A representation theorem for integral rigs and its applications to residuated lattices

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Rigs and really local rigs

Definition

A *rig* is a structure $(A, \cdot, 1, +, 0)$ such that $(A, \cdot, 1)$ and $(A, +, 0)$ are commutative monoids and distributivity holds in the sense that $a \cdot 0 = 0$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in A$.

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Let $\mathcal{E}$ be a category with finite limits. For any rig A in $\mathcal{E}$ we define the subobject $\mathrm{Inv}(A) \to A \times A$ by declaring that the diagram below

$$\begin{array}{ccc} \mathrm{Inv}(A) & \xrightarrow{\;!\;} & 1 \\ \downarrow & & \downarrow{\scriptstyle 1} \\ A \times A & \xrightarrow[\cdot]{} & A \end{array}$$

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is a pullback. The two projections $\mathrm{Inv}(A) \to A$ are mono in $\mathcal{E}$ and induce the same subobject of A.

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A rig morphism $f : A \to B$ between rigs in $\mathcal{E}$ is *local* if the following diagram

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If $\mathcal{E}$ is a topos with subobject classifier $\top : 1 \to \Omega$ then there exists a unique map $\iota : A \to \Omega$ such that the square below

$$\begin{array}{ccc} \mathrm{Inv}(A) & \xrightarrow{!} & 1 \\ \downarrow & & \downarrow \top \\ A & \xrightarrow[\iota]{} & \Omega \end{array}$$

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Lemma

The rig A is really local if and only if the following sequents hold

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\begin{array}{rcl}
0 \in \mathrm{Inv}(A) & \vdash & \bot \\
(x+y) \in \mathrm{Inv}(A) & \vdash_{x,y} & x \in \mathrm{Inv}(A) \quad \vee \quad y \in \mathrm{Inv}(A) \\
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\end{array}
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A functor $F : D^{op} \to \mathbf{Set}$ is an integral rig in $\mathbf{Shv}(D)$ if and only if:

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i) *F is a sheaf respect to the coherent topology.*

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iii) *If $c \leq d$ in D, then $F(d) \to F(c)$ is a morphism of integral rigs.*

Really local integral rigs in $\mathbf{Shv}(D)$

Lemma

An integral rig F in $\mathbf{Shv}(D)$ is really local if and only if, the equalizer of the arrows 0 *and* 1 *is the initial object; and the morphism induced by the coproduct* $r : F + F \to [x + y = 1]$ *is an epimorphism.*

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A more explicit characterization follows:

Lemma

*A sheaf F in **Shv**(D) is really local if an only if:*

i) *For every $d \in D$ and $s, t \in F(d)$ such that $s + t = 1$, there exists $u, v \leq d$ with $u \vee v = d$, such that $s \cdot v = 1_v$ and $t \cdot u = 1_u$.*

ii) *$F(d) = \mathbf{1}$ if and only if $d = 0$.*

Reticulation of an integral rig

Let A and integral rig in **Set** and $x, y \in A$. Define:

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Lemma (Reticulation)

If A is an integral rig the relation $\sim$ is a rig congruence and the quotient $\eta_A : A \to A/{\sim}$ is universal from A to the inclusion **dLat** $\to$ **iRig**. *Moreover, the map $\eta_A : A \to A/{\sim}$ is local.*

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Denote the resulting left adjoint by L : **iRig** $\to$ **dLat** and the associated unit by $\eta_A = \eta : A \to LA$. This unit and its codomain LA may be referred to as the *reticulation* of the rig A.

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Let $F \to A$ a multiplicative submonoid and $x, y \in A$. Define:

$$x \,|_F\, y \quad \text{if and only if} \quad \exists_{w \in F}, \quad wx \leq y$$

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Observe that $|_F$ is a pre-order.

Lemma (Localizations)

If A is integral and $F \to A$ is a multiplicative submonoid then the equivalence relation $\equiv_F$ determined by the pre-order $|_F$ is a congruence and the quotient $A \to A/{\equiv_F}$ has the universal property of $A \to A[F^{-1}]$.

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Lemma (**Pullback-Pushout Lemma**)

Let A an integral rig and $a, b \in A$. The following diagram is a Pushout and also a Pullback in **iRig**.

$$\begin{array}{ccc} A[(a+b)^{-1}] & \longrightarrow & A[a^{-1}] \\ \downarrow & & \downarrow \\ A[b^{-1}] & \longrightarrow & A[(ab)^{-1}] \end{array}$$

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Let $\eta : A \to LA$ the reticulation of A. The assignment $\eta x \mapsto A[x^{-1}]$ defines a presheaf $\overline{A} : LA^{op} \longrightarrow$ **Set** such that

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Proposition

The presheaf $\overline{A}$ is really local in ***Shv***(LA).

