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Expanding FL$_{ew}$ with a Boolean connective

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Abstract We expand FL$_{ew}$ with a unary connective whose algebraic counterpart is the operation that gives the greatest complemented element below a given argument. We prove that the expanded logic is conservative and has the Finite Model Property. We also prove that the corresponding expansion of the class of residuated lattices is an equational class.

1 Introduction

In this paper we study the expansion of the substructural logic FL$_{ew}$, i.e. Full Lambek calculus with exchange and weakening, with a unary connective B whose intended algebraic semantics is as follows: given a bounded integral commutative residuated lattice (or residuated lattice for short) $\mathbf{A}$, Ba is the maximum, if it exists, of the Boolean elements of the universe A below a, which we call *the greatest Boolean below* a, that is,

$$Ba = \max\{b \in A : b \leq a \text{ and } b \text{ is Boolean}\}.$$

In fact, this operator is similar to the so-called Baaz-Monteiro Δ operator, very often used in the context of mathematical fuzzy logic systems that are semilinear expansions of MTL. Baaz [2] studied it in connection with Gödel logic while Hájek [10] investigated Δ in BL logics in general, see also [8, Chapter 2] for a more general perspective. Indeed, in such a context of semilinear logics, i.e. logics that are complete with respect to a class of linearly ordered algebras, the semantics of Δ is exactly the above one for B: in a linearly-ordered MTL-algebra, $\Delta a = 1$ if $a = 1$, and $\Delta a = 0$ otherwise, since the only Boolean elements in a chain are 1 and 0; moreover, from a logical point of view, $\Delta\varphi$ represents the weakest Boolean proposition implying φ.

The operator B can be also related to the join-complement operation D, also known as dual intuitionistic negation, already considered by Skolem [19] in the context of lattices with relative meet-complement, and later independently studied by e.g. Moisil [11] and Rauszer [15] as well, the latter in the context of expansions of Heyting algebras. It turns out that the operation $\neg D$ and its iterations where $\neg$ is the residual negation, has also very similar properties to B, and in some classes of residuated lattices they even coincide.

In this paper we study the operator B in the context of FL$_{ew}$ and axiomatize it. We show that the usual axiomatics of the Δ operator is actually too strong to capture the above intended semantics. In fact, the axiom

$$\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$$

is not sound for B over FL$_{ew}$ any longer. Thus, B is a weaker operator than Δ. However, as we will see, B keeps most of the properties of Δ. In particular, the expansion of FL$_{ew}$ with B is conservative, its corresponding class of algebras is an equational class, and has the same kind of deduction theorem as Δ. Also, B may also be interesting as $\neg B$ has a paraconsistent behaviour. On the negative side, the expansion of a semilinear extension of FL$_{ew}$ with B needs not to remain semilinear.

The paper is structured as follows. In Section 2 we overview well-known facts about residuated lattices and its Boolean elements, as well as basic facts about the

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logic FL_{ew}. Sections 3 and 4 contain an algebraic study of the operator B. In particular, in Section 3 we study basic properties and show, among other things, that the class $\mathbb{RL}^B$ of residuated lattices expanded with B is an equational class and state the modalities, while in Section 4 we compare B with the mentioned Δ and with an operation using the join-complement D. Finally, in Section 5 we focus on logical aspects, introducing the logic FL^B_{we}, i.e. the expansion of FL_{we} with the operator B, and show that is a conservative expansion and has the Finite Model Property, and hence it is decidable. We conclude with some remarks and open problems.

We give appropriate references. However, the paper is self-contained.

2 Preliminaries

2.1 Residuated lattices and Boolean elements

In this section we recall some properties of residuated lattices as well as of their Boolean elements that we will use in the following sections.

Following [9], a bounded, integral, commutative residuated lattice, or residuated lattice for short, is an algebra $\mathbf{A}=(A;\wedge,\vee,\cdot,\to,0,1)$ of type $(2,2,2,2,0,0)$ such that:

- $(A;\wedge,\vee,0,1)$ is a bounded lattice with $0\le a\le 1$, for all $a\in A$,
- $(A;\cdot,1)$ is a commutative monoid (i.e. $\cdot$ is commutative, associative, with unit 1), and
- $\to$ is the residuum of $\cdot$, i.e., $a\cdot b\le c$ iff $a\le b\to c$, for all $a,b,c\in A$,

where $\le$ is the order given by the lattice structure. A negation operator is defined as $\neg x = x\to 0$.

The class of residuated lattices will be denoted by $\mathbb{RL}$. It is well known that $\mathbb{RL}$ is an equational class and that it constitutes the algebraic semantics of the substructural logic FL_{ew} (see Section 2.2).

Example 1 In what follows we will have occasion to refer several times to the residuated lattice structure defined on the five-element lattice of Figure 1 by taking $\cdot=\wedge$ and $\to$ its residuum. With these operations, it actually becomes a five element Gödel algebra, that is, a residuated lattice with $\cdot$ being idempotent and satisfying the pre-linearity law $(a\to b)\vee(b\to a)=1$.

We omit the proof of the following well-known facts, see e.g. [9]

Lemma 1 *Let $\mathbf{A}\in\mathbb{RL}$. For any $a,b,c,d\in A$, the following properties hold:*

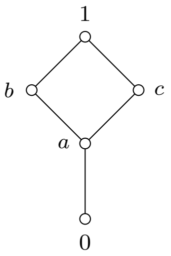

Fig. 1 A five element Gödel algebra

(i) *if $a\vee b=1$, $a\le c$, and $b\le c$, then $c=1$,*
(ii) *if $a\vee b=1$, $a\cdot c\le d$, and $b\cdot c\le d$, then $c\le d$,*
(iii) *if $a\le b$, then $\neg b\le\neg a$,*
(iv) *$a\wedge\neg b\le\neg(a\wedge b)$,*
(v) *$a\cdot\neg b\le\neg(a\to b)$,*
(vi) *if $a\vee b=1$, then $\neg a\le b$,*
(vii) *$a\le\neg\neg a$.*

Special elements in a residuated lattice are those that behave as elements in a Boolean algebra.

Definition 1 *An element a of a residuated lattice $\mathbf{A}$ is called* Boolean *or* complemented *iff there is an element $b\in A$ such that $a\wedge b=0$ and $a\vee b=1$.*

In the rest of this section we state several properties of Boolean elements that will be useful for the remaining parts of the paper. Even if most of them are folklore, we include proofs for all of them for the sake of being self-contained.

An equivalent and simpler condition for an element to be Boolean is the following.

Lemma 2 *An element a in the universe of a residuated lattice is Boolean iff $a\vee\neg a=1$.*

Proof $\Rightarrow$) Suppose there is an element b such that $a\wedge b=0$ and $a\vee b=1$. First, using that $a\wedge b=0$ and $a\cdot b\le a\wedge b$, we have that $a\cdot b=0$. So, $b\le a\to 0$, i.e. $b\le\neg a$. Secondly, we have that $\neg a=\neg a\cdot 1=\neg a\cdot(a\vee b)=(\neg a\cdot a)\vee(\neg a\cdot b)=\neg a\cdot b$. So, $\neg a=\neg a\cdot b$. As $\neg a\cdot b\le b$, it follows that $\neg a\le b$. So, $b=\neg a$. As we have that $a\vee b=1$, it follows that $a\vee\neg a=1$.
$\Leftarrow$) By hypothesis, we have (i) $a\vee\neg a=1$. It is enough to see that $a\wedge\neg a=0$. As $a\cdot\neg a=0$, it is enough to prove that $a\wedge\neg a\le a\cdot\neg a$. We have that $a\wedge\neg a\le\neg a$. So, by monotonicity of $\cdot$, we have (ii) $a\cdot(a\wedge\neg a)\le a\cdot\neg a$. We also have that $a\wedge\neg a\le a$. So, again by monotonicity of $\cdot$, we have $\neg a\cdot(a\wedge\neg a)\le\neg a\cdot a=a\cdot\neg a$. So, it follows (iii) $\neg a\cdot(a\wedge\neg a)\le a\cdot\neg a$. Now, using Lemma 1(ii) with (i), (ii), and (iii), it follows that $a\wedge\neg a\le a\cdot\neg a$, as desired. □

Proposition 1 *Let $\mathbf{A}\in\mathbb{RL}$ and let a be a Boolean element of A. Then, for all $b,c,d\in A$ the following properties hold:*

(i) $a \wedge b = a \cdot b$,
(ii) $a \cdot a = a$,
(iii) $a \wedge \neg a = 0$,
(iv) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
(v) $\neg\neg a = a$,
(vi) $a \to b = \neg a \vee b$,
(vii) $0 = \neg(a \vee \neg a)$,
(viii) *if* $a \leq b \vee c$, $a \wedge b \leq d$, $a \wedge c \leq d$, *then* $a \leq d$,
(ix) *if* $b \vee c = 1$, $a \wedge b \leq d$, $a \wedge c \leq d$, *then* $a \leq d$,
(x) *if* $a \wedge b \leq c$, *then* $a \wedge \neg c \leq \neg b$,
(xi) *if* $a \vee \neg b = 1$, *then* $b \leq a$.

Proof (i) Suppose that (i) $a \vee \neg a = 1$. It is enough to see that $a \wedge b \leq a \cdot b$. We have that $a \wedge b \leq b$. So, by monotonicity of $\cdot$, we have (ii) $a \cdot (a \wedge b) \leq a \cdot b$. We also have that $a \wedge b \leq a$. So, again by monotonicity of $\cdot$, we have $\neg a \cdot (a \wedge b) \leq \neg a \cdot a = 0$. So, it follows (iii) $\neg a \cdot (a \wedge b) \leq a \cdot b$. Now, using Lemma 1(ii) with (i), (ii), and (iii), it follows that $a \wedge b \leq a \cdot b$.

(ii) Using Part (i), $a \wedge a = a \cdot a$. However, $a \wedge a = a$. So, $a \cdot a = a$.

(iii) Using Part(i), it follows that $a \wedge \neg a = a \cdot \neg a = 0$. So, $a \wedge \neg a = 0$.

(iv) In a residuated lattice it holds that $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$. Let a be Boolean. Then, using Part (i) three times, it follows that $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

(v) Suppose that (i) $a \vee \neg a = 1$. It is enough to see that $\neg\neg a \leq a$. We have that (ii) $a \cdot \neg\neg a \leq a$. Also, (iii) $\neg a \cdot \neg\neg a \leq a$, as $\neg a \cdot \neg\neg a = 0$. So, using Lemma 1(ii) with (i), (ii), and (iii), $\neg\neg a \leq a$.

(vi) It is enough to prove (i) $(\neg a \vee b) \cdot a \leq b$ and (ii) if $x \cdot a \leq b$, then $x \leq \neg a \vee b$. To see (i), note that $\neg a \cdot a \leq b$ and $b \cdot a \leq b$, whence $(\neg a \cdot a) \vee (b \cdot a) \leq b$. So, using distributivity of $\cdot$ relative to $\vee$, we get (i). To see (ii), suppose $x \cdot a \leq b$. Then, $x \leq a \to b$. In order to get $x \leq \neg a \vee b$, it is enough to derive (iii) $a \to b \leq \neg a \vee b$ and use transitivity of $\leq$. To get (iii), let us use Lemma 1(ii). As a is Boolean, we have $a \vee \neg a = 1$. Now, $a \cdot (a \to b) \leq b \leq \neg a \vee b$. Also, $\neg a \cdot (a \to b) \leq \neg a \leq \neg a \vee b$. So, using Lemma 1(ii), we get (iii).

