SYSMICS 2016
Universitat De Barcelona
5-9 September 2016

Introduction

Outline of the Talk

In this talk, I wish to demonstrate the significant role of lattice-ordered groups (ℓ-groups) in the study of algebras of logic by focusing on two aspects of their multifaceted influence.

- First, I discuss the role ℓ-groups play in the definition of well-studied classes of ordered algebras.
- Second, I review recent research on residuated lattices that has been inspired by related research in the theory of ℓ-groups.

The Equational Consequence Relation: The Interplay of Algebra and Logic

- $\mathbb{X}$: a fixed countably infinite set of variables
- $\mathcal{L}$: a fixed signature of algebras
- $\mathbf{Fm}(\mathbb{X})$: the term (formula) algebra of signature $\mathcal{L}$ over $\mathbb{X}$
- $Eq(\mathbb{X}) = Fm(\mathbb{X}) \times Fm(\mathbb{X})$: the equations of signature $\mathcal{L}$ with variables in $\mathbb{X}$

Let $\mathcal{U}$ be a class of algebras of signature $\mathcal{L}$. Given $\Sigma \cup \{\varepsilon\} \subseteq Eq(\mathbb{X})$, we say that ε is a $\mathcal{U}$-consequence of Σ provided for every $\mathbf{A} \in \mathcal{U}$ and every homomorphism $\varphi\colon \mathbf{Fm}(\mathbb{X}) \to \mathbf{A}$, if $\Sigma \subseteq \mathrm{Ker}(\varphi)$, then $\varepsilon \in \mathrm{Ker}(\varphi)$.

Lattice-Ordered Groups

A lattice-ordered group (ℓ-group) is an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, \mathrm{e}\rangle$ such that

(i) $\langle G, \wedge, \vee\rangle$ is a lattice;
(ii)$\langle G, \cdot, ^{-1}, \mathrm{e}\rangle$ is a group; and
(iii) multiplication is isotone.

Examples

- $\mathrm{Aut}(\Omega)$ (order-automorphisms of a chain Ω)

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- $\mathrm{Aut}(\Omega)$ (order-automorphisms of a chain Ω)

Holland's Embedding Theorem
Every ℓ-group can be embedded into some $\mathrm{Aut}(\Omega)$.

Residuated Lattices

A residuated lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ such that:

(i) $\langle A, \wedge, \vee\rangle$ is a lattice;
(ii) $\langle A, \cdot, \mathrm{e}\rangle$ is a monoid; and
(iii) the operation $\cdot$ is residuated with residuals $\backslash$ and $/$. This means that, for all $x, y, z \in A$,

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z.$$

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An algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}, \mathrm{f}\rangle$ is said to be a pointed residuated lattice (or an FL algebra) provided: (i) $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, \mathrm{e}\rangle$ is a residuated lattice; and (ii) f is a distinguished element of A.

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The classes $\mathcal{RL}$ (residuated lattices) and $\mathcal{PRL}$ (pointed residuated lattices) are finitely based equational classes. Their defining equations consist of the defining equations for lattices and monoids together with the equations below.

$$\begin{array}{llll}
(\mathrm{RL1}) & x(y \vee z) \approx xy \vee xz & (\mathrm{RL2}) & (y \vee z)x \approx yx \vee zx \\
(\mathrm{RL3}) & x\backslash y \leq x\backslash(y \vee z) & (\mathrm{RL4}) & y/x \leq (y \vee z)/x \\
(\mathrm{RL5}) & x(x\backslash y) \leq y \leq x\backslash xy & (\mathrm{RL6}) & (y/x)x \leq y \leq yx/x
\end{array}$$

Examples I

The variety of ℓ-groups is term equivalent to the subvariety, $\mathcal{LG}$, of $\mathcal{RL}$ defined by the equation $x(x\backslash \mathrm{e}) \approx \mathrm{e}$. The term equivalence is given by

$$x^{-1} = x\backslash \mathrm{e} \text{ and } \mathrm{x/y} = \mathrm{xy}^{-1},\ \mathrm{y}\backslash\mathrm{x} = \mathrm{y}^{-1}\mathrm{x}.$$

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The negative cone of a residuated lattice $\mathbf{L}$ is the residuated lattice $\mathbf{L}^-$ with universe $L^- = \{x \in L \colon x \leq \mathrm{e}\}$, whose monoid and lattice operations are the restrictions to L^- of the corresponding operations in $\mathbf{L}$, and whose residuals $\backslash_-$ and $/_-$ are defined by

$$x\backslash_- y = (x\backslash y) \wedge \mathrm{e} \quad \text{and} \quad \mathrm{y/_-x = (y/x)} \wedge \mathrm{e},$$