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Moreover, in case the above holds, the map $X \to \Lambda$ is unique.

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- A morphism $f : \Lambda_D \to f_*\Lambda_C$ in $\mathbf{Shv}(D)$, such that, for every $d \in D$,

$$f_d : (\downarrow d) \to (f_*\Lambda_C)_d = (\downarrow f(d))$$

is defined as $f_d(x) = f(x)$, for every $x \in (\downarrow d)$.

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A really local representation (of an integral rig) is a pair (D, P) consisting of a bounded distributive lattice D and an integral rig P in $\mathbf{Shv}(D)$ satisfying the equivalent conditions of Lemma.

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$$\begin{array}{ccc} P & \xrightarrow{\;g\;} & f_*(Q) \\ {\scriptstyle \phi_P}\big\downarrow & & \big\downarrow{\scriptstyle f_*(\phi_Q)} \\ \Lambda_D & \xrightarrow[\;f\;]{} & f_*(\Lambda_C) \end{array}$$

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- There exists a unique $\overline{f} : \overline{A} \to Lf_*(\overline{B})$ in $\mathbf{Shv}(LA)$ such that the lower diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \downarrow \\ A[a^{-1}] & \xrightarrow{\overline{f}_{\overline{a}}} & B[f(a)^{-1}] \end{array}$$

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- $\mathcal{R}(f) = (Lf, \overline{f})$ for a morphism of integral rigs $f : A \to B$.

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Theorem (Really Local Representation)

The functor $\Gamma : \mathfrak{I} \longrightarrow \mathbf{iRig}$ *has a full and faithful left adjoint.*

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$$\sigma(a) = \{p \in \sigma(D) \mid p(a) = \top\}$$

for every $a \in D$.

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Let D a bounded distributive lattice. The spectrum of D ($Spec(D)$) is the topological space consisting of the following data:

- Points ($\sigma(D)$): Lattice morphisms $p : D \to 2$.
- Open basic sets:

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Theorem (Classical)

For every bounded distributive lattice D , $\mathbf{LH}/Spec(D) \cong \boldsymbol{Shv}(D)$.

The spectrum of an integral rig

Let $\eta : A \to LA$ the reticulation of A. Observe that there is a bijection $\mathbf{dLat}(LA, 2) \to \mathbf{iRig}(A, 2)$.

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For every integral rig A, $\mathbf{Shv}(LA) \cong \mathbf{LH}/Spec(A)$.

Fibers of the associated sheaf

Observe that, the fiber of the representing sheaf $\overline{A} \in \mathbf{Shv}(LA)$ of A over a point $p : A \to 2$ is

$$(\overline{A})_p = \varinjlim_{px=\top} \overline{A}(\eta x) = \varinjlim_{px=\top} A[x^{-1}]$$

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Remark (Fibers of $\overline{A}$)

Regarding $\overline{A} \in \mathbf{Shv}(LA)$ as a local homeo over $Spec(A)$, implies that the fiber over a point $p : A \to 2$ in $Spec(A)$ coincides with the localization of A at the multiplicative submonoid $p^{-1}(\top) \to A$.

Some Corollaries

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Every integral rig may be represented as the algebra of global sections of a local homeo (over the spectral space $Spec(A)$) whose fibers are really local integral rigs.

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Every integral rig is a subdirect product of really local integral rigs.

Definition

An *MV-rig* is an integral residuated rig $(A, \cdot, 1, +, 0, \multimap)$ such that the following (Wajsberg) condition:

$$(x \multimap y) \multimap y = (y \multimap x) \multimap x$$

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Let **mvRig** the algebraic category of MV-rigs over **Set** and **MV** the category of MV-algebras.

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Let **mvRig** the algebraic category of MV-rigs over **Set** and **MV** the category of MV-algebras.

mvRig are **MV** equivalent.

The representation of MV-algebras

Every MV-algebra M has an associated topological space, whose set of points is given by Z_M and whose topology is determined by the basic open sets of the form $W_a = \{P \in Z_M \mid a \in P\}$, for every $a \in M$. Such space is noted by $Spec_M$.

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$$I_p = \neg(p^{-1}(\top)) = \{\neg x \mid p(x) = \top\} \to A$$

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The spaces $Spec(R)$ y $Spec_M$ are homeomorphic.

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$$(\varphi^* E_M)_p = (E_M)_{\varphi(p)} = M/(I_p) \cong R[Q^{-1}] = \widehat{R}_Q, \quad \text{with} \quad Q = p^{-1}(\top)$$

where $\widehat{R}_Q$ is the fiber of the representation for integral rigs.

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