent to an algebraic one:

(d) *complexity*: every algebra A from the variety $VAR(\Lambda)$ corresponding to Λ can be embedded in a lattice-complete, atomic and completely additive algebra ($\mathcal{CAV}$-BAO), i.e., the dual algebra of some Kripke frame for Λ.

The conditions (a), (b) and (c) may be also reformulated in an algebraic fashion, namely:

(a') $VAR(\Lambda)$ is HSP-generated from the class of its Kripke algebras, i.e., $\mathcal{CAV}$-BAOs.

(b') for every Λ-consistent set of formulas Γ, there exists a $\mathcal{CAV}$-BAO $A \in VAR(\Lambda)$, a valuation Val in A and a principal proper filter $\nabla \subseteq A$ s.t. $Val(\Gamma) \subseteq \nabla$.

(c') for every Γ and φ as in (c), there exists a $\mathcal{CAV}$-BAO $A \in VAR(\Lambda)$ and a valuation Val in A s.t. $Val(\Gamma) = \top$ and $Val(\varphi) \neq \top$.

Now, the natural question is what happens when the class $\mathcal{CAV}$ in formulation of conditions (a') (b'), (c') and (d) is replaced by some other interesting class of algebras $\mathcal{X}$ - denote the resulting completeness notions by $(a\mathcal{X})$, $(b\mathcal{X})$, $(c\mathcal{X})$ and $(d\mathcal{X})$, respectively. Does equivalence $(b\mathcal{X}) \Leftrightarrow (c\mathcal{X}) \Leftrightarrow (d\mathcal{X})$ still hold? Are we able to find an interesting class of algebras $\mathcal{Y}$ s.t. not every logic is $((a\mathcal{Y})$-complete, but all four completeness notions (i.e., $(a\mathcal{Y}), \ldots, (d\mathcal{Y})$) are equivalent?

It is fairly easy to observe by generalizing Wolter's results that

Theorem 1. (i) $(d\mathcal{X}) \Rightarrow (c\mathcal{X}) \Rightarrow (b\mathcal{X}) \Rightarrow (a\mathcal{X})$ holds for any class of algebras.

(ii) $(c\mathcal{X}) \Rightarrow (d\mathcal{X})$ holds for any class $\mathcal{X}$ closed under products.

(iii) $(a\mathcal{X}) \Rightarrow (b\mathcal{X})$ holds for any class $\mathcal{X}$ closed under ultraproducts ($\mathcal{CAV}$ is not such a class, thus the gap between weak and strong Kripke completeness)

The implication $(b\mathcal{X}) \Rightarrow (c\mathcal{X})$ is harder to obtain, though.

Theorem 2 $(b\mathcal{X}) \Rightarrow (c\mathcal{X})$ holds if $\mathcal{X}$ is a class of completely additive algebras ($\mathcal{V}$-BAOs) closed under complete homomorphisms.

To see why one should not expect this implication to hold in general, consider the following example. Shehtman has proven that for any **K4**-logic, weak Kripke completeness $(a\mathcal{CAV})$ implies strong neighbourhood completeness. What he meant by "strong neighbourhood completeness" is $(b\mathcal{CA})$, where $\mathcal{CA}$ is the class of lattice-complete and atomic BAOs. Thus, for example, the Löb logic is strongly locally neighbourhood complete. Nevertheless, it is not complete in the sense $(c\mathcal{C})$ ($\mathcal{C}$ is the class of lattice-complete algebras), and, a fortiori, it is not $(c\mathcal{CA})$ complete: it is not strongly globally complete with respect to neighbourhood frames.

By improving on an old result of Thomason, one may prove

Theorem 3 There is a logic Λ for every class of algebras $\mathcal{X}$ s.t. $\mathcal{C} \supseteq \mathcal{X} \supseteq \mathcal{CAV}$, the consequence relation of monadic second-order logic with binary relation constant is reducible to the Λ-consequence over $\mathcal{X}$ (both local and global, as Λ contains the universal modality).

Thus, in general, consequence over complete algebras must be rather a strong notion. There is, however, a class of algebras whose behaviour is much more decent. The class $\mathcal{AV}$ of atomic and completely additive algebras is closed under ultraproducts, products and complete homomorphic images and thus, by Theorems 1 and 2,

Corollary 4 For atomic and completely additive algebras, weak completeness and strong global completeness are equivalent, i.e., $(a\mathcal{AV}) \Leftrightarrow (d\mathcal{AV})$.

Thus, consequence relation over $\mathcal{AV}$-BAOs is much more tamed than consequence over complete algebras. Nevertheless, the notion of completeness with respect to $\mathcal{AV}$-algebras is nontrivial; actually, by generalizing a result of Blok one may prove

Theorem 5 Every modal logic in a finite modal similarity type which is not an union-splitting of the lattice of all modal logics in the same similarity type, shares its class of $\mathcal{AV}$ algebras with continually many others.

The question whether there exists any **K4**-logic which is incomplete with respect to $\mathcal{AV}$-BAOs is a very interesting open problem.

In the above, we were concerned strictly with modal logic. There is, however, in principle no reason why similar questions should not be studied for other formalisms with algebraic semantics, in particular substructural logics. Indeed, part of our motivation was to abstract these problems from the context of relational semantics, which behave exceptionally well for modal logics and rather poorly for, e.g., substructural logics.

For example, it has been shown recently by Gehrke and Priestley that the variety of MV-algebras is not canonical. For modal logics, the notion of canonicity is stronger than the notion of strong Kripke completeness ($\mathcal{CAV}$-complexity). Indeed, as was proven by Wolter, it is strictly stronger. The tense logic of the reals is strongly $\mathcal{CAV}$-complete, but not canonical. What follows, every algebra in the associated variety is a subalgebra of some complete algebra from the same variety, even though the variety is not closed under canonical extensions. Is the same true about MV-algebras?

Computing coproducts of finitely presented Gödel algebras

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A *Gödel algebra* (a.k.a. an *L-algebra* or a *Gödel-Dummett algebra*) is a Heyting algebra satisfying the prelinearity axiom $(\alpha \to \beta) \vee (\beta \to \alpha) = 1$. The variety of Gödel algebras is locally finite, whence finitely generated Gödel algebras coincide with finite or finitely presented ones. Our main result is an algorithm to compute finite coproducts of finitely generated Gödel algebras.

Let G denote the category of Gödel algebras, and G_{fp} the full subcategory of finitely presented algebras. A *forest* is a finite poset F such that for every $x \in F$, the set $\{y \in F \mid y \leq x\}$ is a chain (i.e. is totally ordered) when endowed with the order inherited from P. A *tree* is a forest with a bottom element. Let F denote the category of forests and open order-preserving maps, and T the full subcategory of trees. (Recall that an order-preserving map $f\colon A \to B$ between posets is *open* iff it carries down-sets to down-sets.) A straightforward development shows that the spectral functor **Spec** yields an equivalence between $\mathsf{G}_{\mathsf{fp}}^{\mathsf{op}}$ and F. Trees correspond to finitely presented Gödel algebras with a unique maximal filter. (Remark: As a matter of convention, we are ordering prime filters by *reverse* inclusion just to make our trees grow upwards.) Standard considerations allow one to reduce computation of (finite) coproducts in G_{fp} to computation of (finite) products in T. We remark in passing that equalisers in T can be effectively computed without difficulties, whence computation of fibred products in T (or fibred coproducts in G_{fp}) follows at once from our main result below.

The core of this piece of work is thus computation of products in T. An *ordered partition* σ is a finite chain of pairwise disjoint nonempty finite sets. We write $\sigma = \{S_1, \ldots, S_m\}$ to mean that S_i precedes S_j iff $i \leq j$. Given ordered partitions $\sigma = \{S_1, \ldots, S_m\}, \tau = \{T_1, \ldots, T_n\}$ with $m \leq n$, we let $\sigma \leq \tau$ iff $S_i = T_i$ for every $i \in \{1, \ldots, m\}$. A *foliage* is a set of mutually incomparable (according to $\leq$) ordered partitions. Given an ordered partition σ, its *support* is $\mathrm{supp}\sigma = \bigcup \sigma$. Similarly, if T is a foliage, we set $\mathrm{supp}T = \bigcup_{\sigma \in T} \mathrm{supp}\sigma$. For σ and τ ordered partitions with disjoint supports, we define a *merged shuffle* of σ and τ to be any ordered partition constructed in a certain (herein not detailed) manner from σ and τ. (Remark: the set of merged shuffles of σ and τ is finite and effectively computable.) If S and T are foliages with $\mathrm{supp}S \cap \mathrm{supp}T = \emptyset$, we call

$$S \times T = \{\theta \mid \theta \text{ is a merged shuffle of some } \sigma \in S, \tau \in T\}$$

the *product* of S and T. It is possible to show that $S \times T$ is a foliage.
Given a foliage T, we set

$$\mathrm{Tree}T = \{\sigma \mid \sigma \text{ is an ordered partition such that } \sigma \leq \tau \in T\}.$$

A moment's reflection shows $\mathrm{Tree}T$ is a tree for any foliage T. Given an ordered partition $\theta = \{B_1, \ldots, B_n\}$ and a set X, we let $\theta - X$ denote the ordered partition $\{B_1 \backslash X, \ldots, B_n \backslash X\} \backslash \{\emptyset\}$, where $\backslash$ is set-theoretic difference. Let S and T be foliages, and set $X = \mathrm{supp}S$, $Y = \mathrm{supp}T$. Assume $X \cap Y = \emptyset$. Let $A = \mathrm{Tree}S$, $B = \mathrm{Tree}T$, and $C = \mathrm{Tree}(S \times T)$. We define a function $\pi_S\colon C \to A$ by $\theta \mapsto \theta - Y$, and similarly $\pi_T\colon C \to B$ by $\theta \mapsto \theta - X$. We call π_S and π_T the *projections induced by* $S \times T$. It turns out that π_S and π_T are morphisms (in fact, epimorphisms) in T.

Main Theorem. Let S and T be foliages such that $\mathrm{supp}S \cap \mathrm{supp}T = \emptyset$. Then

$$\mathrm{Tree}S \stackrel{\pi_S}{\leftarrow} \mathrm{Tree}(S \times T) \stackrel{\pi_T}{\rightarrow} \mathrm{Tree}T$$

is the product of $\mathrm{Tree}S$ and $\mathrm{Tree}T$ in T.

On the basis of this theorem, it is an easy matter to set up an explicit algorithm to compute finite products of trees.

If time allows, we shall offer a small sample of applications of our main result.

Residuated lattices and density

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We describe a uniform strategy for proving that a class of residuated lattices obeying prelinearity is generated by its dense linearly ordered members, and hence that the corresponding fuzzy logics are complete with respect to algebras based on the unit interval [0,1]. The strategy consists of two parts. First, it is shown that derivability in a logic extended with the Takeuti-Titani density rule is equivalent to validity in all dense linearly ordered members of the corresponding class of algebras. A syntactic elimination of the density rule from the logic is then given using hypersequents.

A description of the lattice of the normal extensions of S4

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Our starting point is the well-known theorem of embedding of the intuitionistic propositional logic **Int** into the modal logic **S4**, which was conjectured by K. Gödel in 1933, but obtained the status of a proven statement in a paper by J.C.C McKinsey and A. Tarski only in 1948.

A later study of A. Grzegorczyk showed that that embedding is also true for a proper extension of **S4** known today as **Grz**. Moreover, it follows from results of W. Blok and L. Esakia that that embedding holds for any modal logic between **S4** and **Grz**. Namely, in the most general setting, the embedding theorem can be spelled out as follows.

Theorem (on embedding). Let modal logic M belong to [**S4**, **Grz**]. Then for any set of assertoric formulas Σ and an assertoric formula A, the following equivalence holds:

$$\mathbf{Int} + \Sigma \vdash A \text{ if and only if } M + \Sigma^t \vdash A^t,$$

where A^t is the resulting formula of the placement of modality $\Box$ in front of each subformula of A and Σ^t is the result of this operation applied to each formula in Σ.

However, before this general theorem had been established, L. Maksimova and V. Rybakov had started a comparative investigation of the lattices of (normal) extensions of **Int** and **S4** — *Ext***Int** and *Ext***S4**, respectively, having introduced the following three mappings:

$$\rho : Ext\mathbf{S4} \longrightarrow Ext\mathbf{Int}, \tau : Ext\mathbf{Int} \longrightarrow Ext\mathbf{S4}, \sigma : Ext\mathbf{Int} \longrightarrow Ext\mathbf{Grz},$$

where *Ext***Grz** is the lattice of the extensions of logic **Grz**.

Now the embedding theorem above can be written in terms of these mappings as follows: For every logic $M \in [\mathbf{S4}, \mathbf{Grz}]$ and logic $L \in Ext\mathbf{Int}$, $\rho(M + \tau(L)) = L$, which can be reduced to the equality $\rho \circ \tau(L) = L$. The other side of the coin shows the following property well known today: For every logic $M \in Ext\mathbf{S4}$, $\tau \circ \rho(M) \subseteq M \subseteq \sigma \circ \rho(M)$. We have, hence, $\tau \circ \rho(M) \subseteq M \subseteq \mathbf{Grz} + \tau \circ \rho(M)$, since $\sigma \circ \rho(M) = \mathbf{Grz} + \tau \circ \rho(M)$.

This suggests that for any modal logic $M \in Ext\mathbf{S4}$, the equation $M = M^* + \tau \circ \rho(M)$ is solvable for $M^* \in [\mathbf{S4}, \mathbf{Grz}]$. Indeed, it is easy to see that $M \cap \mathbf{Grz}$ is a solution to this equation. Moreover, given M, all the solutions M^* form a sublattice of *Ext***S4** with $M \cap \mathbf{Grz}$ as its greatest element.

Thus every logic M in *Ext*$\mathbf{S4}$ can be given by the following equality $M = M^* + \tau(L)$, where $M^* \in [\mathbf{S4}, \mathbf{Grz}]$ and $L \in Ext\mathbf{Int}$. It is clear that in the last equality we have $L = \rho(M)$. We call the former equality a τ-representation for the logic M with the *modal component* M^* and the *assertoric component* L. Also, we call $\tau(L)$ a τ-*component* of this τ-representation. Let us fix a τ-representation $M = M^* + \tau(L)$. If $M^* = M \cap \mathbf{Grz}$, we call this τ-representation *saturated*.