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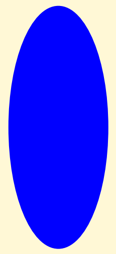
The negative cone of an ℓ-group is an integral cancellative residuated lattice. Further, it satisfies the divisibility laws

$$x(x\backslash y) \approx x \wedge y \approx (y/x)x.$$

Examples II

Every lattice is a sublattice of a cancellative residuated lattice. Let $\mathbf{L}$ be an arbitrary lattice with a top element, and let $\mathbf{L}^*$ be the free monoid over L. We order $\mathbf{L}^*$ as follows:



$W_1 = L$

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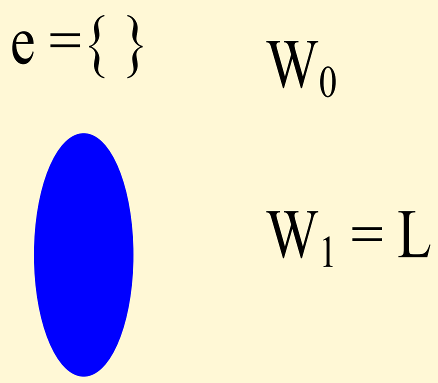

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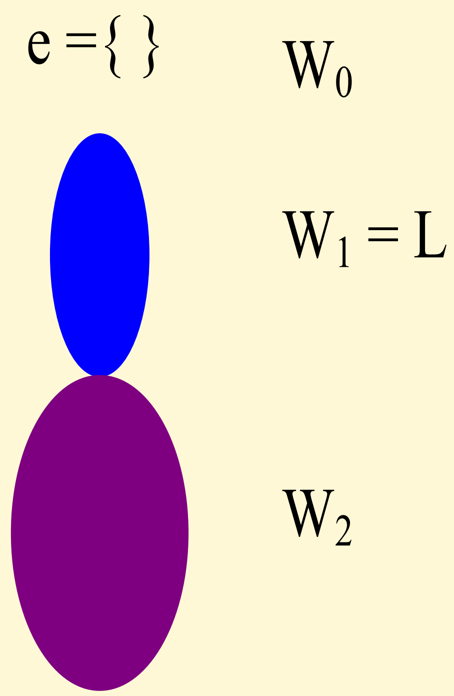

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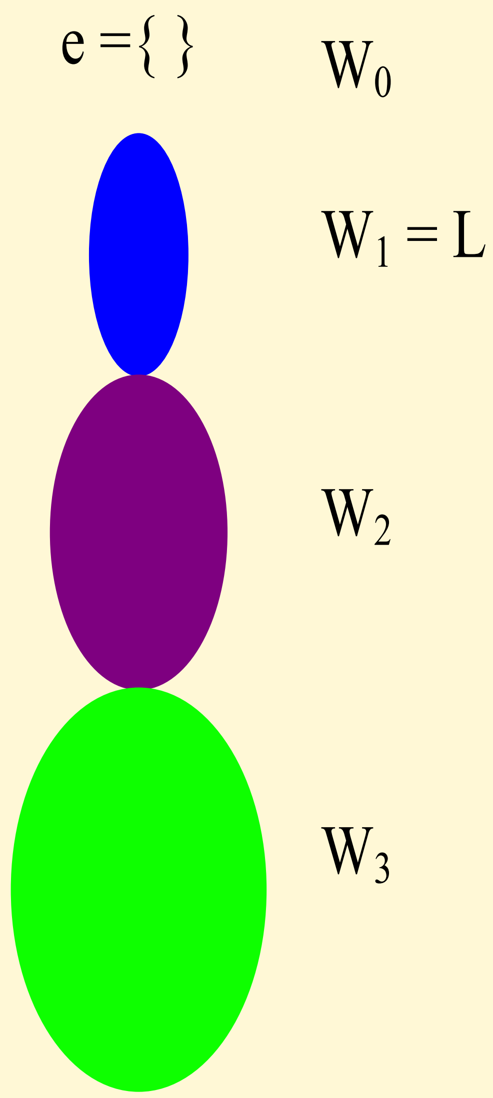