(vii) As we have $\neg c \leq a \vee \neg a$, for any $c \in A$, then, using Lemma 1(iii) and Part (iv), we get $\neg(a \vee \neg a) \leq \neg\neg c \leq c$.

(viii) As $a \leq b \vee c$, we have $a \leq a \wedge (b \vee c)$. Now, using Part (iv), it follows that $a \leq (a \wedge b) \vee (a \wedge c)$. Now, as $a \wedge b \leq d$ and $a \wedge c \leq d$, we have, by a basic property of $\vee$, that $(a \wedge b) \vee (a \wedge c) \leq d$. So, by transitivity of $\leq$, $a \leq d$.

(ix) As $b \vee c = 1$, reason as in Part (viii).

(x) Suppose $a \wedge b \leq c$. Then $a \cdot b \leq c$. By monotonicity of $\cdot$, it follows that $(a \cdot b) \cdot \neg c \leq c \cdot \neg c = 0$. Then, as $\cdot$ is both associative and commutative, $(a \cdot \neg c) \cdot b \leq 0$. So, $a \cdot \neg c \leq \neg b$. Finally, using that a is Boolean, we get $a \wedge \neg c \leq \neg b$.

(xi) Let $a \vee \neg b = 1$. Then, $b \cdot (a \vee \neg b) = b \cdot 1 = b$. By distributivity of $\cdot$ relative to $\vee$, it follows that $(b \cdot a) \vee (b \cdot \neg b) = b$. As $b \cdot \neg b = 0$, we have that $b \cdot a = b$. So, as a is Boolean, $b \wedge a = b$, i.e. $b \leq a$. ☐

Lemma 3 *Let* $\mathbf{A} \in \mathbb{RL}$ *and let* a *and* b *be Boolean elements of* A. *Then,*

(i) $a \wedge b = a \cdot b = \neg(\neg a \vee \neg b)$,
(ii) $(a \cdot b) \vee (\neg a \cdot b) \vee (a \cdot \neg b) \vee (\neg a \cdot \neg b) = 1$.

Proof (i) Firstly, we have that $\neg a \leq \neg a \vee \neg b$. So, using Lemma 1(iii) , it follows that $\neg(\neg a \vee \neg b) \leq \neg\neg a$. Now, using Proposition 1(v) and transitivity of $\leq$, we have that $\neg(\neg a \vee \neg b) \leq a$. Analogously, we get $\neg(\neg a \vee \neg b) \leq b$. Secondly, suppose $c \leq a$ and $c \leq b$, for $c \in A$. Then, using Lemma 1(iii) again, it follows that $\neg a \leq \neg c$ and $\neg b \leq \neg c$. So, $\neg a \vee \neg b \leq \neg c$. So, using Lemma 1(iii) once again, $\neg\neg c \leq \neg(\neg a \vee \neg b)$. Now, using Proposition 1(v) and transitivity of $\leq$, we get $c \leq \neg(\neg a \vee \neg b)$. (ii) It follows immediately using Part (i) and Proposition 1(i).

(ii) Suppose that $a \vee \neg a = b \vee \neg b = 1$. Then, $1 = (a \vee \neg a) \cdot (b \vee \neg b) = (a \cdot b) \vee (\neg a \cdot b) \vee (a \cdot \neg b) \vee (\neg a \cdot \neg b)$. ☐

Proposition 2 *Let* $\mathbf{A} \in \mathbb{RL}$ *and let* a *and* b *be Boolean elements of* A. *Then, (i)* $\neg a$, *(ii)* $a \vee b$, *(iii)* $a \wedge b = a \cdot b$, *(iv)* $a \to b$, *(v)* 0, *and (vi)* 1 *are Boolean.*

Proof (i) Suppose $a \vee \neg a = 1$. Then, as $a \leq \neg\neg a$, we get $\neg\neg a \vee \neg a = 1$.

(ii) Use Lemma 3 (ii) and see that

$a \cdot b \leq a \leq a \vee b \leq (a \vee b) \vee \neg(a \vee b)$,
$\neg a \cdot b \leq b \leq a \vee b \leq (a \vee b) \vee \neg(a \vee b)$,
$a \cdot \neg b \leq a \leq a \vee b \leq (a \vee b) \vee \neg(a \vee b)$, and
$\neg a \cdot \neg b \leq \neg a \wedge \neg b \leq \neg(a \vee b) \leq (a \vee b) \vee \neg(a \vee b)$.
So, $a \vee b$ is Boolean.

(iii) Use Parts (i) and (ii), and Lemma 3(i).

(iv) Use Parts (i) and (ii), and Proposition 1(vi).

(v) Use Parts (i) and (ii), and Proposition 1(vii).

(vi) Use the definition of Boolean element and the fact that $\neg 1 = 0$. ☐

From Proposition 2 it easily follows that, in any residuated lattice $\mathbf{A}$, the set of its Boolean elements $B(A) = \{a \in A : a \text{ is Boolean}\}$ is the domain of a subalgebra of $\mathbf{A}$, which is in fact a Boolean algebra. Indeed, $\mathbf{B(A)} = (B(A); \wedge, \vee, \cdot, \to, 0, 1)$ is the greatest Boolean algebra contained in $\mathbf{A}$. $\mathbf{B(A)}$ is called the Boolean skeleton or the *center* of $\mathbf{A}$.

2.2 On the logic FL_{ew}

The logics we are interested in are extensions or expansions of the logic FL_{ew} described below.

Definition 2 *The language of* FL_{ew} *has four binary connectives,* $\wedge$, $\vee$, $\cdot$, *and* $\rightarrow$, *and two constants,* 0 *and* 1. *The axioms of* FL_{ew} *are:*

$$\begin{array}{ll}
(1) & (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \gamma) \rightarrow (\varphi \rightarrow \gamma)),\\
(2) & (\gamma \rightarrow \varphi) \rightarrow ((\gamma \rightarrow \psi) \rightarrow (\gamma \rightarrow (\varphi \wedge \psi))),\\
(3) & (\varphi \wedge \psi) \rightarrow \varphi \text{ and } (\varphi \wedge \psi) \rightarrow \psi,\\
(4) & \varphi \rightarrow (\varphi \vee \psi) \text{ and } \psi \rightarrow (\varphi \vee \psi),\\
(5) & (\varphi \rightarrow \gamma) \rightarrow ((\psi \rightarrow \gamma) \rightarrow ((\varphi \vee \psi) \rightarrow \gamma)),\\
(6) & (\varphi \cdot \psi) \rightarrow (\psi \cdot \varphi),\\
(7) & (\varphi \cdot \psi) \rightarrow \varphi,\\
(8) & (\varphi \rightarrow (\psi \rightarrow \gamma)) \rightarrow ((\varphi \cdot \psi) \rightarrow \gamma),\\
(9) & ((\varphi \cdot \psi) \rightarrow \gamma) \rightarrow (\varphi \rightarrow (\psi \rightarrow \gamma)),\\
(10) & 0 \rightarrow \varphi \text{ and } \varphi \rightarrow 1.
\end{array}$$

The only rule of FL_{ew} *is* modus ponens*:*

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}.$$

We define $\neg\varphi = \phi \rightarrow 0$ and $\varphi \leftrightarrow \psi = (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$.

The following formulas and rules are derivable in FL_{ew}:

$$\begin{array}{ll}
(11) & \dfrac{\varphi \rightarrow \psi \qquad \psi \rightarrow \gamma}{\varphi \rightarrow \gamma},\\
(12) & \varphi \rightarrow (\psi \rightarrow \varphi),\\
(13) & \varphi \rightarrow \varphi,\\
(14) & \varphi \rightarrow (\varphi \cdot 1) \text{ and } (\varphi \cdot 1) \rightarrow \varphi,\\
(15) & (\varphi \cdot (\psi \cdot \gamma)) \rightarrow ((\varphi \cdot \psi) \cdot \gamma) \text{ and}\\
 & ((\varphi \cdot \psi) \cdot \gamma) \rightarrow (\varphi \cdot (\psi \cdot \gamma)),\\
(16) & (\varphi \cdot (\varphi \rightarrow \psi)) \rightarrow \psi,\\
(17) & (1 \rightarrow \varphi) \rightarrow \varphi,\ \varphi \rightarrow (1 \rightarrow \varphi),\\
(18) & \dfrac{\varphi \qquad \psi}{\varphi \wedge \psi},\\
(19) & (\varphi \vee \neg\varphi) \rightarrow \neg(\varphi \wedge \neg\varphi).
\end{array}$$

The derivations of (11)-(19) are rather easy, and hence they are left to the reader.

We will occasionally consider the following extensions of FL_{ew}.

Definition 3 *Consider the following axiomatic extensions of* FL_{ew}*:*

- *Intutitionistic logic* IL *is* FL_{ew} *plus*
  (Contr) $\varphi \rightarrow (\varphi \cdot \varphi)$.
- *The logic* MTL *is* FL_{ew} *plus the axiom*
  (Prel) $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$.
- SMTL *logic is* MTL *plus the axiom*
  (PC) $\neg(\varphi \wedge \neg\varphi)$
- WNM *logic is* MTL *plus the axiom*
  (WNM) $\neg(\varphi \& \psi) \vee (\varphi \wedge \psi \rightarrow \varphi \& \psi)$
- NM *logic is* WNM *plus the axiom*
  (Inv) $\neg\neg\varphi \rightarrow \varphi$
- BL *logic is* MTL *plus the axiom*
  (Div) $(\varphi \wedge \psi) \rightarrow (\varphi \& (\varphi \rightarrow \psi))$
- Product *logic is* BL *plus (PC) and the axiom*
  (C) $\neg\varphi \vee ((\varphi \rightarrow \varphi \& \psi) \rightarrow \psi)$
- *Gödel logic* G *is* BL *plus* **(Contr)**
- *Łukasiewicz logic* Ł *is* BL *plus* **(Inv)**.

In MTL and its extensions the connective $\vee$ is definable by $\phi \vee \psi := ((\phi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \phi) \rightarrow \phi)$, and in BL and its extensions the connective $\wedge$ is definable as well as $\varphi \wedge \psi := \varphi \& (\varphi \rightarrow \psi)$. Moreover, in IL one can prove the formula $(\phi \cdot \psi) \leftrightarrow (\phi \wedge \psi)$, i.e. connectives $\wedge$ and $\cdot$ are equivalent in IL.

All these logics are algebraizable, and hence they are strongly complete with respect to their corresponding classes of algebras. Namely, FL_{ew} is complete with respect to the variety $\mathbb{RL}$ of residuated lattices, MTL is complete with respect to the variety of pre-linear residuated lattices (MTL-algebras) and IL is complete with respect to the variety of contractive residuated lattices (Heyting algebras). Moreover, all axiomatic extensions of MTL are semilinear logics, that is, they are strongly complete with respect to the corresponding class of linearly ordered algebras. For instance, Gödel logic is complete with respect to the class of linearly ordered Heyting algebras, or Gödel chains.