We intend to use the notions that have been just introduced for presenting some description of *Ext*$\mathbf{S4}$.

Let M_0 be $\mathbf{Grz} \cap \mathbf{S5}$. We denote the set of the extensions of M_0 by $ExtM_0$. For any $M \in ExtM_0$, we call a τ-representation for M a τ_0-representation when M^* is found in $[M_0, \mathbf{Grz}]$.

Theorem 1. Every logic in $ExtM_0$ has a τ_0-representation, which is uniquely determined by its modal and assertoric components and is its saturated τ-representation.

Corollary 1.1. The interval $[M_0, \mathbf{Grz}]$ has a linear order of type $1 + \omega^*$.

Suppose the logics in $[M_0, \mathbf{Grz}]$ are arranged as follows:

$$M_0 \subset \ldots \subset M_3 \subset M_2 \subset M_1 = \mathbf{Grz}.$$

We define $\tau_n(L) = M_n + \tau(L)$ for every $L \in Ext\mathbf{Int}$. We call $\{\tau_n(L) \mid L \in Ext\mathbf{Int}\}$ an *n-th slice* of $ExtM_0$.

Theorem 2. The slices (as defined above) cover all $ExtM_0$ and are mutually disjoint. Each slice is isomorphic to the lattice *Ext*$\mathbf{Int}$. The least element of the *n-th* slice is M_n and its greatest element is the logic of an n-atom interior algebra with two open elements.

Thus, when we fix M_n and let L vary, we get the *n-th* slice. Now let L be fixed and we allow M_n to slide along $[M_0, \mathbf{Grz}]$. We call the resulting set $\{\tau_n(L) \mid n \geq 0\}$ an L-layer. One can notice that each L-layer has a linear order of type $1 + \omega^*$. In virtue of the Homomorphism Theorem, the L-layers form a quotient lattice of $ExtM_0$, isomorphic to *Ext*$\mathbf{Int}$.

By using Esakias duality for Heyting algebras, one can show that the interval $[\mathbf{S4}, \mathbf{Grz}]$ is a union of mutually disjoint smaller intervals, the greatest elements of which belong to $[M_0, \mathbf{Grz}]$. We call the smaller interval with the greatest element M_n an *n-th lower slice*.

Theorem 3. Let M lie in $ExtM_0$ and be given by its τ-representation $M^* + \tau(L)$. Then M belongs to the *n-th* slice if and only if M^* belongs to the *n-th* lower slice.

Let $\mathcal{L}$ be the sublattice of *Ext*$\mathbf{S4}$, consisting of all $\tau(L)$. It is clear that for $n \geq 1$, neither *n-th* slice nor *n-th* lower slice have common elements with $\mathcal{L}$. However, $\mathbf{S5}$ belongs to the 0-*th* slice and $\mathbf{S4}$ belongs to the 0-*th* lower slice.

In conclusion, we propose the following conjectures.

Conjecture 1. The lattice *Ext*$\mathbf{S4}$ consists of only the filter generated by $[M_0, \mathbf{Grz}]$, equal to $ExtM_0$, the ideal generated by the same interval, equal to $[\mathbf{S4}, \mathbf{Grz}]$, and $\mathcal{L}$.

Conjecture 2. The filter and ideal above have exactly two common elements with $\mathcal{L}$ — namely, logics $\mathbf{S5}$ and $\mathbf{S4}$, respectively.

Consequences of omitting types in fuzzy predicate logic with evaluated syntax

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This paper is a contribution to the development of model theory of fuzzy logic in narrow sense with evaluated syntax (EvL in the sequel). It is known that there are many formal systems of fuzzy logic, most of them having traditional syntax and many-valued semantics. Our logic is further generalization, where also syntax is evaluated, i.e. we may consider fuzzy sets of axioms. In the paper we will be interested in n_0-Horizon logic in EvL which is one of the consequences of omitting types in EvL.

Formal theory of fuzzy logic is now a mature theory whose fundamental problems seem already to be solved. Now, it is time to continue its development further and to study possibilities for generalization of all the classical results that may be useful for its main goal — to develop a mathematical theory providing tools for modelling of the vagueness phenomenon. Among such result which was generalized in fuzzy logic with evaluated syntax belongs the omitting types theory which is studied in (7). This theorem makes possible to extend the power of classical logic by characterizing properties that are too complicated to be expressed by one formula but can be expressed using a set of formulas and to construct models for such situations.

Omitting types theory in the classical logic has a lot of consequences (see (1)) and there is a possibility to generalize a lot of them. In this paper we give one application of omitting types theorem which is n_0-Horizon logic which can be used for the construction of non-standard model in EvL.

Fuzzy logic with evaluated syntax is in detail presented in (9) where also its model theory has been founded. It is specific for EvL that the set of truth values must be the Łukasiewicz MV-algebra whose support set is the interval of reals $[0, 1]$.

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When a residuum can be a derived operation of fuzzy logic

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Modus ponens is an essential deduction rule in logic, including fuzzy logics studied in 1. For this, the residuum has to be chosen as the interpretation of implication. On the other hand, attempts are made to build up a fuzzy logic using only a conjunction (or disjunction) and negation as basic connectives [2]. This effort is motivated by possible hardware implementations

- residua are often non-continuous and this makes their representation difficult. The question is when the residuum can be derived from the (involutive) negation and conjunction (t-norm). This is the case of Łukasiewicz logic.

In general, the answer to our question is negative if we allow only finitary operations. Admitting also operations with countably many arguments, the situation changes. Following [3,4,5], the product t-norm, or, more generally, any strict Frank t-norm, admits to derive the respective residuum. This result can been extended to many others - but not all - strict t-norms [6]. (The Hamacher product is a typical counterexample.)

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On the variety of WNM-algebras

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Weak Nilpotent Minimum logic (WNM, for short) was introduced by Esteva and Godo by means of a Hilbert style calculus in the language $\mathcal{L} = \{*, \rightarrow, \wedge, 0\}$ of type $(2, 2, 2, 0)$, where the only inference rule is Modus Ponens and the axiom schemata are the following (taking $\rightarrow$ as the least binding connective):

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

(A2) $\varphi * \psi \rightarrow \varphi$

(A3) $\varphi * \psi \rightarrow \psi * \varphi$

(A4) $\varphi \wedge \psi \rightarrow \varphi$

(A5) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$

(A6) $\varphi * (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$

(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\varphi * \psi \rightarrow \chi)$

(A7b) $(\varphi * \psi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$

(A9) $0 \rightarrow \varphi$

(A10) $\neg(\varphi * \psi) \vee (\varphi \wedge \psi \rightarrow \varphi * \psi)$

being $\neg$ and $\vee$ the following defined connectives:

$\neg\varphi := \varphi \rightarrow 0$;

$\varphi \vee \psi := ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$.

NM logic is the axiomatic extension of WNM obtained by adding the axiom schema of involution: $\neg\neg\varphi \to \varphi$.

As is proved in [2], equivalent algebraic semantics (in the sense of [1]) for those logics are given by the classes of WNM-algebras and NM-algebras, respectively:

Let $\mathcal{A} = \langle A, *, \to, \wedge, \vee, 0, 1\rangle$ be an algebra of type $(2, 2, 2, 2, 0, 0)$. We define a unary operation by $\neg a := a \to 0$. Then, $\mathcal{A}$ is a WNM-algebra if, and only if, it is a bounded residuated lattice satisfying the following equations:

$$(x \to y) \vee (y \to x) \approx 1,$$

$$\neg(x * y) \vee (x \wedge y \to x * y) \approx 1.$$

$\mathcal{A}$ is a NM-algebra if, and only if, in addition it satisfies the equation of involution:

$$\neg\neg x \approx x$$

We will say that $\mathcal{A}$ is a WNM-chain (resp. NM-chain) if, and only if, the lattice order is total. Let $\mathbb{WNM}$ be the variety of all WNM-algebras.

Theorem: [2] WNM-algebras (resp. NM-algebras) are representable as a subdirect product of WNM-chains (resp. NM-chains).

Therefore, axiomatic extensions of WNM correspond to subvarieties of $\mathbb{WNM}$, and all those subvarieties are generated by WNM-chains.

All the subvarieties of NM-algebras were studied by Gispert in [3].

In this talk we will make a first approach to the study of subvarieties of $\mathbb{WNM}$. After considering some general issues about the structure of WNM-chains (defining the notion of isolated and non-isolated involutive element and giving canonical representatives for a class of chains), we will prove that $\mathbb{WNM}$ is locally finite, so all its subvarieties are generated by finite chains. A huge family of those subvarieties will be then axiomatized, namely those generated by a WNM-chain with a finite number of non-isolated involutive elements and those generated by a finite family of those chains. We will also characterize the generic WNM-chains, i.e. those that generate the variety $\mathbb{WNM}$, and finally we will describe the varieties generated by a WNM-t-norm.

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Composition on MV-algebras

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We introduce an extension of MV-algebras obtained by adding a binary operation and a constant, with the aim of modelling composition of functions. The variety of Composition MV-algebra (CMV-algebra, for short) is defined and some results regarding ideals and congruences are stated. Further, we define modules over CMV-algebras and we state some problems.

Principal fuzzy type theories as higher order fuzzy logics

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The successful development of the formal theory of fuzzy logic started in 1979 by the seminal paper of J. Pavelka [10]. He developed the propositional fuzzy logic. Its first order version has been developed by V. Novák in [6] (see also [9]). This logic is based on Łukasiewicz MV-algebra of truth values and has evaluated syntax. Since the book of P. Hájek [4] appeared, a rapid development of various kinds of formal fuzzy logics differing in the used structures of truth values can be noticed (see also [8]).

Recently, fuzzy logic penetrated also to higher order, and a formal theory of *fuzzy type theory* has been developed, first in [7] and in a slightly different form also in [2].

The paper [7] follows the development of the classical type theory, as elaborated by A. Church [3] and L. Henkin [5], and later continued, e.g. in [1].

Because of a large variety of possibilities, a discussion about what is fuzzy logic is now in progress. Based on the expressive power and various experiences, several distinguished kinds of fuzzy logic took privilege over the other ones. Such logics are Łukasiewicz, basic fuzzy logic (BL) and LΠ fuzzy logic. There are several reasons for this fact. However, it turns out that also fuzzy type theory can be developed in parallel ways. The original theory in [7] has been developed on the basis of ???-algebra that is, a residuated lattice with prelinearity and double negation extended, moreover, by a special unary operation of Baaz delta. In this paper we will present all these kinds of fuzzy type theory, namely those based on IMTL_Δ, $\text{Łukasiewicz}_\Delta$, BL_Δ and LΠ algebras. These theories enjoy the generalized completeness property (i.e. completeness w.r.t. generalized models). It should be stressed that the fundamental connective in all of them is a fuzzy equality. Because of essential importance of this connective, the resulting theory is elegant and philosophically interesting.

We will present logical axioms, inference rules, semantics, and some specific properties of all four kinds of fuzzy type theory including the completeness theorems.

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MV algebras and quantum computation

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Quantum computation has recently suggested new forms of quantum logic that have been called *quantum computational logics*. These logics, unlike orthodox quantum logics, identify the meaning of a sentence with a *qubit* or a *quregister* (a system of qubits) or, more generally, with a *qumix* (a mixture of quregisters). The class of all qumixes of the two-dimensional Hilbert space C2 gives rise to an algebra over a subset of the complex numbers, endowed with appropriate operations of inverse and truncated sum. Such an algebra bears striking similarities to Chang's MV algebras - namely, it satisfies all of the usual MV algebraic axioms except that 0 is not a neutral element for truncated sum. It seems therefore worthwhile to try and do, with respect to this standard algebra, exactly what Chang did with respect to the standard MV algebra over the closed real unit interval: abstracting from the properties of such an algebra, one could indeed tentatively axiomatize a class of algebras of the appropriate similarity type, and attempt to prove a standard completeness theorem. In this talk we show that this aim can be attained. We first introduce the variety of *quasi*-MV *algebras* and study its structure theory. We then introduce the subvariety of *flat* quasi-MV algebras, axiomatized by the equation $0 = 1$, and prove that every quasi-MV algebra can be embedded into the direct product of an MV algebra and a flat quasi-MV algebra. Finally, we show the desired standard completeness theorem: an equation of the appropriate type is satisfied in all quasi-MV algebras iff it is satisfied in the standard quasi-MV algebra over the complex numbers.

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The finite model property for knotted extensions of quantized intuitionistic linear logic

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This work is intended as a step towards the development of the Quantized Intuitionistic Linear Logic using the approach of the theory of (involutive) quantales developed by C.J. Mulvey, J.W. Pelletier and J. Rosický and others. Quantales are certain partially ordered algebraic structures which generalize frames (pointless topologies) as well as various lattices of multiplicative ideals from ring theory and functional analysis (C*-algebras, von Neumann algebras).

The logic considered here is the Quantized Intuitionistic Linear Logic (QILL) introduced by N. Kamide – involutive quantales are models of a such logic – extended by a knotted structural rule:

$$\frac{\Gamma, x^n \Rightarrow y}{\Gamma, x^m \Rightarrow y}.$$

Similarly as by C.J. Alten for propositional Intuitionistic Linear Logic, it is proved that the class of algebraic models for such a logic has the finite embeddability property, meaning that every finite partial subalgebra of an algebra in the class can be embedded into a finite full algebra in the class. It follows that each such logic has the finite model property with respect to its algebraic semantics and hence that the logic is decidable.

McNaughton theorem in BL-Logic

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In this contribution we will explicitly work with formulas of the propositional BL-logic [1] and their evaluation in a standard BL-algebra. Our purpose is to characterize functions which are associated with (represented by) formulas of the propositional BL. Let us recall that this problem has been first proved by R. McNaughton in [2] for Łukasiewicz logic and then constructively proved in [4,5]. A description of the free BL-algebra has been obtained in [3] for the case of formulas of the propositional BL with one sentential variable.

In this contribution we will mainly stress the functional aspect of the McNaughton theorem having in mind its relation to characterization of the free BL-algebra. Moreover, the proposed characterization of functions represented by formulas of the propositional BL gives an effective way of computing values of functions and therefore, truth values of the respective formulas.

Let FBL be a set of formulas of the propositional BL with primitive sentential connectives $\&, \to$ and the constant 0. By $\mathcal{L} = \langle [0,1]; \vee, \wedge, *, \to, 0, 1\rangle$ we will denote a standard BL-algebra with a continuous t-norm $*$. BL-algebra $\mathcal{L}$ will be used for evaluation of logical formulas from FBL. Let us fix some continuous t-norm and consider its representation as an ordinal sum of continuous Archemedian t-norms. Based on this, we can prove the following proposition.

Theorem. Let $*$ be a continuous t-norm which determines the following parameters:

- at most countable, linearly ordered family of indices I,
- family of generating functions $\mathcal{F}_I$ consisting of continuous and strictly monotonously increasing functions $f_i : [0,1] \to [0,1]$,
- partition of $[0,1]$ determined by families $\mathcal{A}_I = \{a_i\}_{i\in I}$, $\mathcal{B}_I = \{b_i\}_{i\in I}$,
- transition functions $\varphi_i(x)$.

Then

$$x * y = \begin{cases} \varphi_i^{-1}(f_i^{-1}(\min(f_i(\varphi_i(x)) + f_i(\varphi_i(y)), f_i(1)))), & \text{if } (x,y) \in (a_i,b_i)^2, \\ \min(x,y), & \text{otherwise} \end{cases}$$

Let $D = [0,1] \setminus \bigcup_{i\in I}(a_i,b_i)$. We consider the following set of couples of reals with the lexicographic order:

$$R^- = \bigcup_{a\in D}\{(a,0)\} \cup \bigcup_{i\in I}(\{b_i\} \times (0,-c_i))$$

and the one-to-one mapping $g : [0,1] \to R^-$:

$$g(x) = \begin{cases} (x,0), & \text{if } x \in D, \\ (b_i, -g_i(x)), & \text{if } x \in (a_i,b_i). \end{cases}$$

Let us introduce the operations of truncated sum and truncated subtraction on R^- as follows:

$$(x_1,y_1) \dot{+} (x_2,y_2) = \begin{cases} \min((x_1,y_1),(x_2,y_2)), & \text{if } x_1 \neq x_2, \\ (b_i, y_1+y_2), & \text{if } (x_1 = x_2 = b_i) \,\&\, (y_1+y_2 > -c_i), \\ (a_i, 0), & \text{if } (x_1 = x_2 = b_i) \,\&\, (y_1+y_2 \leq -c_i) \end{cases}$$

and

$$(x_1, y_1) \dot{-} (x_2, y_2) = \begin{cases} (x_1, y_1 - y_2), & \text{if } x_1 = x_2 \;\&\; y_1 < y_2, \\ (1, 0), & \text{if } x_1 > x_2 \text{ or } x_1 = x_2 \;\&\; y_1 \geq y_2, \\ (x_1, y_1), & \text{if } x_1 < x_2. \end{cases}$$

Then the following representation theorem holds true:

Theorem. A function $f : [0,1]^n \to [0,1]$ is represented by a formula from FBL if and only if its isomorphic image under the mapping g is a linear polynomial with integer coefficients over truncated operations $\dot{+}$ and $\dot{-}$.

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The quantale of Galois connections

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While adjoint pairs of maps between partially ordered sets are isotone (order-preserving), the partners of a Galois connection are always antitone (order-reversing). Certainly, an advantage of adjunctions, compared with Galois connections, is that they compose in a natural way. As Blyth and Janowitz put in their monography on Residuation Theory ([2], p.19), "*The reason for our emphasis on residuated mappings rather than Galois connections is the following: two residuated mappings may be composed to yield a new residuated mapping; this is not the case with order-reversing mappings.*". However, as we shall show in this talk, there is also a (less natural) way of composing Galois connections, which allows a nice description of uniform and quasi-uniform structures both in the classical topological setting and in the non-classical (pointfree) setting [3].

Our main result is that, for a frame (locale) L, the set $Gal(L, L)$ of Galois endomaps of L, endowed with that composition, is a quantale [3]. This is proved by first showing that the antitone endomaps of L form a quantale $Ant(L, L)$. Then it is shown that $Gal(L, L)$ is a quotient of $Ant(L, L)$. This quotient is described by a quantic nucleus (in the sense described in the book by Rosenthal [4]). Here, besides recalling the relevant concepts regarding quantic nuclei, we need some auxiliary new results about so-called quantic prenuclei, which generalize a similar concept introduced by Banaschewski [1] for locales.

We also provide discussion for why this quantale is useful in the context of (quasi-)uniform structures.

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Mathematical methods and first-order Gödel logics

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Gödel logics are one of the oldest and most interesting families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel in [6] to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [5] was the first to study infinite valued Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is also called Gödel-Dummett logic or Dummett's LC. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Quantified propositional Gödel logics and first-order Gödel logics are natural extensions of the propositional logics introduced by Gödel and Dummett. In the purely propositional case, the choice of the set of truth-values is immaterial: any infinite set of truth values characterizes the same set of tautologies. This is not longer the case when one considers the propositional consequence relation, and likewise when the language is extended to include quantification over propositions or individuals. For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties result in different sets of valid formulas. Hence it is necessary to consider truth value sets other than the standard unit interval. For first-order Gödel logics we have to consider closed subsets of the [0,1] interval. For every such truth value set V we obtain a Gödel logic consisting of all formulas valid w.r.t. V.

Recently, mathematical methods from different fields, especially from Order Theory, Descriptive Set Theory and Topology have been used to obtain solutions to long standing open problems like the characterization of axiomatizability of Gödel logics [2,9], and the number of different Gödel logics [1,3,8], but also the relation to Kripke frames [4]. We will present these recent results together with the mathematical concepts and methods necessary to obtain them:

Characterization of axiomatizability: A thorough analysis of the completeness proof by Takano [10] shows that under certain, well defined topological properties, it can be extended to truth value sets other than the full [0,1] interval. In fact we can show that iff V contains a non-empty perfect kernel, and either 0 is contained in the perfect kernel or 0 is isolated, the Gödel logic determined by V is finitely axiomatizable. In all the other cases the logic is not even recursively enumerable [2,9].

Number of different Gödel logics (lower bound): That there are at least $\aleph_0$ many different Gödel logics can be obtained in different ways. The first one was done by separating logics with different number of accumulation points [1]. We will present a different method which uses the Cantor-Bendixon rank of countable closed truth value sets to separate the logics induced by these truth value sets [8].

Number of different Gödel logics (upper bound): Inspired by Laver's result [7] on countable linear orderings and their embedabbility which settled Fraïsse's Conjecture, we have extended these methods to deal with countable closed linear orderings and continuous monotone embeddability. We will show that also in this case there are exactly $\aleph_1$ many equivalence classes with respect to this embeddability relation. Using this result we show that there are only $\aleph_0$ many different Gödel logics, a very surprising result, since there are strong reasons why we would have expected uncountable many different Gödel logics [3].

Relation to Kripke Frames: As a bit of a side note we will mention that the class of Gödel logics coincides with the class of logics of countable, linearly ordered Kripke domains with constant domains. This result allows the transfer of results obtained on Gödel logics to Kripke frames [4].

We will close with a presentation of more open problems and present some ideas on how to deal with them.

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Linear representation of relational operations

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By slicing the graph of an n-ary relation along lines defined by all combinations of $n-1$ domain elements, we obtain a matrix with n columns and (in general) infinitely many rows. Components of this matrix are subsets of the domain, and the row indices are $(n-1)$-place vectors of domain elements. We show how the cylindrical algebra operations of substitution, diagonalization and cylindrification can be defined as matrix operations, in particular, as multiplication of the relations matrix representation by distinct matrices corresponding to the various cylindrical operators. Conjunction, disjunction and complementation of relational expressions are also shown to have matrix counterparts.

States on bounded residuated ℓ-monoids

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Bounded commutative residuated ℓ-monoids ($R\ell$-monoids), or commutative integral residuated lattices, generalize, among others, MV-algebras and BL-algebras, i.e. algebraic counterparts of the Łukasiewicz infinite valued logic and the Hájek basic fuzzy logic, respectively. States (i.e. analogues of probability measures) on MV-algebras are averaging the truth-value in the Łukasiewicz logic. But BL-algebras and, more generally, bounded commutative $R\ell$-monoids, in contrast to MV-algebras, do not admit an analogue of a partial addition. Hence there is a serious problem how to define states on those algebras. Georgescu introduced the Bosbach states on (pseudo) BL-algebras which in the case of MV-algebras coincide with the original states. We introduce analogously states on bounded commutative $R\ell$-monoids. We exhibit the state space of such an $R\ell$-monoid proving that the set of extremal states is a non-empty compact Hausdorff topological space homeomorphic with the hull-kernel topology of the set of maximal filters.

For GMV-algebras (= pseudo MV-algebras), non-commutative generalizations of MV-algebras, states are averaging the truth-value in the non-commutative variant of the Łukasiewicz logic. Bounded $R\ell$-monoids (not necessarily commutative) generalize GMV-algebras as well as pseudo BL-algebras. Using the idea of Georgescu, we introduce states also for general bounded $R\ell$-monoids. We show that the existence of states is crucially connected with the existence of normal and maximal filters. Analogously as for commutative case, topological properties of the extremal states are described.

An order-theoretic approach to dynamic epistemic logic and its corresponding sequent calculus

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In an interactive multi-agent system, agents communicate with one another via public and private announcements and this communication changes their information state. In order to be able to reason about information updates in such settings, one has to take into account the dynamic as well as the epistemic aspects of communication. An insightful example is the *muddy children* puzzle where after n children played in the mud k of them have mud on their forehead. They can of course see each other's foreheads but not their own ones. Their father initially announces "at least one of you has mud on his forehead". After that, their father asks $k-1$ times whether they know if they are themselves dirty and $k-1$ times they all simultaneously reply "no". Now the ones which have mud on their forehead will all know this. The traditional approaches to this puzzle, e.g. [8], only consider the epistemic aspect and dismiss the dynamic one, i.e. the announcements and their effects cannot be encoded in these systems. Recent developments have tried to integrate both of these aspects through the traditional rather combinatorial approach to Epistemic logic, i.e. the possible worlds or the Kripke semantics. Dynamic Epistemic Logic [2,5] generalizes all these attempts in a Dynamic Logic [10] (PDL) style logic also based on Kripke semantics. The PDL programs of DEL are 'epistemic' in the sense that they update the information state of agents.

I will present recent joint work with A. Baltag and B. Coecke [3,4], in which we have shown how the Kripke semantics of DEL can be generalized to an order theoretic structure. This structure has a non-boolean resource-sensitive nature and abstracts over epistemic actions of DEL by considering them as fundamental operations of an algebra rather than concrete constructions on a Kripke model. The algebraic approach has also given rise to a sound and complete Lambek-style sequent calculus [11] where agents Γ_A and propositions Γ_M as well as programs Γ_Q are considered in sequents that typically look like $\Gamma_M, \Gamma_A, \Gamma_Q \vdash \delta$, where Γ_X is a finite sequence of propositions, actions or agents depending on the subscript X. We can thus reason about epistemic actions and their updates in a semi-automatic way through substitution in algebra as well as proof search in a sequent calculus. The algebra consists of a pair module-quantale (M, Q) with M and Q both sup-lattices, Q has an additional monoid structure $(Q, \bullet, 1)$ and a right action on the module $- * - : M \times Q \to M$ satisfying some conditions. The pair (M, Q) is called a *system* in the literature and has been theoretically studied in [9,12] and has found applications from behavioral models of concurrent processes in informatics [1] as well as observational models of quantum systems in physics [7].

In our setting, the elements of the module m in M stand for propositions and the elements of the quantale q in Q represents epistemic actions. The action of quantale on the module $m * q$ is epistemic update, but is also the left adjoint to PDL dynamic modality $[q]m$ or *weakest precondition* $(- * q) \dashv [q]-$. The system expresses the dynamic aspect of our setting. The epistemic aspect is taken into account by endowing the system with a family of *system endomorphisms* $f_A : (M, Q) \to (M, Q)$ where $f_A = (f_A^M : M \to M, f_A^Q : Q \to Q)$. These maps interact through the inequality $f_A^M(m * q) <_= f_A^M(m) * f_A^Q(q)$. The endowed system $(M, Q, \{f_A\}_A)$ is called an *epistemic system*. The endomorphisms are called *appearance maps* and represents each agent's appearance of what proposition f_A^M and what action f_A^Q is going on in reality. The left adjoint to these maps stands for the knowledge of each agent $f_A \dashv []_A$. These two adjunctions provide us with a new reasoning method to encode and solve dynamic epistemic scenarios such as the muddy children puzzle presented above. Since our f_A's are join preserving, our knowledge modality, similar to that of DEL, does not have the truth property that says if an agent knows a proposition, then it is true $[]_A m <_= m$. This enables us, following [2], to deal with more interesting scenarios where agents face misinformation actions such as cheating and lying such as a cheating version of the muddy children puzzle.

Different epistemic modalities are obtained in the Sup setting of M either by closure/coclosure operators resulted from composing the adjunction pair $f_A \dashv []_A$, or by forcing conditions on the f_A map. For instance, a positively introspective modality, i.e. a modality that satisfies axiom 4 of $S4$ epistemic logic, can be obtained by either composing f_A and $[]_A$ or the idempotence of f_A. If we assume that M is a boolean algebra, the linear adjoints to f_A and $[]_A$ denoted as f_A^+ and $[]_A^+$ provide us with two more modalities that also constitute an adjoint pair $f_A^+ \dashv []_A^+$. These linear adjoints are De Morgan duals of $[]_A$ and f_A. We thus get the traditional diamond modality as the linear adjoint to f_A, i.e. $<>_A m = f_A^+(m)$. More specifically, we have a representation theorem that constructs a complete Kripke model for DEL through an epistemic system in which M and Q are completely distributive atomistic boolean algebras. Our setting can be applied to encode and analyze *security protocols* [13] and yields a natural dynamic approach to *belief revision* [6].