3 Residuated lattices enriched with B

As explained in the previous section, the set of Boolean elements of a residuated lattice $\mathbf{A}$ form a Boolean algebra denoted *the center or Boolean skeleton of* $\mathbf{A}$. Cignoli and Monteiro considered Boolean elements in Łukasiewicz algebras in [6] and [7]. However, as far as we know, the operator defining the greatest Boolean element below, the operator B studied in this paper, has not yet been studied in the general context of residuated lattices. One relevant exception is the paper [17], where Reyes and Zolfaghari define modal operators $\Box$ and $\Diamond$ in the context of Bi-Heyting algebras that are shown to correspond respectively to the largest and the smallest complemented element below and above, respectively. Thus, the $\Box$ operation coincides with B. In the cited paper, using dual negation (or join-complement) D always in the context of Bi-Heyting algebras, the authors also study a family of modal operators $\Box_n$ and $\Diamond_n$, in a way similar to the one we shall employ in Section 4.2.

We will be considering residuated lattices $\mathbf{A}$ enriched with a unary operator B such that, for all $a \in A$, Ba is the greatest Boolean element below a, as defined in the Introduction. It is immediate to see that B can be characterized by the following three conditions, for any a, b in A:

(BE1) $Ba \leq a$,
(BE2) $Ba \vee \neg Ba = 1$,
(BI) if $b \leq a$ and $b \vee \neg b = 1$, then $b \leq Ba$.

We will denote the class of residuated lattices with B by $\mathbb{RL}^B$. Namely, an $\mathbb{RL}^B$-algebra is an algebra $\mathbf{A} = (A; \wedge, \vee, \cdot, \rightarrow, B, 0, 1)$ such that $(A; \wedge, \vee, \cdot, \rightarrow, 0, 1)$ is a residuated lattice and B satisfies the above three conditions.

First of all, note that B is new, that is, B is not expressible by a $\{\wedge, \vee, \cdot, \rightarrow, 0\}$-term. Indeed, for instance, in the Gödel algebra $\mathbf{G}_2 \times \mathbf{G}_3$ (the direct product of the two-element Boolean algebra with the three-element Gödel algebra) we have, for any $\{\wedge, \vee, \cdot, \rightarrow 0\}$-term t, that $ta \in \{0, a, 1\}$, where $a = (1, 1/2)$ is the join reducible coatom, while $Ba = (1, 0)$ is the join-irreducible atom, which does not belong to $\{0, a, 1\}$.

In the next proposition we see that all operations remain independent.

Proposition 3 *The set of operators $\{\wedge, \vee, \cdot, \rightarrow, 0, B\}$ is independent.*

Proof That $\wedge$ is independent of the rest may be seen by taking the algebra of Example 1 with $\cdot$ such that $b \cdot c = 0$, $b \cdot b = b$, and $c \cdot c = c$, for the coatoms b and c. Then, observe that the subset S with bottom, both coatoms, and top is closed for $\vee$, $\cdot$, $\rightarrow$, 0, and B, but $b \wedge c \notin S$.

That $\vee$ is independent of the rest may be seen by taking the algebra that results from inverting the algebra of Example 1 and considering that the subset S with bottom, both atoms a_1 and a_2 and top is closed for $\wedge$, $\cdot$, $\rightarrow$, 0 and B, but $a_1 \vee a_2 \notin S$.

That $\cdot$ is independent of the rest may be seen by taking the four-element chain such that $a \cdot b = a \cdot a = b \cdot b = a$, for the atom a and the coatom b and considering that the subset S with bottom, the coatom, and the top is closed for $\wedge$, $\vee$, $\rightarrow$, 0, and B, but $b \cdot b \notin S$.

That $\rightarrow$ is independent of the rest may be seen by taking the algebra of Example 1 and considering that the subset S with bottom, the atom a, the coatom b, and top is closed for $\wedge$, $\vee$, $\neg$, and B, but $b \rightarrow a \notin S$.

That 0 is independent of the rest may be seen by taking the two element Boolean algebra and considering that the top is closed for $\wedge$, $\vee$, $\cdot$, $\rightarrow$, and B, but $\neg 1 = 0$. The case of the independence of B has already been considered. ☐

Lemma 4 *Let $\mathbf{A} \in \mathbb{RL}^B$ and let $a \in A$. Then,*
(i) $Ba = a$ iff a is Boolean,
(ii) $Ba = 1$ iff $a = 1$, and
(iii) $BBa = Ba$.

Proof (i) Suppose $Ba = a$. As, using (BE2), we have $Ba \vee \neg Ba = 1$. So, $a \vee \neg a = 1$. For the other conditional, suppose $a \vee \neg a = 1$. Then, as $a \leq a$, using (BI), it follows that $a \leq Ba$. The other inequality follows by (BE2).

(ii) Suppose $Ba = 1$. Using (BE1), it folows that $1 \leq a$. So, $a = 1$. For the other conditional, suppose $a = 1$. Then, $a \leq 1$. Using (BI) and the fact that 1 is Boolean (see Proposition 2(vi)), it follows that $1 \leq Ba$.

(iii) Considering (BE1), it is enough to see that $Ba \leq BBa$, which follows using (BI) and (BE2). ☐

We also have the following properties.

Lemma 5 *Let $\mathbf{A} \in \mathbb{RL}^B$ and let $a, b \in A$. Then,*

(i) *Monotonicity of B: if $a \leq b$, then $Ba \leq Bb$,*
(ii) $B(a \wedge b) = Ba \wedge Bb$,
(iii) $B(a \wedge b) \leq a \cdot b$,
(iv) $B(a \cdot b) = B(a \wedge b)$,
(v) $B(a \cdot b) = Ba \cdot Bb$,
(vi) $Ba \vee Bb \leq B(a \vee b)$,
(vii) $B(a \rightarrow b) \leq Ba \rightarrow Bb$,
(viii) $B0 = 0$,
(ix) $B\neg a \leq \neg Ba$.

Proof (i) Suppose $a \leq b$. Using (BI), it is enough to have $Ba \leq b$ and $Ba \vee \neg Ba = 1$. Now, the former follows by (BE1) and the hypothesis, and the latter is (BE2).

(ii) $B(a \wedge b) \leq Ba \wedge Bb$ follows from $a \wedge b \leq a, b$ using monotonicity of B. The other inequality follows using (BI), (BE1), and (iii) in Proposition 2.

(iii) By (i) in Proposition 1 and part (ii) we have $B(a \wedge b) = Ba \wedge Bb = Ba \cdot Bb$. The goal follows using $Ba \leq a$, $Bb \leq b$, and monotonicity of $\cdot$.

(iv) From $a \cdot b \leq a \wedge b$ by (i), we get $B(a \cdot b) \leq B(a \wedge b)$. For the other inequality, using (BI), it is enough to have $B(a \wedge b) \leq a \cdot b$ and $B(a \wedge b)$ Boolean. Now, the former is (iii) and the latter follows from (BE2).

(v) As Ba is Boolean, by (i) of Proposition 1, we have $Ba \wedge Bb = Ba \cdot Bb$. Moreover, by (ii), $B(a \wedge b) = Ba \wedge Bb$. We get our goal using (iv).

(vi) It follows using (i) (Monotonicity of B).

(vii) We have $a \rightarrow b \leq a \rightarrow b$. Then, $(a \rightarrow b) \cdot a \leq b$. Hence, by monotonicity of B, $B((a \rightarrow b) \cdot a) \leq Bb$. So, using (v), $B(a \rightarrow b) \cdot Ba \leq Bb$. So, $B(a \rightarrow b) \leq Ba \rightarrow Bb$.

(viii) It follows because 0 is Boolean.

(ix) It follows from (vii) and (viii) considering that $\neg a = a \to 0$. ☐

Regarding the inequalities in the previous lemma, that is, *(iii)*, *(vi)*, *(vii)*, and *(ix)*, their reciprocals do not hold. Indeed, inequality $a \cdot b \leq B(a \wedge b)$ fails in the three-element Gödel algebra $\mathbf{G}_3$ taking the top and the middle element. Inequality $B(a \vee b) \leq Ba \vee Bb$ fails in the algebra of Example 1 taking a and b to be the coatoms. Also, inequality $\neg Ba \leq B\neg a$ fails in $\mathbf{G}_3$, taking a to be the middle element. So, also inequality $Ba \to Bb \leq B(a \to b)$ fails.

The operator B may not exist for every element in a residuated lattice. However, B exists in every finite residuated lattice.

Proposition 4 *Let $\mathbf{A} \in \mathbb{RL}$ be finite. Then, B exists in A.*

Proof In a finite residuated lattice $\mathbf{A}$, for any $a \in A$, it actually holds that $Ba = \bigvee\{b \in A : b \leq a \text{ and } b \vee \neg b = 1\}$. It is enough to see that if $b_1 \leq a$, $b_1 \vee \neg b_1 = 1$, $b_2 \leq a$ and $b_2 \vee \neg b_2 = 1$, then (i) $b_1 \vee b_2 \leq a$ and (ii) $(b_1 \vee b_2) \vee \neg(b_1 \vee b_2) = 1$. Now, (i) follows immediately and (ii) follows using (ii) in Proposition 2. ☐

On the other hand, there are infinite residuated lattices where B does not exist. Indeed, we have the following example due to Franco Montagna (see [1]).

Proposition 5 *There is an (infinite) Gödel algebra $\mathbf{A}$ and $a \in A$ such that Ba does not exist, i.e. where B does not exist.*

Proof Let $[0, \frac{1}{2}, 1]_G$ be the three-element Gödel algebra. Let us consider

$A_1 = \{a \in ([0, \frac{1}{2}, 1]_G)^{\mathbb{N}}$ such that $\{i \in \mathbb{N} : a_i = 0\}$ is finite$\}$,
$A_2 = \{a \in ([0, \frac{1}{2}, 1]_G)^{\mathbb{N}}$ such that $\{i \in \mathbb{N} : a_i \neq 0\}$ is finite$\}$, and
$A = A_1 \cup A_2$.

The set A is the domain of a subalgebra of $([0, \frac{1}{2}, 1]_G)^{\mathbb{N}}$. Indeed, if $a, b \in A_1$, then $a \wedge b \in A_1$ and $a \to b \in A_1$, if $a, b \in A_2$, then $a \wedge b \in A_2$ and $a \to b \in A_1$, if $a \in A_1$ and $b \in A_2$, then $a \wedge b \in A_2$ and $a \to b \in A_2$, and if $a \in A_2$ and $b \in A_1$, then $a \wedge b \in A_2$ and $a \to b \in A_1$. Also, $0 \in A_2$. So, A is the domain of a subalgebra $\mathbf{A}$ of $([0, \frac{1}{2}, 1]_G)^{\mathbb{N}}$.

Now, take a to be such that $a_i = 1$ if i is even and $a_i = \frac{1}{2}$ if i is odd. Next, consider the set $\{b \in A : b \leq a$ and b is Boolean$\}$. It consists of all elements b such that $b_i = 0$ for all odd i and for all but finitely many even i, and $b_i = 1$ otherwise. It can be seen that this set has no maximum in A. ☐

Actually, Montagna's example of Proposition 5 can be generalized as follows.