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Local bounded commutative residuated ℓ-monoids

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Commutative residuated ℓ-monoids ($R\ell$-monoids) were introduced (in the dual form) by Swamy as a common generalization of Abelian lattice ordered groups and Heyting algebras. Moreover, bounded commutative $R\ell$-monoids (= commutative integral residuated lattices) are in very close connections with algebras of fuzzy logics, i.e., with BL-algebras, and consequently, with MV-algebras, which can be viewed as particular cases of such $R\ell$-monoids. Many of important properties of BL-algebras are also satisfied in all bounded commutative $R\ell$-monoids. Therefore bounded commutative $R\ell$-monoids could be taken as an algebraic semantics of a more general logic than Hájek's basic fuzzy logic. Therefore it is natural to study filters of those $R\ell$-monoids because from the logical point of view they correspond to sets of provable formulas. A bounded commutative $R\ell$-monoid is called local if it contains a unique maximal filter. We study properties of local $R\ell$-monoids in connection with properties of their filters.

On monadic n×m-valued Łukasiewicz algebras with negation

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In [3], monadic $n \times m$–valued Łukasiewicz algebras with negation (or $MNS_{n\times m}$–algebras) were introduced by adding a unary operator called existential quantifier to $n \times m$–valued Łukasiewicz algebras with negation ([4]). This new variety is a natural generalization of that of monadic n–valued Lukasiewicz algebras ([1]). In this note, different properties of $MNS_{n\times m}$–algebras are studied and some functional representation theorems for these algebras are obtained. In particular, a first representation theorem is described bearing in mind some of the results established in [5]. Besides, another theorem is obtained taking into account Halmos's functional representation theorem for monadic Boolean algebras ([2]). Furthermore, rich $MNS_{n\times m}$–algebras are introduced and characterized, and a third representation for these algebras is obtained. Finally, the relationship between these theorems is shown.

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Classification of functions in cardinal power

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The structure of the poset P^X of all functions from the nonempty set X into a poset P is investigated. In particular, if X is endowed by an ordering relation, then P^X is required to consist of isotone mappings and it is called cardinal power. Under the componentwise defined order, P^X is a poset which is a lattice if P is.

In our approach, the functions from P^X are considered to be fuzzy set in the most general setting: mappings from a set to a poset. Consequently, we investigate the collection of all fuzzy sets on X, with the fixed co-domain P.

Our aim is to classify fuzzy sets in P^X according to the equality of collections of cut sets. It is known that each fuzzy set on X uniquely determines a family of subsets of the domain X, cut sets, indexed by the elements of the co-domain P. However, these families can be equal if considered only as collections of subsets of the domain. Hence, it turns out that P^X can be partitioned into classes of functions (fuzzy sets) with equal collections of cut sets. In addition, this classification can be equivalently formulated in terms of subsets of the co-domain P. In particular, the corresponding results are given for P being a lattice or the unit interval of the real line.

In the second part, P^X is supposed to be a cardinal power. In other words, both X and P are partially ordered and only those fuzzy sets which are isotone functions are considered. In this case, cut sets turn out to be down-sets on X. Classification of fuzzy sets is completely described, and some special cases of posets X and P are additionally considered (the case of lattices, unit intervals, complete posets etc.).

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New results on neighbourhood semantics for modal and intermediate logics

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The results presented here develop the papers [4],[5],[7]. We consider propositional logics of two kinds: normal extensions of **K4** (called just 'modal logics') and intermediate logics; recall that $\mathbf{S4} = \mathbf{K4} + \Box p \to p$, $\mathbf{GL} = \mathbf{K4} + \Box(\Box p \to p) \to \Box p$, $\mathbf{Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$.

A *neighbourhood (***K4***-)frame* consists of a non-empty set X with a unary operation $\Box$ on its subsets such that $\Box X = X$ and for any Y, Z, $\Box(Y \cap Z) = \Box Y \cap \Box Z$, $\Box Y \subseteq \Box\Box Y$. As usual, a *model* over a frame F is a pair (F, φ), where φ is a valuation of propositional variables in F. The definitions of truth and validity are standard.

$\mathbf{ML}(F)$ denotes the *modal logic of* F, i.e. the set of all modal formulas valid in F. Modal logics of this kind are called *N-complete*. For a modal logic L, an *L-frame* is a neighbourhood frame F validating L.

For an **S4**-frame (a topological space) Φ one can also define validity of intuitionistic formulas (via Tarski translation) and obtain the intermediate logic $\mathbf{IL}(\Phi)$.

Note that every neighbourhood frame $(X, \Box)$ is associated with an **S4**-frame $(X, \Box^+)$, where $\Box^+ Y = \Box Y \cap Y$.

1. N-Compactness

Definition 1. A frame $(X, \Box)$ is called *locally* T_1 if the space $(X, \Box^+)$ is locally T_1 (in the sense of [4]), i.e., if every point is closed in some its neighbourhood.

Definition 2. Let S be a set of modal formulas, L a modal logic. S is called *satisfiable* in a frame F if there exists a model M over F and a point $x \in F$ such that $M, x \models A$ for every $A \in S$; S is L-*satisfiable* if S is satisfiable in some L-frame; S is *finitely L-satisfiable* if every its finite subset is L-satisfiable.

Definition 3. A modal logic L is called *N-compact* if every finitely L-satisfiable set of modal formulas is L-satisfiable.

Remark. A different notion of compactness was introduced by S.K. Thomason for the case of Kripke semantics [6]; its analogue for the neighbourhood semantics was studied in [7].

Theorem 1. Let L be a modal logic. If a set of modal formulas S is finitely L-satisfiable in locally T_1-spaces, then S is L-satisfiable.

For the proof an appropriate notion of ultrabouqet is introduced, similar to those in [4], [5].

One can easily show that every neighbourhood **GL**-frame is locally T_1. Hence we obtain

Theorem 2. Every extension of **GL** is N-compact.

Since there exists an embedding of **Grz** into **GL** translating $\Box A$ as $\Box A \wedge A$, this implies

Corollary 3. Every extension of **Grz** is N-compact.

2. N-completeness is stronger than K-completeness: a simple example.

Recall that a logic (modal or intermediate) of a Kripke **S4**-frame is called *K-complete.*

Definition 4. (cf. [4]) A modal logic L is *S-N-complete* if every L-consistent set of formulas is satisfiable in some neighbourhood L-frame. An intermediate logic L is *S-N-complete* if every L-consistent pair of sets of formulas is satisfiable in some neighbourhood L-frame.

Thus a modal logic is S-N-complete iff it is N-complete and N-compact. In [4], [5] it was proved that for intermediate and **S4**-logics K-completeness implies S-N-completeness. The converse is not true:

Theorem 4. There exists a countable locally T_1 topological space $\mathcal{X}$ whose intermediate logic is K-incomplete.

Note that all earlier examples of spaces with K-incomplete modal logics were uncountable [3], [2], [7] and rather comlicated. Our proof of Theorem 4 again uses ultrabouqets. Viz., we take $\mathcal{X} = \bigvee_{\mathcal{U}}(F_n, d_n)$, where F_n is the well-known Fine's frame [1] without the points d_m for $m > n$, $\mathcal{U}$ a non-principal ultrafilter in ω.

The same argument as in Theorem 5.8 from [4] shows that the intermediate logic of any locally T_1-space is S-N-complete. So we obtain

Corollary 5. S-N-completeness does not imply K-completeness for intermediate logics and extensions of **Grz**.

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On the superintuitionistic predicate logics of Kripke frames based on denumerable chains

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We consider superintuitionistic predicate logics (without equality and functional symbols), i.e., extensions of intuitionistic predicate logic **QH** closed under modus ponens, universalization and predicate substitution. We consider the standard predicate Kripke semantics. For a class Y

of posets let $\mathbf{L}Y$ (or $\mathbf{L}^cY$) be the predicate logic characterized by the class of all Kripke frames (or, respectively, all Kripke frames with constant domains) with the structures of possible worlds from Y.

The predicate logics $\mathbf{L}W_n$ and $\mathbf{L}^cW_n$ of n-element chains W_n are finitely axiomatizable [2]. The logics $\mathbf{L}W_\omega = \bigcap_n \mathbf{L}W_n$ and $\mathbf{L}^cW_\omega = \bigcap_n \mathbf{L}^cW_n$, where W_ω is an ω^*-chain, are not RE [3,4]; clearly, they are Π^0_2-arithmetical.

Let η be the set of rational numbers.

Let us consider the following formulas:

$$\begin{aligned} Z &: (Q \to R) \vee (R \to Q) ; \\ D &: \forall x(P(x) \vee Q) \to \forall x P(x) \vee Q ; \\ K &: \forall x \neg\neg P(x) \to \neg\neg \forall x P(x) . \end{aligned}$$

A modification of Non-Axiomatizability Theorem, stated in [3], gives the following description of the predicate logics $\mathbf{L}Y$ and $\mathbf{L}^cY$ for classes Y of denumerable chains.

Theorem Let Y be a class of denumerable (or finite) chains, and assume that some chain from Y contains an infinite cone (otherwise, if all cones are finite, clearly $\mathbf{L}Y = \mathbf{L}W_n$ and $\mathbf{L}^cY = \mathbf{L}^cW_n$ for some $n \leq \omega$). Then:

(1) If every chain from Y has a top element and some chain from Y contains an η-subchain, then $\mathbf{L}Y = [\mathbf{QH} + Z\&K]$ and $\mathbf{L}^cY = [\mathbf{QH} + D\&Z\&K]$.

(2) If some chain from Y contains a cofinal η-subchain, then $\mathbf{L}Y = [\mathbf{QH} + Z]$ and $\mathbf{L}^cY = [\mathbf{QH} + D\&Z]$.

(3) In all other cases the logics $\mathbf{L}Y$ and $\mathbf{L}^cY$ are Π^1_1-hard.

This theorem strengthens the result of Baaz et al [1]. Namely, they considered closed subsets X of the real interval $[0,1]$ as Heyting algebras, and proved that their superintuitionistic predicate logics $\mathbf{L}[X]$, except for the logics $[\mathbf{QH} + D\&Z]$, $[\mathbf{QH} + D\&Z\&K]$, and $\mathbf{L}^cW_n$ $(n \leq \omega)$, are not RE. Also A. Beckmann proved (a paper in preparation) that any logic $\mathbf{L}[X]$ of this kind equals $\mathbf{L}^cW$ for some denumerable (or finite) chain W.

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Free finitely generated equivalential algebras

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By an *equivalential algebra* we mean a grupoid $\mathbf{A} = (A, \leftrightarrow)$ that is a subreduct of a Brouwerian semilattice with the operation $\leftrightarrow$ given by $x \leftrightarrow y = (x \to y) \wedge (y \to x)$. This notion was introduced by Kabziński and Wroński in [2] as an algebraic counterpart of the equivalential fragment of propositional intuitionistic logic. The class $\mathcal{E}$ of all equivalential algebras is equationally definable by the following identities: $xxy = y$, $xyzz = xz(yz)$, $xy(xzz)(xzz) = xy$. (We adopt the convention of associating to the left and ignoring the symbol of equivalence operation.) It is easy to show that the term $1 := xx$ is the constant unit term in $\mathcal{E}$. The equivalential algebras form a paradigm of congruence permutable Fregean varieties, in the sense that every such variety has a binary term that turns every of its member into an equivalential algebra [1]. Recall that a variety $\mathcal{V}$ of algebras with a distinguished constant 1 is called *Fregean* if it is 1-*regular* and *congruence orderable*, i.e., $\Theta_{\mathbf{A}}(1,a) = \Theta_{\mathbf{A}}(1,b)$ implies $a = b$ for all $a, b \in A$ and $\mathbf{A} \in \mathcal{V}$. The mapping $\mathrm{Con}(\mathbf{A}) \ni \theta \to 1/\theta \in \Phi(\mathbf{A}) = \{1/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$ is a natural isomorphism of lattices.

The variety $\mathcal{E}$ is locally finite, however, the cardinality of the n-generated free algebra $\mathbf{F}_{\mathcal{E}}(n)$ is known only for $n = 1, 2, 3$, and is equal to 2, 9, 4415434, respectively; see [5]. We present a recursive construction of $\mathbf{F}_{\mathcal{E}}(n)$ based on the representation theorem, which is valid for an arbitrary finite algebra $\mathbf{A}$ from a congruence permutable Fregean variety $\mathcal{V}$. This theorem generalises the well-known fact that for a finite Brouwerian semilattice there is a bijection from the algebra to the family of upwards closed (under inclusion) sets of its meet-irreducible filters. In the general case only certain upwards closed sets represent the elements of algebra $\mathbf{A}$. To characterise them we introduce an equivalence relation $\sim$ on the set of completely meet-irreducible filters $\mathrm{Fm}(\mathbf{A})$ assuming that $\varphi \sim \psi$ if and only if the prime intervals $I[\varphi, \varphi^{+}]$ and $I[\psi, \psi^{+}]$ are projective in $\Phi(\mathbf{A})$ for $\varphi, \psi \in \mathrm{Fm}(\mathbf{A})$, where η^{+} denotes the unique cover of $\eta \in \mathrm{Fm}(\mathbf{A})$. Note that $\varphi \sim \psi$ implies $\varphi^{+} = \psi^{+}$. For the variety $\mathcal{E}$ the reverse implication is also true, whereas for an arbitrary arithmetic Fregean variety we have $\varphi \sim \psi$ if and only if $\varphi = \psi$. We show that each equivalence class, supplemented with a unit element, is closed under the natural Boolean group operation $\varphi \cdot \psi := (\varphi \div \psi)' \cap \varphi^{+}$. This construction leads to the notion of *hereditary sets* of meet-irreducible filters. We show that there is a one-to-one correspondence between the elements of a finite algebra and the class of hereditary subsets of the set of its meet-irreducible filters. Moreover, the equivalence operation in this algebra can easily be recovered from this representation. Thus, to construct the n-generated free algebra in a congruence permutable locally finite Fregean variety $\mathcal{V}$ it suffices to describe the frame $(\mathrm{Fm}(\mathbf{F}_{\mathcal{V}}(n)), \subset, \sim, \cdot)$. In fact, for two extreme cases of equivalential algebras and Brouwerian semilattices the construction is easier because the relation $\sim$ is known. (The construction for Brouwerian semilattices can be found, e.g., in [3].) In the recursive construction of the set $\mathrm{Fm}(\mathbf{F}_{\mathcal{V}}(n))$ the number of levels corresponds to the number of free generators of the algebra. The members of the k-th level ($k = 1, \ldots, n$) are precisely those elements $\varphi \in \mathrm{Fm}(\mathbf{F}_{\mathcal{V}}(n))$ for which the height of the quotient algebra $\mathbf{F}_{\mathcal{V}}(n)/\varphi$ is equal to $k + 1$.