Proposition 6 *Let $\mathbb{V}$ a variety of MTL-algebras such that there is a linearly ordered algebra $\mathbf{A} \in \mathbb{V}$ with a proper filter F (i.e. $\{1\} \subsetneq F \subsetneq A$). Then, $\mathbb{V}$ contains an infinite algebra where B does not exist.*

Proof Let $\mathbf{D} \in \mathbb{V}$ a chain and F be a filter of $\mathbf{A}$ satisfying the hypothesis of the proposition. Let us define $F^{\neg} = \{x \in D \mid \exists y \in F, x \leq \neg y\}$ and let $C = F \cup F^{\neg}$. It is easy to check that C is the domain of a subalgebra of $\mathbf{D}$. Finally define the following sets:

$A_1 = \{a \in C^{\mathbb{N}}$ such that $\{i \in \mathbb{N} : a_i \in F\}$ is finite$\}$,
$A_2 = \{a \in C^{\mathbb{N}}$ such that $\{i \in \mathbb{N} : a_i \in F^{\neg}\}$ is finite$\}$,
$A = A_1 \cup A_2$.

One can check that again A is the domain of a subalgebra of $\mathbf{C}^{\mathbf{N}}$, taking into account that if $x \in F$ and $y \in F^{\neg}$, then $x \wedge y, x * y, x \to y \in F^{\neg}$, and if $x, y \in F^{\neg}$, then $x \to y \in F$.

Thus, A is a subalgebra and taking an element a such that $a_i = 1$ if i is even and $a_i = b$, for a given $b \in F \setminus \{1\}$, then the same argument as in Montagna's example proves that Ba does not exist. ☐

For readers familiar with the main systems of mathematical fuzzy logic and their algebraic semantics (see [8]), we provide the following corollary with further examples of subvarieties of residuated lattices containing algebras where B does not exist.

Corollary 1 *In the following varieties of MTL-algebras, there is an infinite algebra where B does not exist:*

- *the variety generated by any continuous t-norm,*
- *the varieties generated by either the NM t-norm or a WNM t-norm.*

Proof In all these varieties there is an algebra $\mathbf{A}$ satisfying the conditions of Proposition 6.

If the t-norm is either a Gödel, Product, or a WNM t-norm (including NM), then take as $\mathbf{A}$ the standard chain and as F the positive elements respect to $\neg$, i.e., the elements such that $\neg x \leq x$. If the t-norm is Łukasiewicz, then take $\mathbf{A}$ as the Chang algebra and F as the set of its positive elements. Finally, if the continuous t-norm is a proper ordinal sum, then take $\mathbf{A}$ as the standard chain and $F = [a, 1]$, where $a \in (0, 1)$ is the end point of a component. It is clear that in all cases F is a proper filter and thus Proposition 6 applies. ☐

Some papers (e.g. [4]) consider the notion of *compatible operation*. Operation B is not compatible, that is, the congruences of $\mathbb{RL}$ and $\mathbb{RL}^B$ are not the same. To see this, take the three-element Heyting or Gödel algebra $\mathbf{G}_3$ with universe $\{0, m, 1\}$ and the equivalence relation given by $\theta = \{(0,0),(m,m),(m,1),(1,m),(1,1)\}$. It holds that θ is a $\mathbb{RL}$-congruence, but not an $\mathbb{RL}^B$-congruence, as $Bm = 0$ and $B1 = 1$.

Taking *modality* to mean a finite combination of the unary operators $\neg$ and B, the next statement shows how many different modalities there are in $\mathbb{RL}^B$.

Proposition 7 *In $\mathbb{RL}^B$ there are nine different modalities. They may be ordered as follows: on the one hand, the positive modalities $B \leq Id \leq \neg\neg \leq \neg B \neg$ (with also $B \leq B\neg\neg \leq \neg\neg$) and, on the other hand, the negative modalities $B\neg \leq \neg \leq \neg B\neg\neg \leq \neg B$. See Figure 2.*

Proof The inequalities are immediate. The reverse inequalities can be seen not to be the case by considering either the only atom in the three-element Gödel algebra $\mathbf{G}_3$ or any of the two non-comparable elements of the Heyting algebra obtained by adding a top element to the Boolean algebra of 4 elements. There are no other modalities, because if we apply operations $\neg$ and B to the given nine modalities, we do not get anything new, as $\neg\neg B = B$, $BB = B$, and $B\neg B = \neg B$. ☐

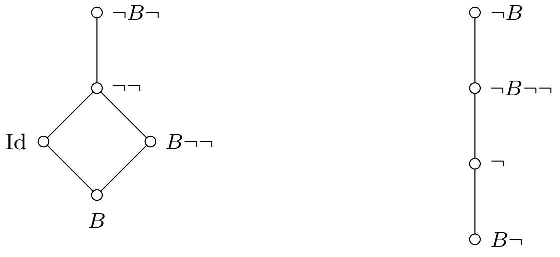

Fig. 2 The positive and negative modalities of $\neg$ and B

3.1 An equational class

It is natural to inquire whether the class $\mathbb{RL}^B$ of residuated lattices with the B operator is in fact an equational class. To this end, we start by focusing our attention on the following equations, where we will be using $x \preccurlyeq y$ as an abbreviation for $x \vee y \approx y$:

(BI1) $Bx \preccurlyeq B(x \vee y)$,
(BI2) $B1 \approx 1$.

Lemma 6 *The equations (BI1) and (BI2) are valid in $\mathbb{RL}^B$.*

Proof Both equations follow immediately from lemmas 4(ii) and 5(vi), respectively. ☐

We are also interested in the equation

(BI3) $B(x \vee \neg x) \preccurlyeq Bx \vee B\neg x$,

but it is not easy to see that it also holds in $\mathbb{RL}^B$. Towards this goal, we first state and prove the following result.

Lemma 7 *In $\mathbb{RL}^B$ the following equation holds:*

$$(B(x \vee \neg x) \wedge x) \vee \neg(B(x \vee \neg x) \wedge x) \approx 1.$$

Proof Using Lemma 1(i), and using T for the left hand side of the given equation, it is enough to get

(i) $B(x \vee \neg x) \vee \neg B(x \vee \neg x) \approx 1$,
(ii) $B(x \vee \neg x) \preccurlyeq T$, and
(iii) $\neg B(x \vee \neg x) \preccurlyeq T$.

Part (i) is immediate because of (BE2).

To see (ii), using Proposition 1(viii), note that we have that $B(x \vee \neg x) \preccurlyeq x \vee \neg x$, (immediate using (BE1)), $B(x \vee \neg x) \wedge x \preccurlyeq T$ (also immediate), and $B(x \vee \neg x) \wedge \neg x \preccurlyeq T$, which follows from $B(x \vee \neg x) \wedge \neg x \preccurlyeq \neg(B(x \vee \neg x) \wedge x)$, which holds because of Lemma 1(iv).

To see (iii), note that $B(x \vee \neg x) \wedge x \leq B(x \vee \neg x)$. So, using Lemma 1(iii), it follows that $\neg B(x \vee \neg x) \preccurlyeq \neg(B(x \vee \neg x) \wedge x)$. And so, $\neg B(x \vee \neg x) \preccurlyeq T$. ☐

Proposition 8 *The equation* (BI3) *is valid in $\mathbb{RL}^B$.*

Proof Using Proposition 1(ix), it is enough to check the following three conditions:

(i) $Bx \vee \neg Bx \approx 1$,
(ii) $B(x \vee \neg x) \wedge Bx \preccurlyeq Bx \vee B\neg x$, and
(iii) $B(x \vee \neg x) \wedge \neg Bx \preccurlyeq Bx \vee B\neg x$.

Now, (i) is immediate due to (BE2) and (ii) is also immediate considering that $B(x \vee \neg x) \wedge Bx \preccurlyeq Bx$. Regarding (iii), it is clear that it follows from $B(x \vee \neg x) \wedge \neg Bx \preccurlyeq B\neg x$, which in turn follows, using BI, from

(iv) $B(x \vee \neg x) \wedge \neg Bx \preccurlyeq \neg x$ and
(v) $(B(x \vee \neg x) \wedge \neg Bx) \vee \neg(B(x \vee \neg x) \wedge \neg Bx) \approx 1$.

Now, (v) is immediate using Proposition 2(i) and (iii). Concerning (iv), using Proposition 1(x), it follows from $B(x \vee \neg x) \wedge x \preccurlyeq Bx$, which immediately follows, using BI, from $B(x \vee \neg x) \wedge x \preccurlyeq x$ and Lemma 7. So, we are done. ☐

The following theorem answers positively the question posed above.

Theorem 1 $\mathbb{RL}^B$ *is an equational class. An equational basis relative to* $\mathbb{RL}$ *is the following set of equations :*

(BE1) $Bx \preccurlyeq x$,
(BE2) $Bx \vee \neg Bx \approx 1$,
(BI1) $Bx \preccurlyeq B(x \vee y)$,
(BI2) $B1 \approx 1$,
(BI3) $B(x \vee \neg x) \preccurlyeq Bx \vee B\neg x$.

Proof It is enough to prove (BI) using the given equations. Suppose (i) $b \leq a$ and (ii) $b \vee \neg b = 1$. From (i), using (BI1), it follows (iii) $Bb \leq Ba$. From (ii), using (BI2), it follows $B(b \vee \neg b) = 1$, which, using (BI3), implies that $Bb \vee B\neg b = 1$, which, using (iii) implies $Ba \vee B\neg b = 1$, which in turn, using (BE1), implies that $Ba \vee \neg b = 1$. Now, by the fact that $b \cdot 1 = b$, we get (iv) $b \cdot (Ba \vee \neg b) = b$. From (iv), using distributivity, we get $(b \cdot Ba) \vee (b \cdot \neg b) = b$. However, $b \cdot \neg b = 0$. Consequently, $b \cdot Ba = b$. So, as b is Boolean, using (i) in Proposition 1, $b \wedge Ba = b$, i.e. $b \leq Ba$. ☐

It is also natural to inquire whether the given equations are independent.

Proposition 9 *The set* {(BE1), (BE2), (BI1), (BI2), (BI3)} *is independent.*

Proof To see that (BE1) is independent of the rest, take the three-element Gödel algebra $\mathbf{G}_3$ and define $B0 = 0$, and $Ba = 1$, if a is not 0.

To see that (BE2) is independent of the rest, take the four-element Gödel chain $\mathbf{G}_4$ and define $B0 = 0$, $B1 = 1$, $Ba = 0$, for the only atom a of the chain, and $Bc = c$, for the remaining element c.

To see that (BI1) is independent of the rest, take the Gödel algebra $\mathbf{G}_3 \times \mathbf{G}_3$. If a is any of the four Boolean elements, then put $Ba = a$, else put $Ba = 0$.

To see that (BI2) is independent of the rest, take again $\mathbf{G}_3$, but now define $Ba = 0$, for every a.

Finally, to see that (BI3) is independent of the rest, take the four-element Boolean $\mathbf{G}_2 \times \mathbf{G}_2$ algebra and define, for any a, if $a = 1$, then $Ba = 1$, else $Ba = 0$. ☐

3.2 Subdirectly irreducible $\mathbb{RL}^{\mathbb{B}}$-algebras

In this section we show the subdirectly irreducible members of $\mathbb{RL}^{\mathbb{B}}$ are those whose Boolean elements are only the top and bottom elements.

Definition 4 *Let* $\mathbf{A} \in \mathbb{RL}^{\mathbb{B}}$. *A set* F *contained in* A *is said to be a* $\mathbb{RL}^{\mathbb{B}}$*-filter iff for all* $a, b \in A$ *it satisfies*

(1) $1 \in F$,
(2) *if* $a \in F$ *and* $a \leq b$, *then* $b \in F$,
(3) *if* $a, b \in F$, *then* $a \cdot b \in F$,
(4) *if* $a \in F$, *then* $Ba \in F$.