Using this method we can describe the finitely generated free algebras and determine the free spectra of varieties of linear equivalential algebras $\mathcal{E}_{\omega}$ and linear equivalential algebras of finite height $\mathcal{E}_h$ ($h \in N$) corresponding, respectively, to the equivalential fragments of intermediate Gödel-Dummett logic and intermediate finite-valued logics of Gödel. In this case we can represent meet-irreducible filters in the n-generated free algebra as non-empty chains in the power set of $\{1, \ldots, n\}$. Combining this fact with the representation theorem, we obtain a closed-form formula for the free spectrum of the variety $\mathcal{E}_3$ and recurrence formulas for $\mathcal{E}_h$ ($h \geq 4$) and $\mathcal{E}_{\omega}$. In particular, we show that the double logarithm of the number of elements of the free spectrum of $\mathcal{E}_{\omega}$ behaves as $n \ln n$ for large n.

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Axiomatizations of quantum actions

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As formalized by von Neumann and others, Quantum Mechanics is based on a few postulates, among which are the following: the possible *states* of a physical system can be represented as *one-dimensional linear subspaces* in an infinite-dimensional Hilbert space $\mathcal{H}$; the possible *experimental (or testable) properties* of a system correspond to all *closed linear subspaces*; a *successful measurement* (of a physical property P) corresponds to a *projector* P_W in $\mathcal{H}$ (onto the subspace W corresponding to P); in the absence of measurements, the possible evolutions of a physical system are given by (reversible) linear maps U, called *unitary transformations*. Every testable property P has an orthocomplement $\sim P$ (given by the orthogonal space $W^{\perp}$ of the space W corresponding to P): this encodes a sense of "necessary failure" (impossibility of a successful measurement) of P.

Traditional quantum logic takes "the logic of quantum mechanics" to refer to the (non-distributive) *lattice structure* of the family $\mathcal{L}$ of all *testable properties*. The long-standing program of this tradition has been to give a *"Hilbert-complete" axiomatization* of this logic, i.e. one that is *complete with respect to infinite-dimensional Hilbert spaces*. A modal axiomatization (in terms of orthoframes) has been proposed, but this was not Hilbert-complete: the (Hilbert-valid) law of "Weak Modularity" not only fails to be valid on orthoframes, but it corresponds to no first-order frame condition [3]. The standard quantum-logical setting is the *orthomodular quantum logic*, which is also Hilbert-incomplete (see [2] for a counter-example). C. Piron [5,6] proposed an algebraic axiomatization (the so-called *Piron lattices*), for which he proved a Representation Theorem w.r.t. *generalized* Hilbert spaces. This result was improved by Solèr and Mayet [4,7], who added another axiom, obtaining an (abstract) Hilbert-completeness result. (However, this axiomatization is not obviously first-order and contains some axioms of an artificial, un-intuitive character.)

So although this program has (partially) succeeded, I argue here that the result could certainly be improved and put into a new light by moving to a *dynamic-logical* setting, in which *physical actions* (and not only static physical properties) are logically represented. This is the content of my joint work with A. Baltag in [1] on the dynamic logic of quantum actions. In the paper, we present two (equivalent) complete axiomatizations for the "logic of quantum actions" (LQA): one in terms of *quantum transition systems* (QTS), and one in terms of *quantum dynamic algebras* (QDA). In this talk, I concentrate on the first setting.

A QTS is a Kripke frame of the form $\mathcal{F} = (\Sigma, \{\stackrel{P?}{\rightarrow}\}_{P\in\mathcal{L}}, \{\stackrel{U}{\rightarrow}\}_{U\in\mathcal{U}}, C \subseteq \Sigma)$ consisting of: *states* $s \in \Sigma$; binary relations for "test" actions $\stackrel{P?}{\rightarrow} \subseteq \Sigma \times \Sigma$, labelled by "testable properties" P coming from a set $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$; binary relations for "unitary evolutions" $\stackrel{U}{\rightarrow} \subseteq \Sigma \times \Sigma$; and a finite set $C \subseteq \Sigma$ of special states. These are subject to a number of conditions, given in [1]. The axioms correspond to simple *frame conditions*, with a natural dynamic/operational interpretation. In [1] we proved an (abstract) Hilbert-completeness result for this axiomatization.

Next, I present some recent (unpublished) joint work with A. Baltag, on a *finitary modal logic* (PDL-style) and *proof system* for the "logic of quantum actions" LQA. The syntax consists of propositional *formulas* and of *programs*, defined by mutual induction:

$\varphi ::= p \quad | \quad c \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad [\pi]\varphi$

$\pi ::= \varphi? \quad | \quad U \quad | \quad U^{\dagger} \quad | \quad \pi \cup \pi \qquad \pi ; \pi$

Formulas are interpreted over QTS's using Boolean operations and weakest preconditions. Programs are interpreted using "tests" $P?$, evolutions U (and their inverses $U^{\dagger}$), relational union $R \cup R'$ and the relational composition $R; R'$. The proof system is *sound and expressive* w.r.t. the above-mentioned frame conditions: i.e. the axioms are valid on a Kripke frame $\mathcal{S}$ iff $\mathcal{S}$ is bisimilar to a QTS. We *conjecture that this proof system is complete w.r.t. QTS's (and thus also w.r.t. Hilbert spaces).* Unlike other quantum-logical approaches, (the "static", propositional fragment of) our logic is "classical", i.e. Boolean; all the "quantum" effects are captured by the *dynamic side* of our system. The "moral" is that quantum physics does not require any modification of the classical laws of Propositional Logic (governing "static" information), but only a *non-classical dynamics of information.*

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LP logic with fixed point operator

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1 Introduction

The theory of Fixed Point appears in many different fields of Mathematics, standing at the core of Computer Science and being involved in many foundational aspects of Logic. Starting from a proposal by Aho and Ulmann [AU79] several extensions of First Order Logic with Fixed Point Operator have been so far studied. The first and most known is the Least Fixed Point, but after that, different formalizations such as Inductive (aka Inflationary), Partial or Non Deterministic FPO, quickly appeared. As the expressive power of those operators (over finite structures) showed to be closely related (often, equivalent) to important open problems in Complexity Theory, their importance increased over the years.

The purpose of this study is to investigate fixed points in Many Valued Logic, starting from ŁΠ Logic. This Logic is the union of two important multi valued Logics, Łukasiewicz and Product Logic. It has been extensively studied in [EG99], [Mon00], [EGM01], having acquired importance for many reasons: it has been used for formalizing probability [EGH00] and seems to be a suitable setting to handle fuzzy-controls rules. In terms of expressiveness it subsumes different many valued Logics: it includes the three most important t-norm based many valued Logic, i.e. Łukasiewicz, Product and Gödel Logic as well as Rational Pavelka Logic.

Here we present the equivalent algebraic semantic of ŁΠ, namely ŁΠ algebras. (This definition differs from the original one from [EGM01] and was introduced in [C05]):

Definition An ŁΠ algebra is a structure $\mathcal{L} = \langle L, \oplus, -_L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$ where:

1. $\mathcal{L}' = \langle L, \oplus, -_L, 0_L \rightarrow \rangle$ is a MV algebra
2. $\mathcal{L}'' = \langle L, \Rightarrow_\Pi, *_\Pi, 0_L, 1_L \rangle$ is a Π algebra
3. $x *_\Pi (y \ominus z) = (x *_\Pi y) \ominus (x *_\Pi z)$
4. If $(x \Rightarrow_L y) = 1_L$ then $(x \Rightarrow_\Pi y) = 1_L$

2 Results

In this study we try to put together these very expressive tools in what we called μŁΠ Logic. The aim is two folds. On one hand adding a Fixed Point Operator is motivated in an algebraic perspective, since it adds new properties to ŁΠ algebras, leading to structures similar to Real Closed Fields. On the other hand being able to use induction *inside* Multi Valued Logic could open new topics in Approximate Reasoning.

2.1 Algebraic Completeness

The first step in order to algebraically study this Logic was to introduce the following class of algebras.

Definition The class of μŁΠ algebras is axiomatized by the following quasiequations: All the axioms and rules from ŁΠ algebras, plus, for any t terms not containing the symbol $\rightarrow_\Pi$, the following schemas:

1. $\mu.x(t(x)) = x$
2. $(\bigwedge_{i \leq n} \Delta(p_i \Leftrightarrow q_i)) \leq (\mu.x(t(p_1, \dots, p_n)) \Leftrightarrow \mu.x(t(q_1, \dots, q_n)))$
3. If $t(p) = p$ then $\mu.x(t(x)) \leq p$

Despite this presentation, as the algebras contain a discriminator, it can be proved that the quasivariety is indeed a variety. Moreover this variety stays to μŁΠ Logic in the same way as ŁΠ algebras stay to ŁΠ logic.

Theorem 2.1 *μŁΠ logic is algebraically complete, i.e. if φ is a formula in the language of μŁΠ logic, the following are equivalent*

(i) φ is provable in μŁΠ

(ii) For each linearly ordered μŁΠ algebra $\mathcal{A}$, $\mathcal{A} \models \varphi$

(iii) For each μŁΠ algebra $\mathcal{A}$, $\mathcal{A} \models \varphi$

2.2 Categorical Equivalence

Theorem 2.2 *The category of linearly ordered μŁΠ algebras and the category of linearly ordered Real Closed Fields are equivalent.*
Our efforts at this moment are, in fact, toward a generalization of this equivalence so that we can replace the linearly ordered algebras with the whole category of μŁΠ algebras.

2.3 Standard Completeness

Theorem 2.1 can be extend to something more important in Fuzzy Logic,

Theorem 2.3 *μŁΠ is standard complete, i.e. a formula φ is a μŁΠ tautology if, and only if, it is true on the μŁΠ algebra on $[0, 1]$.*

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On the assertional logic of the generic pointed discriminator variety

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with Bob Bignall (Monash University, Australia)
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For a quasivariety K over a language type Λ with a constant term $\mathbf{1}$, the $\mathbf{1}$-*assertional logic of* K, in symbols $\mathcal{S}(\mathsf{K}, \mathbf{1})$, is the consequence relation $\vdash_{\mathcal{S}(\mathsf{K},\mathbf{1})}$ from sets of Λ-terms to Λ-terms determined by the equivalence

$$\Gamma \vdash_{\mathcal{S}(\mathsf{K},\mathbf{1})} \varphi \quad \text{if and only if} \quad \{\psi \approx \mathbf{1} : \psi \in \Gamma\} \models_{\mathsf{K}} \varphi \approx \mathbf{1}.$$

A *distributive symmetric local skew lattice* is an algebra $\langle A; \wedge, \vee\rangle$ of type $\langle 2, 2\rangle$ that is a non-commutative analogue of a distributive lattice. An *implicative BCSK-algebra* is an algebra $\langle A; \Rightarrow, \rightarrow, 1\rangle$ of type $\langle 2, 2, 0\rangle$ that is a 'non-commutative' analogue of an implication algebra. A *skew Boolean* $\cup$*-algebra* is an algebra $\langle A; \wedge, \vee, \Rightarrow, \rightarrow, 1\rangle$ of type $\langle 2, 2, 2, 2, 0\rangle$ that is the conjunction (in a precise technical sense) of a distributive symmetric local skew lattice with an implicative BCSK-algebra. The *skew Boolean propositional calculus*, in symbols $\mathcal{SBPC}$, is the $\mathbf{1}$-assertional logic of the variety of all skew Boolean $\cup$-algebras. The deductive system $\mathcal{SBPC}$ arises naturally in algebraic logic as a non-Fregean analogue of the axiomatic extension of Hilbert's positive logic by the Peirce law.

The (*ternary*) *discriminator* on a set A is the function $t : A^3 \rightarrow A$ defined for all $a, b, c \in A$ by

$$t(a,b,c) := \begin{cases} c & \text{if } a = b \\ a & \text{otherwise.} \end{cases}$$

A (*ternary*) *discriminator variety* is a variety V for which there exists a ternary term $t(x, y, z)$ that realises the discriminator on every subdirectly irreducible member of V. A *pointed discriminator variety* is a discriminator variety with a constant term. The *generic pointed discriminator variety*, in symbols $\mathsf{PD}_{\mathbf{1}}$, is the discriminator variety generated by the class of all algebras $\langle A; t, 1\rangle$, where t is the ternary discriminator on A and 1 is a nullary operation. By a result (implicitly) due to McKenzie, every discriminator variety is term equivalent to a subvariety of the generic pointed discriminator variety with compatible operations, for a suitable notion of compatible operation.

A deductive system is said to be a *pointed discriminator logic* if it is algebraizable in the sense of Blok and Pigozzi and its equivalent algebraic semantics is a pointed discriminator variety. Examples of pointed discriminator logics abound in the literature and include classical propositional logic; the modal logic **S5**; the n-dimensional cylindric logics; the n-valued Post logics; the n-valued Łukasiewicz logics; and the tetravalent modal logic of Font and Rius.

In this talk, we present a simple finite axiomatisation of $\mathcal{SBPC}$ and show that it is, to within formula equivalence, exactly the **1**-assertional logic of the generic pointed discriminator variety $\mathsf{PD}_{\mathbf{1}}$. In the main result of the talk, we further show that every pointed discriminator logic is, up to definitional expansion, an expansion of $\mathcal{SBPC}$ by extensional logical connectives. As a by-product of the general theory, we obtain insights into the structure and behaviour of pointed discriminator logics, and are able to clarify connections between (pointed) discriminator varieties and the *fixedpoint discriminator varieties* of Blok and Pigozzi. We illustrate the theory with applications to several well known pointed discriminator logics, including the normal modal logic **S5** and the three-valued logic of Łukasiewicz and Tarski.

Modal correspondence for topological semantics

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We prove topological analogues of two classical results in modal logic—the Goldblatt-Thomason Theorem and the van Benthem Characterization Theorem. That is, on one hand we provide the necessary and sufficient conditions for a first-order definable class of topological spaces to be definable (a) in the basic modal language; (b) in the hybrid modal language. On the other hand, we show that a formula $\alpha(x)$ in the first-order topological correspondence language is equivalent to a standard translation of a modal formula iff $\alpha(x)$ is invariant under topo-bisimulations. An application of the Goldblatt-Thomason Theorem is given by showing that the class of Hausdorff spaces is not definable in the hybrid modal language.

We work with the topological semantics for modal logic where the modal diamond is understood as the closure operator of a topological space. The question of modal definability with respect to this semantics was discussed in [5], where the notion of the Alexandroff extension of a topological space was introduced. However, the main result of [5], The topological Goldblatt-Thomason theorem was proved only for the classes of spaces closed under the formation of Alexandroff extensions. Here we substitute this somewhat impractical precondition with the more intuitive notion of the elementary class of topological spaces. To this end, we consider the two-sorted first-order language L_t defined in [4] (see also [2, pp.7–8] for a concise description of L_t). The language L_t is tailored to 'speak' about topological structures. We call a class of topological spaces *elementary* if it is definable by a sentence of L_t. Recall that a map between topological spaces is called *interior* if it is continuous and open. Alexandroff extension of a space X will be denoted by X^*.

Theorem 1 An elementary class K of topological spaces is modally definable iff it is closed under taking opens subspaces, interior images, topological sums and it reflects Alexandroff extensions.

As demonstrated in [5], extending the modal language with nominals increases the expressive power. In particular, in the hybrid modal language $H(@)$ (see [3] for the definition) lower separation axioms T_0 and T_1 become expressible. We can characterize precisely the elementary classes definable in $H(@)$ with the help of the following notion adapted from [3]:

Definition: Let T and S be topological spaces. S is called *topological ultrafilter morphic image* of T if there is a surjective interior map $f : T \to S^*$ such that $|f^{-1}(u)| = 1$ for every principal ultrafilter $u \in S^*$ (one can say figuratively 'f is injective on principal ultrafilters').

Theorem 2 An elementary class K of topological spaces is definable in $H(@)$ iff it is closed under topological ultrafilter morphic images and taking open subspaces.

We denote the hybrid language with universal modalities by $H(E)$.

Theorem 3 An elementary class K of topological spaces is definable in $H(E)$ iff it is closed under topological ultrafilter morphic images.

As an application of these theorems we show that:

Theorem 4 The class of T_2 spaces in not definable in $H(@)$ and $H(E)$.

Similarly to the relational semantics, we can translate the modal formulas into L_t-formulas provided L_t is enriched with the unary predicates—one for each propositional letter. Let such enrichment of L_t be called the Topological Correspondence Language. Using the notion of a topo-bisimulation from [1] one can characterize precisely the modal fragment of the topological correspondence language.

Theorem 5 A first-order formula $\alpha(x)$ of the topological correspondence language is invariant under topo-bisimulations iff it is equivalent to a standard translation of a modal formula.

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Construction of different types of fuzzy sets with a given family of cuts

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In our work the following problem has been solved: For a given a set X, a collection F of subsets of X and a lattice L: whether there exists a lattice valued fuzzy set on X with the co-domain L, such that F is equal to the family of cut sets of the constructed fuzzy set. The well known theorem of synthesis of fuzzy sets starts from a given set and a collection of subsets and construct a fuzzy set with the required family of cuts having the lattice co-domain constructed out of the given collection of cuts (i.e., the lattice is not given in advance). This is the difference of the Theorem of synthesis with our present investigations (for some previous results see [1,2]).

Further, we discuss the analogue problem for the strong cuts and other types of cut sets and also for lattice valued intuitionistic fuzzy sets. In the latter case there are two related families of cut sets, so the problem is more complicated. Finally, we are solving the mentioned problems for fuzzy sets whose co-domain is a partially ordered set.

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Algebraic approach to canonical formulas in $\mathrm{ExtFL_{ew}}$ and NExtK

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The aim of the talk is to present algebraic setting for investigations into the so-called *canonical formulas* over a given logic from $\mathrm{ExtFL_{ew}}$ or NExtK. The notion is understood as in the Chapter 9 of *Modal Logic* by A. Chagrov and M. Zakharyaschev. Canonical sets of formulas are proposed for some classes of normal modal logics and substructural logics without contraction.

0. Main definition

We say that a set of formulas Γ is sufficiently rich (SRich for short) over a variety $\mathcal{W}$ if for any subvariety $\mathcal{V} \subseteq \mathcal{W}$ there exists a subset $\Delta \subseteq \Gamma$ determining $\mathcal{V}$ relatively to $\mathcal{W}$. (It is assumed that algebras from $\mathcal{W}$ contain a constant 1 and the subvariety determined by a formula α is the class of all algebras verifying the identity $\alpha \approx 1$.)

Definition 1. Let Γ be SRich over L_0, $L_0 = \mathsf{Log}(\mathcal{W})$. Γ is a set of **canonical formulas** over L_0 if there exists an operator $\mathsf{O} : \mathcal{W} \to \mathcal{P}(\mathcal{W})$ such as for any formula $\alpha \in \mathbf{ForL}$ the following holds:

- Formulas from Γ describe finite algebras: $L_0+\alpha = L_0+\gamma(\mathbf{A}_1)+\ldots+\gamma(\mathbf{A}_k)$, $\{\mathbf{A}_1, \ldots, \mathbf{A}_k\} = \mathcal{K}_\alpha \subseteq \mathcal{W}_{fin}$.
- Algebras from $\mathcal{K}_\alpha$ characterize countermodels of α. For any $\mathbf{B} \in \mathcal{W}$, $\mathbf{B}$ refutes α iff $\mathbf{A}_i \in \mathsf{O}(\mathbf{B})$ for an $\mathbf{A}_i \in \mathcal{K}_\alpha$.
- If $\mathbf{A} \in \mathcal{K}_\alpha$ then $\gamma(\mathbf{A})$ says: "I am refuted in $\mathbf{B}$ iff $\mathbf{A}$ belongs to $\mathsf{O}(\mathbf{B})$".
- There are suitable algorithms producing $\mathcal{K}_\alpha$ and $\gamma(\mathbf{A}_i)$ given α and $\mathbf{A}_i$.

As implied by the definition, the job of defining canonical formulas has two parts: semantic - defining the class $\mathcal{K}_\alpha$ and syntactic - defining actual canonical formulas on algebras from $\mathcal{K}_\alpha$.

1. Semantic part

For any formula α, the subvariety of $\mathcal{W}$ determined by this formula can be defined as the largest subvariety of $\mathcal{W}$ not containing algebras which are *critical* over it. Although the class $\mathsf{Crit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ cannot be directly used as $\mathcal{K}_\alpha$ since it may be infinite and contain infinite algebras, it can be used to define such a set.

Each algebra in $\mathsf{Crit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ is generated by a valuation refuting α. Taking into account the elements being values of α-subformulas under such valuations (and adding the biggest element) we obtain a finite set of finite partial algebras $\mathsf{PCrit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$ which may play the role of $\mathcal{K}_\alpha$.

2. Syntactic part

The definition of formulas is based on Jankov's characteristic formulas originally defined for finite subdirectly irreducible Heyting algebras. $[v]^\alpha$ denotes a partial algebra determined by a valuation v and subformulas of α and $\Delta(\mathbf{P})$ is the *diagram* of a partial algebra $\mathbf{P}$ describing how (partial) operations act on the universe of $\mathbf{P}$. $\mathcal{M}$ and $\mathcal{R}$ are varieties of normal modal algebras and residuated lattices respectively.

Definition 2. Let $\mathcal{W}$ be a subvariety of $\mathcal{M}$ or $\mathcal{R}$, $n < \omega$. Let $\mathbf{P} = [v]^\alpha$ for a $[v]^\alpha \in \mathsf{PCrit}_{\mathcal{W}}(\mathsf{Mod}_{\mathcal{W}}(\alpha))$. **The characteristic formula of order n** for $\mathbf{P}$ is defined as follows:

$$\chi^{(n)}(\mathbf{P}) = \Delta^{(n)}(\mathbf{P}) \to x_{v(\alpha)}.$$

If $\mathcal{W} \subseteq \mathcal{M}$ then $\Delta^{(n)}(\mathbf{P}) = \Delta(\mathbf{P}) \wedge \ldots \wedge \Box^n \Delta(\mathbf{P})$. If $\mathcal{W} \subseteq \mathcal{R}$ then $\Delta^{(n)}(\mathbf{P}) = (\Delta(\mathbf{P}))^n$.

The set of all characteristic formulas over partial algebras for all "exponents" will be denoted by $\mathbf{Char}^{(\omega)}\mathbf{ForPar}(\mathcal{W})$.

3. Canonical formulas

Theorem 1. Let $\mathcal{W} \subseteq \mathcal{M}$ or $\mathcal{W} \subseteq \mathcal{R}$.

- $\mathbf{Char}^{(\omega)}\mathbf{ForPar}(\mathcal{W})$ is SRich over $\mathcal{W}$
- If $\mathcal{W}$ has EDPC then there is n such as $\mathbf{Char}^{(\mathbf{n})}\mathbf{ForPar}(\mathcal{W})$ is SRich over $\mathcal{W}$
- If $\mathcal{W}$ has EDPC and $Log(\mathcal{W})$ is decidable then, for appropriate n, $\mathbf{Char}^{(\mathbf{n})}\mathbf{ForPar}(\mathcal{W})$ is a set of canonical formulas over $\mathcal{W}$.

Equivalence of consequence relations: an order-theoretic and categorical perspective, I

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The aim of my talk, as well as that of the talk by Nikolaos Galatos (see page 26), is to propose an order-theoretic and categorical framework for various constructions and concepts connected with the study of logical consequence relations. Our approach places under a common umbrella a number of existing results regarding the equivalence of consequence relations and provides a roadmap for future research in this area.

A consequence relation is defined relative to an algebraic signature $\mathcal{L}$. The set Fm of $\mathcal{L}$-formulas is the universe of the term algebra $\mathbf{Fm}$ of signature $\mathcal{L}$ over a countably infinite set of variables. Throughout this abstract, we identify the set Eq of $\mathcal{L}$-equations with the universe of the algebra $\mathbf{Fm} \times \mathbf{Fm}$, and denote by $\mathbf{\Sigma}$ the monoid of substitutions of $\mathbf{Fm}$.

It is proved in [BP] that a substitution invariant, finitary consequence relation $\vdash$ on Fm is algebraizable if and only if there exists an algebraic consequence relation $\models$ on Eq such that the lattices, $\mathbf{Th}_{\vdash}$ and $\mathbf{Th}_{\models}$, of the theories corresponding to $\vdash$ and $\models$ are isomorphic under a map that commutes with inverse substitutions. A second equivalent condition to the algebraizability of $\vdash$ is the following. There exist (i) an algebraic consequence relation $\models$ on Eq and (ii) finitary maps $\tau : Fm \to \wp(Eq)$, and $\rho : Eq \to \wp(Fm)$ that commute with substitutions, such that for all $\Psi \cup \{\phi\} \in \wp(Fm)$ and $\varepsilon \in Eq$

1. $\Psi \vdash \phi$ iff $\tau[\Psi] \models \tau(\phi)$
2. $\varepsilon \dashv\models \tau\rho(\varepsilon)$

In [BJ99] and [BJ05], W. J. Blok and B. Jónsson use the first property above as the definition of equivalence of two consequence relations. More precisely, they declare two consequence relations equivalent provided the lattices of their theories are isomorphic under a map that commutes with inverse substitutions. Moreover, they establish a result that subsumes the cited result in [BP].

Our setting is more general and places the aforementioned considerations on solid algebraic and categorical ground. Starting with the concrete situation above, we note that there exists a natural action of $\mathbf{\Sigma}$ on $\mathbf{Fm}$ that extends to an action of the corresponding power sets. The power set $\wp(\Sigma)$ is a ringlike object – in which set-union plays the role of addition and complex product serves as multiplication. On the other hand, $\wp(Fm)$ is a structure corresponding to an abelian group, with set-union playing again the role of addition. The latter action possesses the critical property of being residuated, which, in this particular instance, means that it preserves arbitrary unions in each coordinate. Analogous comments hold for the action of $\mathbf{\Sigma}$ on $\mathbf{Eq}$.

This concrete situation leads naturally to the general concept of a (left) module. The *scalars* $\mathbf{A}$ of such a structure are the elements of a residuated partially ordered monoid. The *vectors* $\mathbf{P}$ form a partially ordered set. The scalar multiplication $\star : \mathbf{A} \times \mathbf{P} \to \mathbf{P}$ is a bi-residuated map (i.e., a residuated map in each coordinate) that satisfies the usual properties of a monoid action. Given a partially ordered residuated monoid $\mathbf{A}$, all $\mathbf{A}$-modules constitute the objects of a category, ${}_{\mathbf{A}}\mathcal{M}$, whose morphisms are residuated maps that preserve scalar multiplication.

For a fixed $\mathbf{A}$, the category ${}_{\mathbf{A}}\mathcal{M}$ provides an ideal environment to abstract the aforementioned concepts and identify their categorical properties. For example, consequence relations on an object $\mathbf{P}$ correspond bijectively to the epimorphic images of $\mathbf{P}$. Thus, consequence relations may be identified with objects of this category. Not surprisingly then, we stipulate that two consequence relations are equivalent if the $\mathbf{A}$-modules corresponding to them are isomorphic. On the other hand, we can define equivalence of consequence relations by abstracting the second condition for algebraizability stated above. The main result of this work, to be described in Nikolaos Galatos's abstract, identifies categorically the modules for which the two definitions coincide. This result subsumes the cases considered in [BJ99], as well as those involving the equivalence of consequence relations on sequents.