Proposition 10 *Let* $\mathbf{A} \in \mathbb{RL}^{\mathbb{B}}$. *The lattice of* $\mathbb{RL}^{\mathbb{B}}$*-congruences is isomorphic to the set of* $\mathbb{RL}^{\mathbb{B}}$*-filters. Indeed, let* $f : Con(A) \longrightarrow Fil(A)$ *be defined by: if* $\equiv$ *is a* $\mathbb{RL}^{\mathbb{B}}$*-congruence, then* $f(\equiv)$ *is the* $\mathbb{RL}^{\mathbb{B}}$*-filter* $F_{\equiv} = \{a \in A : a \equiv 1\}$. *Then, the function* f *is an isomorphism such that if* F *is a* $\mathbb{RL}^{\mathbb{B}}$*-filter, then* $f^{-1}(F)$ *is the* $\mathbb{R}^{\mathbb{B}}$*-congruence* $\equiv_F$ b *defined by* $a \equiv_F b$ *iff* $a \to b, b \to a \in F$.

Proof It is obvious that $F_{\equiv}$ is a $\mathbb{RL}^{\mathbb{B}}$-filter. In order to prove that $\equiv_F$ is a congruence we need to prove that if $a \equiv_F b$, then $Ba \equiv_F Bb$, since the other conditions are known to be true for any residuated lattice. So, suppose $a \to b$ and $b \to a \in F$. Then, by the fourth condition in the definition of filter, $B(a \to b) \in F$. Now, using (vi) in Lemma 5 and the second condition in the definition of filter, it follows that $Ba \to Bb \in F$. Analogously, we obtain that $Bb \to Ba \in F$. Finally, it is also obvious that $f^{-1} \circ f = Id$. ☐

From there we can characterize a family of $\mathbb{RL}^{\mathbb{B}}$-filters.

Proposition 11 *Let* $\mathbf{A} \in \mathbb{RL}^{\mathbb{B}}$. *For any element* $a \in B(A)$, *then* $F_a = [a, 1] = \{x \in A : a \leq x \leq 1\}$ *is a* $\mathbb{RL}^{\mathbb{B}}$*-filter.*

Proof It is obvious that F_a satisfies the first two conditions of a $\mathbb{RL}^{\mathbb{B}}$-filter. The third is an easy consequence of the fact that if $a \in B(A)$, then $a * x = a \wedge x$ and thus if $x, y \in F_a$, then $a = a \wedge y \leq x * y$ and thus $x * y \in F_a$. Finally, if $x \in F$, then $a = Ba \leq Bx$. ☐

From now on we will denote by F_a the principal filter defined by $a \in B(A)$. In order to characterize the subdirectly irreducible $\mathbb{RL}^{\mathbb{B}}$-algebras we will use the result of [21, Theorem 97]: an algebra $\mathbf{A}$ is subdirectly decomposable iff there exists a family of non-trivial congruences σ_i such that their intersection is the identity. In our case, this means that $\mathbf{A}$ is subdirectly irreducible iff there is a unique coatom in the lattice of $\mathbb{RL}^{\mathbb{B}}$-congruences of $\mathbf{A}$.

Proposition 12 *Let* $\mathbf{A} \in \mathbb{RL}^{\mathbb{B}}$. *Then,* $\mathbf{A}$ *is subdirectly irreducible iff* $B(A) = \{0, 1\}$.

Proof Observe first that if F is a $\mathbb{RL}^{\mathbb{B}}$-filter of A, then F contains a Boolean element a (by the third condition of $\mathbb{RL}^{\mathbb{B}}$-filter) and thus F contains F_a. Thus, to obtain the intersection of the non-trivial $\mathbb{RL}^{\mathbb{B}}$-filters of $\mathbf{A}$ it is enough to compute the intersection of the filters F_a. However, this intersection is not the identity iff there exist a unique Boolean element a such that a is a coatom of $B(A)$. So, being $B(A)$ a Boolean algebra, this implies that $B(A) = \{0, 1\}$. ☐

4 Comparing B with other operations

The operation B is strongly related to other operations considered in the literature, e.g. the Monteiro-Baaz $\varDelta$ operation and operations defined using the join-complement D. This section is devoted to explore these relationships.

4.1 Comparing B with $\varDelta$

The operation $\varDelta$ was already considered by Monteiro in his paper about symmetric Heyting algebras in 1980 (see [13]). Monteiro considered the same definitions of possibility and necessity operations given by Moisil in [11] (see p. 67 in [13]). However, instead of using Moisil's notation, Monteiro used ∇ and $\varDelta$ to denote them, respectively. When doing so, he did not explicitly mention Moisil. However, many works by Moisil appear in the list of references of [13], including [11] and [12]. Monteiro also considered the $\varDelta$ operator in the setting of linear symmetric Heyting algebras and studied the properties of $\varDelta$ in the totally linear case as well [13, Ch. 5, Sect. 3]. In 1996, independently, Baaz in [2] considered an expansion of Gödel logic with a connective he also called $\varDelta$ satisfying certain axioms and the rule $\varphi/\varDelta\varphi$. Although he did not cite Monteiro, the proposed axioms are equivalent to the properties that Monteiro proved for his $\varDelta$ operator in the framework of totally linear symmetric Heyting algebras. Baaz also provided a deduction theorem using $\varDelta$: $\varGamma, \varphi \vdash \psi$ iff $\varGamma \vdash \varDelta\varphi \to \psi$. In 1998, Hájek considered Baaz's $\varDelta$ in the context of BL-algebras and BL logic (see pp. 57-61 [10]). He gave for $\varDelta$ exactly the same axioms as Baaz presented in [2] for Gödel logic. He observed that all $\varDelta$ axioms make it behave like a necessity operator, with the exception of the axiom $\varDelta(\varphi \vee \psi) \to (\varDelta\varphi \vee \varDelta\psi)$, that is characteristic of possibility operations (see Remark 2.4.7 of [10]). The $\varDelta$ operation has also been considered in the more general context of Mathematical fuzzy logic, see several chapters in the handbook [8]. More recently, in [1] the authors study in depth, among other things, the expansion of FL_{ew} with the $\varDelta$ operator and show that is conservative.

In MTL, $\varDelta$ can always be defined over chains, namely as $\varDelta 1 = 1$ and $\varDelta x = 0$ for all $x \neq 0$, and thus, $\varDelta$ and B over MTL-chains coincide. But there are (non-linearly) MTL-algebras where $\varDelta$ does not exist. Nevertheless, this is not a problem because MTL is semilinear, and the semantics of $\varDelta$ over chains is clear. However, there is not a clear semantical interpretation of the axioms of $\varDelta$ in the general context of residuated lattices.

In the context of a residuated lattice, the operator $\varDelta$ is introduced e.g. in [1] by the same equations as in MTL or BL (cf.[10, p. 58]):

($\varDelta$E1) $\varDelta x \preccurlyeq x$,
($\varDelta$E2) $\varDelta x \vee \neg\varDelta x \approx 1$,
($\varDelta$I1) $\varDelta(x \vee y) \preccurlyeq \varDelta x \vee \varDelta y$,
($\varDelta$I2) $\varDelta 1 \approx 1$,
($\varDelta$I3) $\varDelta x \preccurlyeq \varDelta\varDelta x$,
($\varDelta$I4) $\varDelta(x \to y) \preccurlyeq \varDelta x \to \varDelta y$,

where, again, $x \preccurlyeq y$ abbreviates $x \vee y \approx y$. Note that $\varDelta$I3 may be derived from the rest: it is enough to check that an operator satisfying the rest of the equations, satisfies all the equations in Theorem 1, and hence the quasi-equation (BI) as well; then use (iii) of Lemma 4. Also, regarding their defining equations, the only difference between $\varDelta$ and B is that $\varDelta$ satisfies $\varDelta(x \vee y) \preccurlyeq \varDelta x \vee \varDelta y$, whereas B only satisfies the particular case $y = \neg x$, that is, B only satisfies $B(x \vee \neg x) \preccurlyeq Bx \vee B\neg x$.

We will denote by $\mathbb{RL}^{\varDelta}$ the class of residuated lattices expanded with $\varDelta$.

It will be useful to bear in mind the following fact.

Lemma 8 *Let* $\mathbf{A} \in \mathbb{RL}^{\varDelta}$ *and* $a \in A$. *Then,* $\varDelta a = a$ *iff* a *is Boolean.*

Proof Supposing $\varDelta a = a$, using ($\varDelta$E2) it follows that a is Boolean. On the other hand, suppose a is Boolean. Considering ($\varDelta$E1), it is enough to prove that $a \leq \varDelta a$. By Lemma 1(ii), it is enough in turn to prove $\varDelta a \vee \varDelta\neg a = 1$, $a \cdot \varDelta a \leq \varDelta a$, and $a \cdot \varDelta\neg a \leq \varDelta a$. The first condition holds using ($\varDelta$I1) and ($\varDelta$I2), since a is Boolean. The second condition is immediate. For the third, observe that $a \cdot \varDelta\neg a \leq a \cdot \neg a = 0 \leq \varDelta a$.

Actually, $\varDelta$ is somewhat stronger than B in the following sense.

Proposition 13 *Let* $\mathbf{A} \in \mathbb{RL}$. *If* $\varDelta$ *exists in* $\mathbf{A}$, *then so does* B, *with* $B = \varDelta$.

Proof Considering Theorem 1, all we have to see is that $\varDelta$ satisfies the equational basis given for B. This is immediate excepting (BI1). Let us see that the equation $\varDelta x \preccurlyeq \varDelta(x \vee y)$ also holds. As we have $x \to (x \vee y) \approx 1$, using ($\varDelta$I2) and ($\varDelta$I4) we get $1 \preccurlyeq \varDelta x \to \varDelta(x \vee y)$, which gives $\varDelta x \preccurlyeq \varDelta(x \vee y)$. □

On the other hand, we have the following result.

Proposition 14 *There exist finite residuated lattices where* B *exists, but* $\varDelta$ *does not.*

Proof Consider the coatoms b and c in the Gödel algebra of Example 1. Using Proposition 4, it is clear that B exists, as the algebra is finite. To see that $\varDelta$ does not exist, note that ($\varDelta$E1) and ($\varDelta$E2) imply that $\varDelta b = \varDelta c = 0$. So, $\varDelta b \vee \varDelta c = 0$. However, $\varDelta(b \vee c) = 1$, due to ($\varDelta$I2). Then, ($\varDelta$I1) is not satisfied. □

This example makes clear the basic difference between Δ and B when we define them over MTL-algebras. It is well known that MTL^{Δ}, the expansion of MTL with Δ, is semilinear, i.e. each algebra of the variety is a subdirect product of lineraly ordered MTL^{Δ}-algebras. Moreover, we have seen that Δ and B coincide over chains. Thus, the example proves that MTL^{B}, the expansion of MTL with B, is not semilinear. In fact, this was already clear from Proposition 12, since there exist subdirectly irreducible MTL^{B}-algebras (like the one defined in the previous example) that are not linearly ordered.

4.2 Comparing B with an operation using the join-complement

The join-complement operation D has a long history. In 1919 Skolem considered lattices expanded with both meet and join relative complements (see §2 of [19] or pp. 77-85 of [20]). He just worked from an algebraic point of view. He noted that existence of both top 1 and bottom 0 is implied. Also, he briefly considered the meet and join-complements, for which, for an arbitrary argument a, he used the notations $\frac{0}{a}$ and $1 - a$, respectively.