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The modal logic of minimal topological spaces

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In this paper we axiomatically define the modal logic MT, the modal analog of Smetanich intermediate logic. We prove that MT is complete with respect to one-step, weak transitive frames. Moreover we show that MT has the finite model property. We also show that MT is the logic of minimal topological spaces, where $\Diamond$ modality is interpreted as derived set operator.

The axioms of modal system MT are:

1) all classical axioms,
2)$\Box(p \to q) \to (\Box p \to \Box q)$,
3)$\Box p \wedge p \to \Box\Box p$,
4)$(p \wedge \Diamond(q \wedge \Box\neg p)) \to \Box(q \vee \Diamond q)$.

The rules of inference are: modus-ponens, substitution and necessitation.

Our approach is algebraic. We investigate the variety $\mathbb{MT}$ of $MT-$algebras.

Definition 1 We say that a pair (B, d) is $MT-$algebra if B is a boolean algebra and $d : B \to B$ a unary operator on B with the following properties:1) $d(0) = 0$,

2) $d(p \vee q) = d(p) \vee d(q)$,
3) $dd(p) \leq d(p) \vee p$,
4) $p \wedge d(q - d(p)) \leq \tau(q \vee d(q))$. where $\tau p = -d(-p)$

$MT-$algebras form algebraic semantics of the modal system MT. So we can switch from logical study of the system MT to algebraic study of the variate $\mathbb{MT}$.

Theorem 2 The variety $\mathbb{MT}$ is locally finite.

An immediate consequence of theorem 2 is that $\mathbb{MT}$ is finitely generated. So we reduce the study of the variety $\mathbb{MT}$ to the study of it's finite members.

Now we introduce one-step, weak transitive relational structures, which turn out to be dual objects of $MT-$algebras in the finite case. First we give the definition:

Definition 3 We say that $R \subseteq W \times W$ is weak transitive if
$(\forall x, y, z)((xRy \wedge yRz \wedge x \neq z) \Rightarrow (xRz))$.

Definition 4 We say that $R \subseteq W \times W$ is one-step relation if
1)$(\forall x, y, z)((xRy \wedge yRz) \Rightarrow (yRx \vee zRy))$
2)$(\forall x, y, z)((xRy \wedge xRz) \Rightarrow (yRz \vee zRy \vee x = z))$.

Here formula 1) says that frame (W, R) has height less or equal to two. And 2) says that branches are not allowed.

Theorem 5 below links up algebraic semantics of MT with it's Kripke semantics.

Theorem 5 There is a one-to-one correspondence between finite $MT-$algebras and finite weak transitive, one-step relational structures.

Corollary 6 MT has the finite model property

As $\mathbb{MT}$ is finitely generated, only finite subdirectly irreducible algebras are enough to generate it. By the above mentioned correspondence rooted (point-generated) frames of MT correspond to subdirectly irreducible algebras.

We can think of rooted frame (W, R) as of two clusters W_1, W_2 where:

$$w \in W_1 \wedge w' \in W_2 \Rightarrow wRw'$$

Next theorem tells us that only those subdirectly irreducible algebras are enough to generate variety $\mathbb{MT}$, for which the corresponding frames are irreflexive .

Theorem 7 Every finite rooted, weak transitive, one-step frame is a p-morphic image of a rooted, weak transitive, one-step, irreflexive frame.

Theorem 8 There is a one-to-one correspondence between finite rooted weak transitive one-step, irreflexive frames and finite topological spaces with minimal topology.

Corollary 9 MT is the modal logic of minimal topological spaces.

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Axiomatizability by sentences of the form $\forall \exists ! \bigwedge p = q$

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Given an equational class V, several important subclasses of V can be defined by sentences of the form $\forall \exists ! \bigwedge p = q$. For example, if V is the class of all semigroups with unit, then the subclass of all groups can be defined in this way, and if V is the class of all bounded distributive lattices, then the subclass of all Boolean lattices is also axiomatizable in this way. We consider the following general problem:

Problem: Given a variety V characterize the subclasses of V which can be axiomatized by a set of sentences of the form $\forall \exists ! \bigwedge p = q$.

We will present a solution to this problem for certain varieties of distributive lattice expansions.

The (internal) logical system of residuated lattices, its fragments and their involutive extensions

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In the literature there are several ways to consider logical systems: sets of formulas, consequence relations between sets of formulas and formulas (called deductive systems in [3]), consequence relations between sets of sequents and sequents (called Gentzen systems in [14]), etc. In this contribution we restrict our interest to Gentzen systems and to the natural generalization of deductive systems for logical systems lacking some structural rules, i.e., consequence relations between finite sequences of formulas and formulas (what it is analogous to consider sets of sequents). We will use the word deductive system for this generalization. The purpose of this contribution is to analyze the fragments of the Gentzen systems determined by the Gentzen calculi, including the cut, $\mathbf{FL_{ew}}$ [8,9] and its involutive extension (we take as primitive connectives $\vee, \wedge, *, 0, 1, \rightarrow$ and $\neg$) and to analyze some deductive systems associated with them. We focus in two deductive systems: those which in the terminology of [2] are called the external deductive system associated with $\mathbf{FL_{ew}}$ and the internal deductive system associated with $\mathbf{FL_{ew}}$ (see [14,4]).

It is well known that the Gentzen system determined by $\mathbf{FL_{ew}}$ is algebraizable [1]. In the contribution we discuss what happens with its fragments. The most interesting cases are the fragment given by $*$ and the fragment given by $*$ together with $\neg$. Although these two fragments are not algebraizable it results that they become algebraizable by using the new sense described by Pigozzi in [12] being their counterparts, respectively, the partially ordered variety of monoids and the partially ordered quasivariety of pseudocomplemented (with respect to the fusion) monoids. We stress that for the case of the involutive extension the fragment given by $*$ together with $\neg$ is already algebraizable in the normal sense being the equivalent quasi-variety semantics the quasi-variety of Grišin algebras [5].

For the case of the external deductive system associated with $\mathbf{FL_{ew}}$ all structural rules hold, so what we obtain is also a deductive system in the sense of [3]. It is well known that this system (also known as Monoidal Logic [7], H_{BCK} [11], and $IPC^*\backslash c$ [1]) is algebraizable being its equivalent variety semantics the variety of commutative integral bounded residuated lattices [10]. In this contribution we prove that this deductive system cannot be axiomatized by using Tarski-style conditions [15], i.e., by using conditions imposed on the consequence operator which involves exactly one connective. This result contrasts with the result of Grzegorczyk [6] stating that intuitionistic logic (i.e., the external deductive system associated with $\mathbf{FL_{ewc}}$) is axiomatized by using a finite number of Tarski-style conditions.

Besides the previous deductive system we consider the internal deductive system associated with $\mathbf{FL_{ew}}$, which essentially corresponds to the sequents that are derivable in $\mathbf{FL_{ew}}$. This system trivially has a Tarski-style axiomatization, simply write as Tarski-style conditions each one of the rules in the calculus $\mathbf{FL_{ew}}$; and this is also true for all its fragments (cf. [6,13]). We check that the internal system is a proper subsystem of the external one and that it is properly substructural (because it does not satisfy the contraction rule). In this occasion it does not have any sense to discuss algebraizability because contraction does not hold, but at least it is well known an algebraic completeness theorem.

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Metrics on universes of propositions determining continuous t-norms

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In order to interprete the conjunction in logics whose formulas are modelled by values from the real unit interval, usually t-norms are used. It is obvious that the axioms of t-norms reflect properties of the conjunction in two-valued logic: associativity, commutativity, neutrality of the truth constant 1, and monotonicity. It is, however, less obvious why these axioms are found adequate in the intended context of multivalued reasoning. For they neither reflect some clearly defined intuition, nor are they modelled upon some specific mathematical structure. The usage of a specific t-norm is, consequently, hard to justify, even if the t-norm is not the result of an infinite ordinal sum construction.

While we do not intend here to enter into the difficult discussion about the proper intuition connected to fuzzy logics, we propose a motivation for the most prominent fuzzy connective on the base of a mathematical structure. Namely, the context of similarity relations (see e.g. [KMP]) gives a justification for specific t-norms. We shall outline this idea shortly, which is inspired by the article [Höh] of U. Höhle.

Let $\mathcal{P}$ be a boolean algebra, and think of the elements of $\mathcal{P}$ as propositions which arise simultaneously in some context. Furthermore, let d be a metric on $\mathcal{P}$ bounded from above by 1, and assume that $d(a,b)$ tells us to what extent the two propositions $a \in \mathcal{P}$ and $b \in \mathcal{P}$ differ from each other.

Certainly, d gives rise to the unsharp property that two elements are similar; simply set $p(a,b) = 1 - d(a,b),\ \ a,b \in \mathcal{P}$. Moreover, w.r.t. a t-norm $\odot\colon\ [0,1]^2 \to [0,1]$, p is a *similarity relation* if (i) $p(a,b) = 1$ iff $a = b$, (ii) $p(a,b) = p(b,a)$, and (iii) $p(a,b) \odot p(b,c) \leq p(a,c)$ for

$a, b, c \in \mathcal{P}$; see e.g. [KMP]. (In [KMP], the notion "T-equality" is used, in [Höh] "separated M-valued equality".)

It is easy to check that $p = 1 - d$ is always a similarity relation w.r.t. the Łukasiewicz t-norm. But p might be a similarity relation w.r.t. other t-norms as well; if, for instance, d is an ultrametric, E is a similarity relation also w.r.t. the minimum t-norm. Cf. the mentioned paper [Höh].

Our concern is to associate to d a single, canonical t-norm, and we found the following definition most natural. We say that the metric d on $\mathcal{P}$ *determines* the t-norm $\odot$ if $E = 1 - d$ is a similarity relation w.r.t. $\odot$ and if among all such t-norms $\odot$ is, up to isomorphism, the weakest (i.e. largest) one.

It may be the case that d does not determine a t-norm. But there are quite natural sufficient conditions for d to determine a t-norm. It is only at this point the boolean algebra structure of $\mathcal{P}$ comes into play. d is requested to be associated to a submeasure μ on $\mathcal{P}$ (cf. e.g. [Fre]), which is inner regular and fulfils a certain homogeneity condition.

On the other hand, we get in the indicated way all continuous t-norms. Let $\odot$ be a continuous t-norm. Then there is a boolean algebra $\mathcal{P}$ and a submeasure μ on $\mathcal{P}$ such that the metric d_μ associated with μ determines $\odot$. The boolean algebra may be chosen to be the algebra of subsets of a countable set.

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Automorphisms of powers of linearly ordered sets

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Let $\mathbb{I} = ([0,1], \vee, \wedge, 0, 1)$, where $\vee$ and $\wedge$ are max and min, respectively. This algebra is the basic building block of fuzzy set theory and logic. Likewise, the algebra $\mathbb{I}^{[2]} = ([0,1]^{[2]}, \vee, \wedge, 0, 1)$, where $[0,1]^{[2]} = \{(a,b) : a, b \in [0,1],\ a \leq b\}$, $\vee$, $\wedge$ are given coordinate-wise, and 0 and 1 are the bounds on $[0,1]^{[2]}$, is the relevant algebra for interval-valued fuzzy set theory and logic. A good share of the theory of fuzzy sets is concerned with endowing $\mathbb{I}$ with additional structure such as t-norms, t-conorms, and negations other than the usual max and min operations, and usual negation.

In addressing the situation for $\mathbb{I}^{[2]}$, a basic problem was deciding on the appropriate definitions [1]. For example, what should a t-norm on $\mathbb{I}^{[2]}$ be? In this regard, knowing the automorphisms of $\mathbb{I}^{[2]}$ was fundamental for representation theorems for norms, conorms, and negations. The basic theorem is that any automorphism of $\mathbb{I}^{[2]}$ is of the form $(a,b) \to (\varphi(a), \varphi(b))$ where φ is an automorphism of $\mathbb{I}$ [1].

Our initial motivation was to extend this theorem from $\mathbb{I}^{[2]}$ to $\mathbb{I}^{[n]}$ for any positive integer n. But more generally, we will replace $[0,1]$ by a bounded linearly ordered set S, and consider the automorphisms of both $S^{[n]}$ and S^n. The result for $S^{[n]}$ is the same as for $\mathbb{I}^{[2]}$ except for an anomaly for certain combinations of finite S and integers n.

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The word problem for involutive residuated lattices

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An involutive residuated lattices is an algebraic structure $\langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$, where $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and d is a cyclic dualizing element, i.e. for all $x \in L$, $d/x = x\backslash d$ and $d/(x\backslash d) = x = (d/x)\backslash d$. An algebraic structure $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is said to be a residuated lattice, if $\langle L, \wedge, \vee\rangle$ is a lattice, $\langle L, \cdot, e\rangle$ is a monoid and, for all $x, y, z \in L$,

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

Examples of involutive residuated lattices are Boolean algebras, MV-algebras, lattice-ordered groups and certain reducts of relation algebras.

It will be shown that the word problem for involutive residuated lattices and for finite involutive residuated lattices is undecidable. The proof relies on the fact that the monoid reduct of a group can be embedded as a monoid into an involutive residuated lattice. Thus, results about groups by P. S. Novikov and about finite groups by A. M. Slobodskoi can be used.

Similarity, metric, topology

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This paper presents a logic $\mathcal{SL}$ intended for reasoning about *similarity*. $\mathcal{SL}$ extends the logic of metric and topology from [2] with the *closer operator* $\leftleftarrows$ using which one can define concepts like *reddish* as $\{red\} \leftleftarrows \{green, \dots, black\}$: 'a colour is reddish iff it is more similar (with respect to the RGB, HSL or some other explicit or implicit colour model) to the prototypical colour *red* than to the prototypical colours *green*, ..., *black*.'