In 1942 Moisil defined possibility as $\neg\neg$ and necessity as DD in a setting where he had both intuitionistic negation $\neg$ and its dual D (see §4 of [11] or p. 365 in [12]). He did not mention Skolem. In 1949 Ribenboim proved that distributive lattices with D form an equational class (see [18]). In fact, the meet is not needed, as the class with join and join-complement D is already an equational class. In 1974 Rauszer, mainly considering algebraic aspects, studied a logic with conjunction, disjunction, conditional and its dual (see [15]). She also included both intuitionistic negation $\neg$ and its dual D, though these can be easily defined. Her axiomatization included the expected axioms plus the rules *modus ponens* and $\varphi/\neg D\varphi$. She also provided a deduction theorem using the formula $(\neg D)^n$. She neither mentioned Skolem nor Moisil.

In the context of a join semi-lattice $\mathbf{A}$, it is possible to postulate the existence of the join-complement $Da = \min\{b \in A : \text{for all } c \in A,\ c \leq a \vee b\}$, for $a \in A$. This is equivalent to the following two conditions:

(DI) $b \leq a \vee Da$, for all $a, b \in A$,
(DE) for any $a, b \in A$, if for all $c \in A$, $c \leq a \vee b$, then $Da \leq b$.

In a join semi-lattice the existence of D implies the existence of both top $1 = a \vee Da$, for any a, and bottom $0 = D(a \vee Da)$, for any a. Moreover, D can be equationally characterized by the following three equations, where, again, we use $x \preccurlyeq y$ as an abbreviation for $x \vee y \approx y$:

(DI) $y \preccurlyeq x \vee Dx$,
(DE1) $D(x \vee Dx) \preccurlyeq y$,
(DE2) $Dy \preccurlyeq x \vee D(x \vee y)$.

In what follows, we will denote by $\mathbb{RL}^{D}$ the class of residuated lattices expanded with an operation D satisfying these equations. Obviously, by definition, $\mathbb{RL}^{D}$ is an equational class. Notice that in a residuated lattice $\mathbf{A}$, having in the signature the symbols 0 and 1 for the bottom and top elements, respectively, the above definition of D can be simplified to $Da = \min\{b \in A : a \vee b = 1\}$, and the condition (DE) simplifies to be

(DE′) for any $a, b \in A$, if $a \vee b = 1$, then $Da \leq b$.

Moreover the equations (DI) and (DE1) can also be simplified to:

(DI′) $x \vee Dx \approx 1$,
(DE1′) $D1 \approx 0$.

Remark 1 *Note that, while $x \preccurlyeq \neg\neg x$ holds in $\mathbb{RL}$, from* (DE′) *and* (DI′) *it follows that $DDx \preccurlyeq x$ holds in $\mathbb{RL}^{D}$. Note also that in a Heyting algebra D is the dual of $\neg$, since in that case $\neg$ coincides with the meet complement.*

As in the case of the operator B, D may not exist in some residuated lattices, but it always exists in finite residuated lattices.

Proposition 15 *Let $\mathbf{A}$ be a finite residuated lattice. Then D exists in A.*

Proof It is enough to prove that $\bigwedge\{b \in A : a \vee b = 1\}$ exists in A. For that, it is enough to see that if $a \vee b_1 = 1$ and $a \vee b_2 = 1$, then $a \vee (b_1 \wedge b_2) = 1$. Now, from the antecedent it follows that $(a \vee b_1) \cdot (a \vee b_2) = 1$ and using twice the distributive law of $\cdot$ with respect to $\vee$, we have that $(a \cdot a) \vee (a \cdot b_2) \vee (b_1 \cdot a) \vee (b_1 \cdot b_2) = 1$. Any subterm t of the left-hand term is such that $t \leq a \vee (b_1 \wedge b_2)$. □

Following [15] and [17], we consider now the compound operation $\neg D$ and its relation to B. First, let us state the following fact.

Lemma 9 *Let $\mathbf{A} \in \mathbb{RL}^{\mathbb{D}}$ and $a, b \in A$. Then,*
(i) $\neg Da \leq a$,
(ii) *if $a \leq b$, then $Db \leq Da$ and $\neg Da \leq \neg Db$.*

Proof (i) follows from $a \vee Da = 1$ using Lemma 1(vi).

(ii) Assume $a \leq b$. Then, $1 = a \vee Da \leq b \vee Da$. Hence, by (DE) we have $Db \leq Da$. Now, apply Lemma 1(iii) to get $\neg Da \leq \neg Db$. □

In [5, Section 5] the authors prove a result about iterations of the operation $\neg D$ in the context of meet-complemented distributive lattices with D. Once trivially adapted to $\mathbb{RL}^{\mathbb{D}}$, it is the following fact.

Proposition 16 (i) *For any natural $n > 0$, let $\mathbb{RL}^{\mathbb{D}_n}$ be the subvariety of $\mathbb{RL}^{\mathbb{D}}$ defined by adding to those of $\mathbb{RL}^{\mathbb{D}}$ the following equation:*

$$(\neg D)^{n+1}x \approx (\neg D)^n x.$$

Then, the sequence of varieties $\mathbb{RL}^{\mathbb{D}_1} \subset \mathbb{RL}^{\mathbb{D}_2} \subset \ldots \subset \mathbb{RL}^{\mathbb{D}_n} \subset \ldots$ is strictly increasing.

(ii) *There are algebras of $\mathbb{RL}^{\mathbb{D}}$ where none of the equations given in* (i) *hold.*

Next, consider the following example of a Heyting algebra where B exists but D does not.

Example 2 B exists in the Heyting algebra $1 + (\mathbb{N}\times\mathbb{N})^{\partial}$ of Figure 3, where $(\mathbb{N}\times\mathbb{N})^{\partial}$ is obtained 'turning upside down' the partial order $\mathbb{N}\times\mathbb{N}$. In that Heyting algebra, $Ba = 1$ if $a = 1$ else $Ba = 0$. However, D does not exist for the elements $(0, n)$ and $(n, 0)$, with non-zero n.

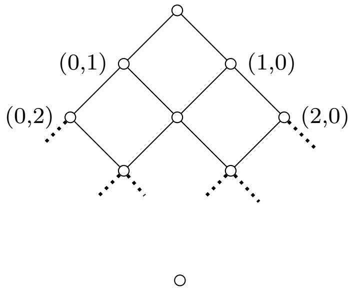

Fig. 3 The residuated lattice $1 + (\mathbb{N}\times\mathbb{N})^{\partial}$, where B exists but D does not

There are also $\mathbb{RL}$-algebras where D exists, but B does not (see the end of Section 2 of [5] for an example of a Heyting algebra). Note that in Franco Montagna's example, neither B nor D exist.

In the following case, existence of D implies existence of B.

Proposition 17 *Let $\mathbf{A} \in \mathbb{RL}$ and $\neg a \vee \neg\neg a = 1$, for all $a \in A$. Then, if D exists in $\mathbf{A}$, then B also exists in $\mathbf{A}$, with $B = \neg D$.*

Proof Let us take an $a \in A$. Then, $\neg Da$ exists in A. We have to see (i) $\neg Da \leq a$, (ii) $\neg Da \vee \neg\neg Da = 1$, and (iii) if $b \leq a$ and $b \vee \neg b = 1$, then $b \leq \neg Da$. Now, (i) holds as seen in Lemma 9(i) and (ii) follows from the hypothesis that $\neg a \vee \neg\neg a = 1$, for any $a \in A$. To see (iii), suppose (iv) $b \leq a$ and (v) $b \vee \neg b = 1$. We have that (iv) implies $Da \leq Db$ as seen in Lemma 9(ii) and (v) implies, using (DE), that $Db \leq \neg b$. So, by $\leq$ transitivity it follows that $Da \leq \neg b$. Then, in a residuated lattice we have $b \leq \neg Da$, as desired. ☐

Remark 2 *Given the conditions of Proposition 17, taking any of the coatoms in the algebra of Example 1, it is easy to check that the equation $Dx \approx \neg Bx$ does not hold. Also, the reciprocal of Proposition 17 is not the case, as the algebra in Example 2 satisfies the equation $\neg x \vee \neg\neg x \approx 1$ and B exists in that algebra, but D does not exist.*

Taking into account that De Morgan laws are satisfied in any MTL algebra, we can easily obtain the following consequence of the previous proposition.

Corollary 2 *Let $\mathbf{A}$ be a SMTL-algebra, i.e. an MTL algebra such that for all $a \in A$, $a \wedge \neg a = 0$. Then, for any $a \in A$, if Da exists, then so does Ba, and $Ba = \neg Da$.*

Lemma 10 *Let $\mathbf{A} \in \mathbb{RL}^{\mathbb{D}}$ and $a \in A$. Then, the following are equivalent:*

(i) *a is Boolean,*
(ii) *$\neg Da = a$,*
(iii) *$Da = \neg a$.*

Proof (i) $\Rightarrow$ (ii) Suppose $a \vee \neg a = 1$. Then, using (DE), $Da \leq \neg a$. Then, $a \leq \neg Da$. Now, by Lemma 9(i) we have $\neg Da \leq a$. So, $\neg Da = a$.

(ii) $\Rightarrow$ (iii) Suppose $\neg Da = a$. Then, $\neg\neg Da = \neg a$. As $Da \leq \neg\neg Da$, we have that $Da \leq \neg a$. Now, by (DI), $a \vee Da = 1$. So, also $\neg a \leq Da$. Then, $Da = \neg a$.

(iii) $\Rightarrow$ (i) Suppose $Da = \neg a$. As using (DI) we have $a \vee Da = 1$, it follows that $a \vee \neg a = 1$. ☐

As a direct consequence we have the following fact.

Corollary 3 *Let $\mathbf{A}$ be a residuated lattice where both B and D exist. Then, $Da \leq \neg Ba$, for all $a \in A$. Equivalently, $Ba \leq \neg Da$, for all $a \in A$.*

Proof Let $a \in A$. We have that $Ba \leq a$. Hence, using Lemma 9(ii), $Da \leq DBa$. Now, since Ba is Boolean, using Lemma 10 it follows that $Da \leq \neg Ba$. ☐

Remark 3 *The equality $B \approx \neg D$ does not hold. Indeed, consider the join-irreducible coatom c in the Heyting algebra in Figure 4, where $0 = Bc < \neg Dc$.*

Lemma 11 *Let $\mathbf{A}$ be a $\mathbb{RL}^{\mathbb{D}}$-algebra and $a, b \in A$. We have that if $b \leq a$ and $b \vee \neg b = 1$, then $b \leq \neg Da$.*

Proof Suppose $b \leq a$. Then, $\neg Db \leq \neg Da$. Now, using the hypothesis $b \vee \neg b = 1$ and Lemma 10, we have that $\neg Db = b$. So, $b \leq \neg Da$. ☐

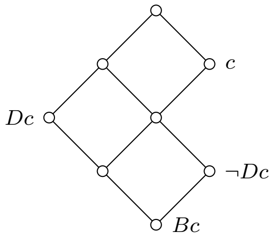

Fig. 4 Behaviour of B and D in a coatom of a residuated lattice

Proposition 18 *Let* $\mathbf{A}$ *be a* $\mathbb{RL}^{\mathbb{D}}$*-algebra, let* $a \in A$*, and let us have a finite number of elements* $b \in A$ *such that* $b \leq a$*. Then,* $Ba = (\neg D)^n a$*, for some* $n \in \mathbb{N}$*.*

Proof In the case a is Boolean, $Ba = (\neg D)^0 a$. In the case a is not Boolean, take $\neg Da$. Now, Lemma 11 says we will not be missing Boolean elements. Repeating the procedure we will find the first Boolean below a. □

Proposition 19 *(i) In every algebra of* $\mathbb{RL}^{\mathbb{D}_n}$*, for any natural number* $n \geq 0$*,* B *exists, with* $B = (\neg D)^n$*.*

(ii) There are $\mathbb{RL}^{\mathbb{D}}$*-algebras where* B *does not exist.*

Proof (i) It is enough to see that $(\neg Dx)^n$ satisfies (BE1), (BE2), and (BI). It always satifies (BE1), as it is easily seen by induction using $\neg Da \leq a$ and $\leq$-transitivity. We get (BE2) applying Lemma 10 on the hypothesis that $(\neg D)^{n+1} = (\neg D)^n$. Finally, to get (BI), suppose both (i) $b \leq a$ and (ii) $b \vee \neg b = 1$. From (i) it follows (iii) $\neg Db \leq \neg Da$. From (ii), using Lemma 10, we get (iv) $\neg Db = b$. From (iii) and (iv) we get $b \leq \neg Da$. Repeat the argument n times to get $b \leq (\neg D)^n a$.

(ii) cf. end of Section 2 in [5]. □

In the next proposition we will use the following De Morgan properties for $\neg$ and D. In the proof we use the abreviation $\{x_i\}$ for $\{x_i \in A : i \in I\}$.

Lemma 12 *Let* $\mathbf{A}$ *be a complete* $\mathbb{RL}^{\mathbb{D}}$*-algebra. Then, both (i)* $\neg \bigvee\{x_i\} = \bigwedge\{\neg x_i\}$ *and (ii)* $D\bigwedge\{x_i\} = \bigvee\{Dx_i\}$*.*

Proof We prove only (ii), (i) is already known. By (DI) we have that $x_j \vee Dx_j = 1$. Then, $x_j \vee \bigvee\{Dx_i\} = 1$ for all $j \in I$ and so $\bigwedge\{x_i\} \vee \bigvee\{Dx_i\} = 1$. So, by (DE) we obtain that $D\bigwedge\{x_i\} \leq \bigvee\{Dx_i\}$.

For the other inequality, using (DI′), we have that $\bigwedge\{x_i\} \vee D\bigwedge\{x_i\} = 1$. Now, from $\bigwedge\{x_i\} \leq x_j$ and $x_j \leq x_j \vee D\bigwedge\{x_i\}$ we obtain $\bigwedge\{x_i\} \leq x_j \vee D\bigwedge\{x_i\}$, and taking into account that $D\bigwedge\{x_i\} \leq x_j \vee D\bigwedge\{x_i\}$, we have that $\bigwedge\{x_i\} \vee D\bigwedge\{x_i\} \leq x_j \vee D(\bigwedge\{x_i\}$. Now, by (DI′) we obtain that $\{x_i\} \vee D\bigwedge\{x_i\} = 1$, which, by (DE), implies $Dx_j \leq D\bigwedge\{x_i\}$, for all $j \in I$. Thus, we finally get $\bigvee\{Dx_i\} \leq D\bigwedge\{x_i\}$. □

In the next proposition we use the notation $\mathbb{N}$ and $\mathbb{N}^+$, for the set of natural numbers including 0 and excluding 0, respectively.

Proposition 20 *Let* $\mathbf{A}$ *be a complete* $\mathbb{RL}^{\mathbb{D}}$*-algebra. Then,* Ba *exists, with* $Ba = \bigwedge\{(\neg D)^n a : n \in \mathbb{N}\}$*, for any* $a \in A$*.*

Proof Considering the definition of B, it is enough to prove, for $a \in A$, (i) $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} \leq x$, (ii) $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} \vee \neg \bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} = 1$, and (iii) if $b \leq a$ and $b \vee \neg b = 1$, then $b \leq \bigwedge\{(\neg D)^n a : n \in \mathbb{N}\}$. Now, (i) follows, because $a \in \{(\neg D)^n a : n \in \mathbb{N}\}$, as $a = (\neg D)^0 a$. Regarding (ii) and using Lemma 10, it is enough to prove that $\neg D\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} = \bigwedge\{(\neg D)^n a : n \in \mathbb{N}\}$. As it is always the case, for any $b \in A$, that $\neg Db \leq b$, it suffices to prove that $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} \leq \neg D\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\}$. Now, using both properties of Lemma 12, we have that the right hand side of the just given inequality is equal to $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}^+\}$. It is clear that $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} \leq \bigwedge\{(\neg D)^n a : n \in \mathbb{N}^+\}$, because $\bigwedge\{(\neg D)^n a : n \in \mathbb{N}\} \leq (\neg D)^m a$, for $m \in \mathbb{N}^+$. Regarding (iii), suppose (iv) $b \leq a$ and $b \vee \neg b = 1$, the last of which implies, by Lemma 10, that (v) $\neg Db = b$. In order to prove that $b \leq \bigwedge\{(\neg D)^n a : n \in \mathbb{N}\}$, it is enough to prove that $b \leq (\neg D)^n a$, for all $n \in \mathbb{N}$, which easily follows by induction, as $b \leq a = (\neg D)^0 a$, by (iv), and supposing that $b \leq (\neg D)^n a$, it follows, using Lemma 9, that $\neg Db \leq (\neg D)^{n+1} a$, and, using (v), $b \leq (\neg D)^{n+1} a$, as required. □

In Example 2 we saw that the existence of B in a residuated lattice does not force the existence of D. Now, let us see that operation Δ is stronger than B in this respect.

Proposition 21 *Let* $\mathbf{A} \in \mathbb{RL}$*. If* Δ *exists in* $\mathbf{A}$*, then also* D *exists in* $\mathbf{A}$*, with* $D = \neg\Delta$*. We also have* $\Delta = \neg D$*.*

Proof Suppose $\mathbf{A}$ is a residuated lattice where Δ exists. Then, also $\neg\Delta$ exists. We have to prove that $Da = \neg\Delta a$, for any $a \in A$. We have that $a \vee \neg\Delta a = 1$, as $\Delta a \vee \neg\Delta a = 1$ and $\Delta a \leq a$, it . Now, suppose $a \vee b = 1$. Then, $\Delta(a \vee b) = \Delta 1 = 1$. It is also the case that $\Delta(a \vee b) = \Delta a \vee \Delta b$. So, $\Delta a \vee \Delta b = 1$. Using Lemma 1(vi), $\neg\Delta a \leq \Delta b$. Moreover, $\Delta b \leq b$. So, $\neg\Delta a \leq b$.

Let us see that we also have $\Delta a = \neg Da$, for all a. From the first part it follows that $\neg Da = \neg\neg\Delta a$. However, from ($\Delta$E2), using Lemma 1(vi), it follows that $\neg\neg\Delta a \leq \Delta a$. As due to Lemma 1(vii) we have $\Delta a \leq \neg\neg\Delta a$, it follows that $\Delta a = \neg Da$. □

Remark 4 *The reciprocal of Proposition 21 is not the case, as* D *exists in Example 1, but* Δ *does not.*

Corollary 4 *Let $\mathbf{A} \in \mathbb{RL}$. If Δ exists in $\mathbf{A}$, then also B and D exist, and we have $\Delta = B = DD = \neg D$.*

Proof Considering Propositions 13 and 21, it is enough to prove that $\Delta = DD$, which is obtained checking that $\neg\Delta\neg\Delta = \Delta$. As for any $a \in A$, Δa is Boolean due to (ΔE2), using Lemma 8 it is enough to check that $\neg\neg\Delta a = \Delta a$, which follows again from (ΔE2) and Lemma 1(vi). ☐

5 The logic $\mathbf{FL}_{ew}^{B}$

In this section we introduce an expansion of FL_{ew} with a unary connective B, whose intended algebraic semantics is the variety of $\mathbb{RL}^{\mathbb{B}}$-algebras studied in the previous sections.

Indeed, we define FL_{ew}^{B} as the expansion of FL_{ew} with the following axiom schemas:

(B1) $B\varphi \to \varphi$,
(B2) $B\varphi \vee \neg B\varphi$,
(B3) $B(\varphi \vee \neg\varphi) \to (B\varphi \vee B\neg\varphi)$,
(B4) $B(\varphi \to \psi) \to (B\varphi \to B\psi)$.

and with the following additional rule:

(B) From φ derive $B\varphi$.

We will simply denote the notion of (finitary) derivation in FL_{ew}^{B} by $\vdash$, without danger of confusion.

Note that we have the following facts.

Lemma 13 *(i) $\vdash B\varphi \to BB\varphi$ and*
(ii) $B\varphi \to \psi \vdash B\varphi \to B\psi$.

Proof For (i) check the following derivation:

1.	$B\varphi \vee \neg B\varphi$	(B2)
2.	$B(B\varphi \vee \neg B\varphi)$	1, rule (B)
3.	$BB\varphi \vee B\neg B\varphi$	(B3), 2, mp
4.	$BB\varphi \to (B\varphi \to BB\varphi)$	FL_{ew}
5.	$B\neg B\varphi \to \neg B\varphi$	(B1)
6.	$\neg B\varphi \to (B\varphi \to BB\varphi)$	FL_{ew}
7.	$B\neg B\varphi \to (B\varphi \to BB\varphi)$	5, 6, FL_{ew}
8.	$B\alpha \to BB\alpha$	3, 4, 7, FL_{ew}

(ii) follows easily using (i). ☐

Clearly, FL_{ew}^{B} is a *Rasiowa implicative* logic (cf. [16]). Then, it follows that it is algebraizable in the sense of Blok and Pigozzi [3]. It is also straightforward to check that its equivalent algebraic semantics is in fact the variety of $\mathbb{RL}^{B}$-algebras. Algebraizability immediately implies strong completeness of FL_{ew}^{B} with respect to $\mathbb{RL}^{B}$.

Theorem 2 *For every set $\Gamma \cup \{\varphi\}$ of formulas, $\Gamma \vdash \varphi$ iff for every $\boldsymbol{A} \in \mathbb{RL}B$ and every $\boldsymbol{A}$-evaluation e, $e(\varphi) = 1$, whenever $e[\Gamma] \subseteq \{1\}$.*

In FL_{ew}^{B} the usual form of the deduction theorem does not hold. Indeed, we may have that $\varphi \vdash B\varphi$, but $\nvdash \varphi \to B\varphi$, as can be easily seen to fail in the three-element Gödel algebra $\{0, \frac{1}{2}, 1\}$, where $B(1) = 1$ and $B(\frac{1}{2}) = B(0) = 0$: for any evaluation e in this algebra, if $e(\varphi) = 1$, then $e(B\varphi) = 1$, but for $e(\varphi) = \frac{1}{2}$ we have $e(B\varphi) = 0$, and thus $e(\varphi \to B\varphi) = 0$.

Actually, FL_{ew}^{B} enjoys the same form of deduction theorem holding for logics with the Δ operator (cf. [2, Proposition 2.2]).