Syntax and semantics. The *terms* C of $\mathcal{SL}$ are defined by taking

$$C \quad ::= \quad \{\ell\} \quad | \quad A \quad | \quad \neg C \quad | \quad C_1 \sqcap C_2 \quad | \quad C_1 \leftleftarrows C_2 \quad | \quad \exists^{<a} C \quad | \quad \exists^{\leq a} C,$$

where the ℓ are individual constants, the A are atomic terms, and $a \in \mathbb{Q}^+$. $\mathcal{SL}$-terms C are interpreted as subsets $C^{\mathfrak{I}} \subseteq \Delta^{\mathfrak{I}}$ in models of the form

$$\mathfrak{I} \quad = \quad \langle \Delta^{\mathfrak{I}}, d^{\mathfrak{I}}, \ell_1^{\mathfrak{I}}, \dots, A_1^{\mathfrak{I}}, \dots \rangle,$$

where $\mathfrak{D} = \langle \Delta^{\mathfrak{I}}, d^{\mathfrak{I}} \rangle$ is a *distance space* in the sense that $d^{\mathfrak{I}}$ is a map from $\Delta^{\mathfrak{I}} \times \Delta^{\mathfrak{I}}$ to $\mathbb{R}^+$ such that, for all $x, y \in \Delta^{\mathfrak{I}}$, $d^{\mathfrak{I}}(x, y) = 0$ iff $x = y$. If $d^{\mathfrak{I}}$ is symmetric and satisfies the triangle inequality then $\mathfrak{D}$ is clearly a *metric space*; we say then that $\mathfrak{I}$ is a *metric model*. Now, $A_i^{\mathfrak{I}} \subseteq \Delta^{\mathfrak{I}}$, the interpretation of the Boolean operators in $\mathfrak{I}$ is as usual, $\{\ell\}^{\mathfrak{I}} = \{\ell^{\mathfrak{I}}\}$, and

$$\begin{aligned}(C_1 \leftleftharpoons C_2)^{\mathfrak{I}} &= \{x \in \Delta^{\mathfrak{I}} \mid d^{\mathfrak{I}}(x, C_1^{\mathfrak{I}}) < d^{\mathfrak{I}}(x, C_2^{\mathfrak{I}})\} \\ (\exists^{<a} C)^{\mathfrak{I}} &= \{x \in \Delta^{\mathfrak{I}} \mid (\exists y \in C^{\mathfrak{I}})\ d^{\mathfrak{I}}(x, y) < a\} \\ (\exists^{\leq a} C)^{\mathfrak{I}} &= \{x \in \Delta^{\mathfrak{I}} \mid (\exists y \in C^{\mathfrak{I}})\ d^{\mathfrak{I}}(x, y) \leq a\}\end{aligned}$$

In other words, $C_1 \leftleftharpoons C_2$ is the set containing those objects of $\Delta^{\mathfrak{I}}$ that are 'more similar' or 'closer' to C_1 than to C_2. $\exists^{<a} C$ is the (open) a-neighbourhood of C in $\Delta^{\mathfrak{I}}$.

Example. Observe that if $\mathfrak{D} = \langle \Delta^{\mathfrak{I}}, d^{\mathfrak{I}} \rangle$ is a metric space then the operator $\Box$ defined by taking $\Box C = (\top \leftleftharpoons \neg C)$ is the *interior* operator in the induced topology, and $\exists C = (C \leftleftharpoons \bot)$ defines the existential modality (i.e., the dual of the universal modality $\forall$) over $\Delta^{\mathfrak{I}}$. Thus, our logic contains contains full $\mathbf{S4}_u$, and so can be used for spatial representation and reasoning.

Besides the class $\mathbf{M}$ of all metric models we will be considering its three subclasses:

- the class $\mathbf{D}$ of *discrete* models where, for every $x \in \Delta^{\mathfrak{I}}$ and every nonempty $Y \subseteq \Delta^{\mathfrak{I}}$, there is $y \in Y$ such that $d^{\mathfrak{I}}(x, Y) = d^{\mathfrak{I}}(x, y)$,
- the class $\mathbf{SD}$ of '*superdiscrete*' models where, for all nonempty $X, Y \subseteq \Delta^{\mathfrak{I}}$, there are $x \in X$ and $y \in Y$ such that $d^{\mathfrak{I}}(X, Y) = d^{\mathfrak{I}}(x, y)$, and
- the class $\mathbf{SR}$ of models over superdiscrete subspaces of $\mathbb{R}^2$.

Given a class $\mathbf{C} \subseteq \mathbf{M}$, denote by $\mathsf{Log}\,\mathbf{C}$ ($\mathsf{Log}_{\leftleftharpoons}\mathbf{C}$) the set of all $\mathcal{SL}$-terms (containing neither nominals nor operators of the form $\exists^{<a}$ and $\exists^{\leq a}$) C that are *valid* in all models $\mathfrak{I} \in \mathbf{C}$ in the sense that $C^{\mathfrak{I}} = \Delta^{\mathfrak{I}}$.

The main result of this paper is the following theorem:

(i) $\mathsf{Log}\,\mathbf{SD}$ *is ExpTime-complete and enjoys the finite model property.*

(ii) $\mathsf{Log}\,\mathbf{D}$ *is ExpTime-complete and does not have the finite model property.*

(iii) $\mathsf{Log}_{\leftleftharpoons}\mathbf{M}$ *is decidable, ExpTime-hard and does not have the finite model property.*

(iv) $\mathsf{Log}_{\leftleftharpoons}\mathbf{SR}$ *is undecidable.*

The lack of the finite model property in (ii) can be established using the term $A \sqcap \forall[(A \to (B \leftleftharpoons C)) \sqcap (B \to (C \leftleftharpoons A)) \sqcap (C \to (A \leftleftharpoons B))]$.

The ExpTime-completeness results in (i) and (ii) also hold for the classes of models based on distance spaces with or without symmetry or the triangle inequality (note, however, that some of these logics coincide, e.g., the classes of discrete and superdiscrete models over arbitrary distance spaces define the same logic).

The lower ExpTime bound is established by reduction of the global consequence relation for the modal logic $\mathbf{K}$ which is known to be ExpTime-complete (which can actually be done in the sublanguages of $\mathcal{SL}$ with only $\leftleftharpoons$ or only $\exists^{<1}$). The upper bound is proved by a rather involved reduction to the emptiness problem for tree automata with one complemented pair [1].

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Transparent unification for equivalential algebras and some related varieties

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Let $\mathbb{T}$ be the set of terms of a fixed similarity type generated by a denumerably infinite set of variables X. Then $\mathbb{T}$ can be viewed as the absolutely free algebra of the considered type freely generated by X. Endomorphisms of $\mathbb{T}$ that are identical on all but finitely many variables are called *substitutions*. Substitutions are preordered by so called *subsumption preorder*: $\varepsilon \leq \varepsilon' :\Leftrightarrow \lambda\varepsilon = \varepsilon'$, for some substitution λ. Let $\mathbb{K}$ be a class of algebras similar to $\mathbb{T}$ and $p, q \in \mathbb{T}$. A substitution ε such that $\mathbb{K} \models \varepsilon(p) \approx \varepsilon(q)$ is called a $\mathbb{K}$-*unifier* of p and q and $\mathbf{U}_{\mathbb{K}}(p,q)$ stands for the set of all $\mathbb{K}$-unifiers of p and q. We say that p, q are $\mathbb{K}$-*unifiable* if $\mathbf{U}_{\mathbb{K}}(p,q) \neq \emptyset$. If every non-empty set of the form $\mathbf{U}_{\mathbb{K}}(p,q)$ has a least element w.r.t. subsumption preorder (*a most general unifier*), the class $\mathbb{K}$ is called *unitary*. Unitarity matters for so called rules of resolution widely used in automated theorem provers.

By a *transparent* $\mathbb{K}$-*unifier* of p and q we mean a unifier $\varepsilon \in \mathbf{U}_{\mathbb{K}}(p,q)$ such that for every variable x, $\mathbb{K} \models p \approx q \Rightarrow x \approx \varepsilon(x)$. If a transparent $\mathbb{K}$-unifier of p and q exists whenever $\mathbf{U}_{\mathbb{K}}(p,q) \neq \emptyset$ then we say that $\mathbb{K}$ *has transparent unifiers*. Note the following observations:

(i) Every transparent unifier is most general;

(ii) Transparent $\mathbb{K}$-unifiers are the same as transparent $q\mathbb{K}$-unifiers, where $q\mathbb{K}$ is the quasivariety generated by $\mathbb{K}$;

(iii) If $\mathbb{L} \subseteq \mathbb{K}$ then transparent $\mathbb{L}$-unifiers include all transparent $\mathbb{K}$-unifiers and it may happen that $\mathbf{U}_{\mathbb{L}}(p,q)$ is non-empty but $\mathbf{U}_{\mathbb{K}}(p,q) = \emptyset$;

Note that the class of all algebras of given similarity type always is unitary but if the considered type is non-trivial (i.e. some operations have positive arity) then such class does not have transparent unifiers. Thus, the property of having transparent unifiers is strictly stronger then unitarity. The advantage of this property is the fact that in most cases it easily carries over from a quasivariety to all its subclasses which must therefore be unitary.

Let $\mathbb{H}$ be the variety of Heyting algebras (see [1]) with basic operations: $\vee, \wedge, \to, \mathbf{0}$ (*join, meet, relative pseudocomplementation, zero*). The operations of *pseudocomplementation* ($\neg$), *equivalence* ($\leftrightarrow$) and *unit* ($\mathbf{1}$) can be introduced as term-operations by the usual definitions: $\neg a := a \to \mathbf{0}$, $a \leftrightarrow b := (a \to b) \wedge (b \to a)$ and $\mathbf{1} := \neg\mathbf{0}$. Given a set τ of term-operations of Heyting algebras, by $\mathbb{H}_\tau$ we denote the variety of type τ generated by τ-reducts of members of $\mathbb{H}$. Thus: $\mathbb{H}_{\{\wedge,\to\}}, \mathbb{H}_{\{\to\}}$ and $\mathbb{H}_{\{\leftrightarrow\}}$ are varieties of *Brouwerian semilattices, Hilbert algebras* and *equivalential algebras* respectively. The first two are well known. Equivalential algebras are introduced in [4]. They are locally-finite, congruence-permutable but – unlike the first two – they are not congruence-distributive and do not have congruence extension property (see [3]). There are $2^{\aleph_0}$ distinct varieties of equivalential algebras (see [5]). General importance of equivalential algebras has become apparent during the study of so called *Fregean varieties* (see [2]). An ingredient of Fregeanity is the following condition of congruence-orderability: a class $\mathbb{K}$ with a distinguished constant $\mathbf{1}$ is *congruence-orderable* if for every $a, b \in \mathfrak{A} \in \mathbb{K}$, $\Theta_{\mathfrak{A}}(\mathbf{1}, a) = \Theta_{\mathfrak{A}}(\mathbf{1}, b)$ implies $a = b$. Let us recall:

Theorem 1(see [2]) *Every congruence-orderable variety with permutable congruences has a binary term* $\mathsf{e}(x,y)$ *turning its members into equivalential algebras possibly expanded with some* e-*compatible operations.*

The main result here, is the following:

Theorem 2 *All varieties of equivalential algebras have transparent unifiers.*

As to other reducts of Heyting algebras let us note:

Theorem 3 (i) *All subvarieties of* $\mathbb{H}_{\{\to\}}, \mathbb{H}_{\{\wedge,\to\}}$ *and* $\mathbb{H}_{\{\wedge,\to,\neg\}}$ *have transparent unifiers.* (ii) *If* $\mathbb{K}$ *is a non-trivial subvariety of* $\mathbb{H}_\tau$*, where* $\tau \in \{\{\to,\neg\},\{\wedge,\neg\}\}$*, then* $\mathbb{K}$ *is unitary iff it is term-equivalent to Boolean algebras.*

The fact that among varieties $\mathbb{H}_{\{\to\}}, \mathbb{H}_{\{\to,\neg\}}, \mathbb{H}_{\{\wedge,\to,\neg\}}$ only the first and the last are unitary seems to be rather intriguing and thus, we pose:

Problem *Find algebraic properties of varieties which are responsible for the property of having transparent unifiers.*

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On the representation of Łukasiewicz-Moisil algebras by intuitionistic fuzzy sets

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The aim of the paper is the elaboration of a representation theory of involutive θ-valued Łukasiewicz-Moisil algebra, the concept of intuitionistic fuzzy sets playing the role that the notion of field of sets plays for the representation of Boolean algebras. This theory provides both a semantic interpretation of Łukasiewicz many valued logic and logical basis to intuitionistic fuzzy set theory.

In 1941 [4] Moisil introduced the notion of Łukasiewicz-Moisil algebra under the name Łukasiewician algebras, as an algebraic counterpart of the Łukasiewicz many-valued logics. The theory of Łukasiewicz-Moisil algebras has developed both as a tool for studying certain non-classical logics and as an algebraic theory having its own interest; besides, it is now considered one of the fundamental formalizations of fuzzy logic. The reader is referred e.g. to [10], [11].

Not long after Zadeh published his important paper $\ll$ fuzzy sets $\gg$ [14] Atanassov published the paper $\ll$ intuitionistic fuzzy sets $\gg$ [11]. The concept of intuitionistic fuzzy sets is a generalization of the concept of fuzzy sets.

In this paper we want to study, in a general way the relationships between the involutive θ-valued Łukasiewicz-Moisil algebra and algebras of intuitionistic fuzzy sets.

In Section 1, we will remind the reader what an involutive θ-valued Łukasiewicz-Moisil algebras is [4], we recall the definition of intuitionistic fuzzy sets [11] and we will define the intuitionistic weak α-cut and intuitionistic strong α-cut. Section 2 describes our main results.

On generalized Ockham-Nelson algebras with a quantifier

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In this note, a new equational class of algebras, denominated as generalized Ockham–Nelson algebras with a quantifier (or gQON–algebras) is defined and investigated. These algebras are a common abstraction of both generalized N–lattices, introduced by A. V. Figallo ([1]) in 1990, and OQ-algebras which we began to study in [2].

Our main interest is to describe the congruence lattice for these algebras. In order to do this, a duality between gQON–algebras and Ockham spaces with additional conditions is determined. Next, the notion of $gN-$subset of the associated space of a gQON–algebra is introduced; thus becoming a useful tool in characterizing the congruences of these algebras. In fact, an anti-isomorphism between the congruence lattice of a gQON–algebra and the lattice of $gN-$subsets which verify an additional property is obtained. Finally, it is worth mentioning that monadic generalized Ockham–Nelson algebras are also introduced and some results on them are determined.

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