Theorem 3 *$\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash B\varphi \to \psi$.*

Proof $\Rightarrow$) We prove by induction on every formula χ_i ($1 \leq i \leq n$) of the given derivation of ψ from $\Gamma \cup \{\varphi\}$ that $\Gamma \vdash B\varphi \to \chi_i$. If $\chi_i = \varphi$, then the result follows due to axiom schema (B1). If χ_i belongs to Γ or is an instance of an axiom, then the result follows using *modus ponens* and the derivability of the schema $\chi_i \to (B\varphi \to \chi_i)$. If χ_i comes by application of *modus ponens* on previous formulas in the derivation, then the result follows, because from $B\varphi \to \chi_k$ and $B\varphi \to (\chi_k \to \chi_i)$ we may derive $(B\varphi \& B\varphi) \to (\chi_k \& (\chi_k \to \chi_i))$ and then also $B\varphi \to \chi_i$, using transitivity of $\to$ applied to the derivable formulas $B\varphi \to (B\varphi \& B\varphi)$ and $(\chi_k \& (\chi_k \to \chi_i)) \to \chi_i$. Finally, if $\chi_i = B\chi_k$ comes using rule (B) from formula χ_k, then from $B\varphi \to \chi_k$ we may derive $B\varphi \to B\chi_k$ using Lemma 13(ii).

$\Leftarrow$) To the derivation given by the hypothesis add a step with φ. In the next step put $B\varphi$, which follows from the previous formula using rule (B). Finally, derive ψ using *modus ponens*. ☐

Thanks to this B-deduction theorem, the logic FL_{ew}^{B} has the following property: if we expand FL_{ew}^{B} with any further rule $\varphi_1, \ldots, \varphi_n/\alpha$, then it is possible to dispose of the rule just adding the axiom $(B\varphi_1 \wedge \cdots \wedge B\varphi_n) \to \varphi$. This property is also the case for the logics $\mathrm{FL}_{ew}^{\Delta}$ and FL_{ew}^{D}.

Proposition 22 *FL_{ew}^{B} is a conservative expansion of FL_{ew}.*

Proof Use Proposition 4 and the Finite Model Property of FL_{ew} (see [14]). ☐

One could analogously define the expansion of MTL (which is in turn the extension of FL_{ew} with the prelinearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$) with B, with the same additional axioms and rule, yielding the logic MTL^{B}, which is again algebraizable and strongly complete with respect to variety $\mathbb{MTL}^{B}$ of MTL^{B}-algebras.

However, unlike the case of expansion with Δ, MTL^B is not a semilinear logic, that is, it is not complete with respect to the class of MTL^B-chains. The reason for this, as it is easily seen in the example of Figure 1, is that the $\vee$-form of rule (B), "from $\psi \vee \varphi$ derive $\psi \vee B\varphi$", is not derivable in MTL^B. Indeed, taking elements b and in the five Gödel algebra of the example it is clear that $B(b) = B(c) = 0$ while $B(b \vee c) = B(1) = 1$.

As a final result, we can show that FL$^B_{ew}$ inherits from FL$_{ew}$ the Finite Model Property (FMP). Before proving this, we introduce some preliminary notation.

For a logic $L \in \{Fl_{ew}$ or $FL^B_{ew}\}$, let us denote by $Fm(L, Var)$ the set of L-formulas built from a set Var of propositional variables. Now let us define the enlarged set of propositional variables $Var^* = Var \cup \{\text{“}B\varphi\text{”} \mid B\varphi \in Fm(FL^B_{ew}, Var)\}$, where "$B\varphi$" is intended to denote a fresh propositional variable, one for each formula $B\varphi \in Fm(FL^B_{ew}, Var)$. Then, we can define a one-to-one translation of every formula $\varphi \in Fm(FL^B_{ew}, Var)$ into a formula $\varphi^* \in Fm(FL_{ew}, Var^*)$, by just inductively defining:

- $0^* = 0$,
- if $\varphi = p \in Var$, then $\varphi^* = p$,
- if $\varphi = B\psi$, then $\varphi^* = \text{“}B\psi\text{”}$,
- if $\varphi = \psi \odot \chi$, then $\varphi^* = \psi^* \odot \chi^*$, for $\odot \in \{\wedge, \vee, \&, \rightarrow\}$.

If Γ is a set of formulas, we will write $\Gamma^* = \{\varphi^* \mid \varphi \in \Gamma\}$. Note that for any $\psi \in Fm(FL_{ew}, Var^*)$, there is a formula $\varphi \in Fm(FL^B_{ew}, Var)$ such that $\varphi^* = \psi$.

Moreover, we need the following result that will allow us to reduce proofs in FL^B_{ew} to proofs in FL_{ew}.

Lemma 14 *Let T be the set of all instances of axioms of FL^B_{ew}. For each set $\Gamma \cup \{\varphi\} \subseteq Fm(FL^B_{ew}, Var)$, it holds that*

$$\Gamma \vdash_{FL^B_{ew}} \varphi \textit{ iff } \Gamma^* \cup Cg^* \cup T^* \vdash_{FL_{ew}} \varphi^*,$$

where $Cg = \{B\varphi \leftrightarrow B\psi \mid \Gamma \vdash_{FL^B_{ew}} \varphi \leftrightarrow \psi\}$.

The proof is quite straightforward and analogous to those of similar results that can be found in the literature in slightly different contexts.

Theorem 4 *FL^B_{ew} enjoys the FMP, that is, if $\Gamma \nvdash_{FL^B_{ew}} \varphi$, then there is a finite $\mathbf{A} \in \mathbb{RL}^{\mathbb{B}}$ and an $\mathbf{A}$-evaluation e such that $e(\Gamma) = 1$ and $e(\varphi) < 1$.*

Proof If $\Gamma \nvdash_{FL^B_{ew}} \varphi$, by Lemma 14, it holds that $\Gamma^* \cup Cg^* \cup T^* \nvdash_{FL_{ew}} \varphi^*$, and by strong completeness and FMP of FL_{ew}, there is a finite algebra $\mathbf{C} \in \mathbb{RL}$ and $\mathbf{C}$-evaluation v such that $v(\Gamma^* \cup Cg^* \cup T^*) = 1$ and $v(\varphi^*) < 1$. Then, the result will follow from the following facts:

Claim 1: $G = \{v(\text{“}B\varphi\text{”}) \mid B\varphi \in Fm(FL^B_{ew}, Var)\}$ is a set of Boolean elements of $\mathbf{C}$.

Proof of the claim: It is enough to check that $v((B\varphi)^*) \vee \neg v((B\varphi)^*) = v((B\varphi)^* \vee \neg(B\varphi)^*) = v((B\varphi \vee \neg B\varphi)^*) = 1$, where the latter holds because $B\varphi \vee \neg B\varphi$ is the axiom (BE2) of FL^B_{ew}. ⊣

Claim 2: Let $\mathbf{A}$ be the $\mathbb{RL}$-algebra generated by the set $X = \{v(\varphi) \mid \varphi \in Fm(FL_{ew}, Var^*)\}$, which is finite since $\mathbf{A}$ is a subalgebra of $\mathbf{C}$. Then, B exists in $\mathbf{A}$ and $B(\mathbf{A}) = G$. Therefore, $\mathbf{A}$ is indeed an $\mathbb{RL}^{\mathbb{B}}$-algebra.

Proof of the claim: That $\mathbf{A}$ is finite is obvious, and thus, by Proposition 5, B exists. On the other hand, the elements of G keep being Boolean in $\mathbf{A}$. Hence, the only missing thing to check is that any Boolean element of $\mathbf{A}$ already belongs to G. This is also clear since Boolean elements are closed by propositional combinations with connectives. ⊣

Claim 3: Let us define the $\mathbf{A}$-evaluation (taking $\mathbf{A}$ as $\mathbb{RL}^{\mathbb{B}}$-algebra) $e : Var \rightarrow A$ defined by $e(p) = v(p)$. Then, for any φ, $e(\varphi) = v(\varphi^*)$, in particular, $e(B\varphi) = v(\text{“}B\varphi\text{”})$.

Proof of the claim: We prove that $e(\varphi) = v(\varphi^*)$ by structural induction.

- if φ is a propositional variable, it holds by construction
- if $\varphi = \psi \odot \chi$ for $\odot \in \{\wedge, \vee, \&, \rightarrow\}$, by induction hypothesis we have $e(\psi) = v(\psi^*)$ and $e(\chi) = v(\chi^*)$, and hence $e(\varphi) = e(\psi \odot \psi) = e(\psi) \odot e(\chi) = v(\psi^*) \odot v(\chi^*) = v(\psi^* \odot \chi^*) = v((\psi \odot \chi)^*) = v(\varphi^*)$.
- If $\varphi = B\psi$, then we have to prove that $v(\text{“}B\psi\text{”}) = B(e(\psi))$, the latter being equal to $e(B\psi)$ by definition. Therefore, we have to prove in turn that the three defining conditions (BE1), (BE2), and (BEI) are satisfied by $v(\text{“}B\psi\text{”}) = v((B\psi)^*)$ to be the greatest Boolean below $e(\psi)$, assuming by induction that $v(\psi^*) = e(\psi)$.

(BE1) Since $B\psi \rightarrow \psi$ is axiom (BE1) of FL^B_{ew}, we have that $1 = v((B\psi \rightarrow \psi)^*) = v((B\psi)^*) \rightarrow v(\psi^*) = v((B\psi)^*) \rightarrow e(\psi)$. Hence, $v((B\psi)^*) \leq e(\psi)$.

(BE2) is clear from Claim 1.

(BI) We have to check that if $b \in B(A)$ is such that $b \leq e(\psi) = v(\psi^*)$, then $b \leq v((B\psi)^*)$. If $b \in B(A)$, by construction of $\mathbf{A}$, then there exists a formula χ such that $b = v((B\chi)^*)$. On the other hand, by (ii) of Lemma 12, we know that $B\chi \rightarrow \psi, B\chi \vee \neg B\chi \vdash B\chi \rightarrow B\psi$. Thus, we also know that if $v((B\chi)^*) \leq v(\psi)^*$ and $v((B\chi)^*) \vee \neg v((B\chi)^*) = 1$, then $v((B\chi)^* \leq v((B\psi)^*)$. Now, the two conditions are satisfied, hence we have $b = v((B\chi)^* \leq v((B\psi)^*)$.

This closes the proof of Claim 3. $\dashv$

Finally, from these claims it readily follows that $e(\Gamma) = v(\Gamma^*) = 1$ and $e(\varphi) = v(\varphi^*) < 1$, as required. $\square$

6 Conclusions and dedication

In this paper we have conisered the expansion of FL_{ew} with the operator B, that in algebraic terms provides the greatest Boolean below a given element of a residuated lattice. Among other things, we have axiomatized it and shown that the resulting logic is a conservative expansion enjoying the Finite Model Property. The axioms for B turn out to be very close to those of the Monteiro-Baaz Δ operator, in fact only one axiom is a weaker version of the one for Δ. Even if the properties are very similar, that small difference causes, e.g. that in the context of MTL, the expansion with B is not any longer a semilinear logic, in contrast to the expansion with Δ.

As a matter of fact, we have chosen this topic for our humble contribution to honour the memory of our beloved and late friend Franco Montagna, because it was suggested by Franco to the first author during the preparation of their joint manuscript [1], together with Amidei, where they study the expansion of FL_{ew} and other substructural logics with Δ.

Compliance with Ethical Standards

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