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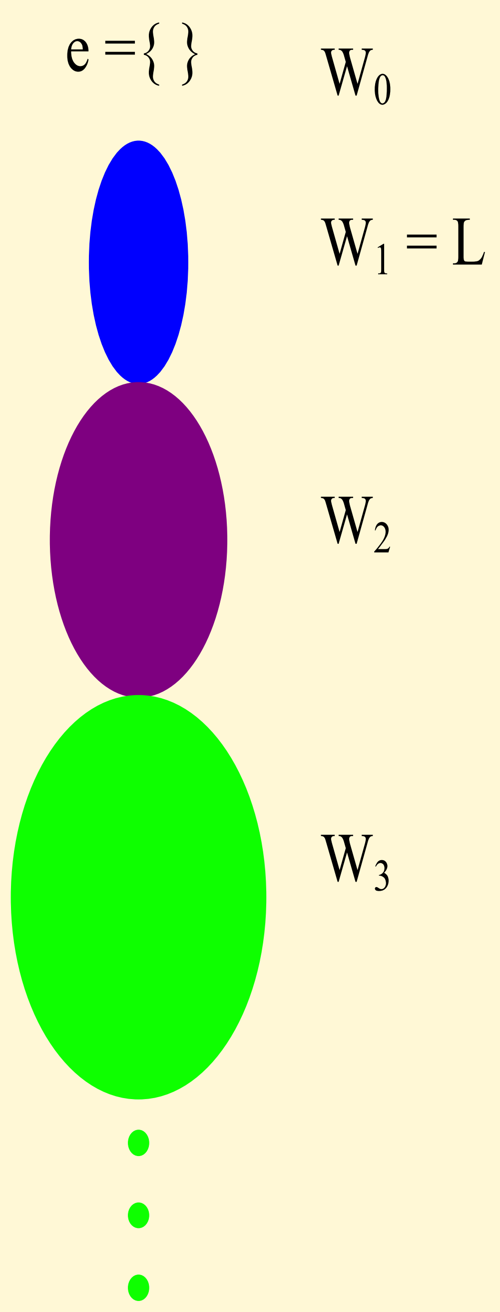

Examples III

Richard Dedekind

Robert P. Dilworth

Garrett Birkhoff

Ideal lattices of rings: $I \cdot J = \{\sum_{k=1}^{n} a_k b_k | a_k \in I; b_k \in J; n \in \mathbb{Z}^+\}$

Notation: If $x \backslash y = y / x$, we write $x \to y$ for the common value.

Heyting algebras: $xy \approx x \wedge y$ and $x \wedge \mathrm{f} \approx \mathrm{f}$.

Boolean algebras: $xy \approx x \wedge y$, $(x \to y) \to y \approx x \vee y$ and $x \wedge \mathrm{f} \approx \mathrm{f}$.

MV algebras: $xy \approx yx$, $(x \to y) \to y \approx x \vee y$ and $x \wedge \mathrm{f} \approx \mathrm{f}$.

ΨMV algebras (pseudo-MV algebras): $y/(x \backslash y) \approx x \vee y$, $(y/x) \backslash y \approx x \vee y$, and $x \wedge \mathrm{f} \approx \mathrm{f}$.

Categorical Equivalences

GBL and GMV algebras

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Lemma: $\mathcal{GMV} \subseteq \mathcal{GBL}$

Theorem: A residuated lattice is $\mathbf{L}$ is a GBL (respectively, GMV) algebra if and only if it has a direct sum decomposition $\mathbf{L} = \mathbf{A} \bigoplus \mathbf{B}$, where $\mathbf{A}$ is an ℓ-group and $\mathbf{B}$ is an integral GBL (respectively, GMV) algebra.

A nucleus on a residuated lattice $\mathbf{L}$ is a closure operator γ on $\langle L, \leq \rangle$ that satisfies the inequality $\gamma(a)\gamma(b) \leq \gamma(ab)$, for all $a, b \in L$.

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Proposition

Let γ be a nucleus on a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \cdot, /, \mathrm{e} \rangle$. Then the structure $\gamma[\mathbf{L}] = \langle \gamma[L], \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(\mathrm{e}) \rangle$ – where $x \vee_\gamma y = \gamma(x \vee y)$ and $x \circ_\gamma y = \gamma(xy)$, for all $x, y \in \gamma[L]$, is a residuated lattice.

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Proposition

If $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, \mathrm{e} \rangle$ is a residuated lattice and η a co-nucleus on it, then the structure $\eta[\mathbf{L}] = \langle \eta[L], \wedge_\eta, \vee, \cdot, \backslash_\eta, /_\eta, \mathrm{e} \rangle$ – where $x \wedge_\eta y = \eta(x \wedge y)$, $x /_\eta y = \eta(x/y)$ and $x \backslash_\eta y = \eta(x \backslash y)$, for all $x, y \in \eta[L]$ – is a residuated lattice.

GMV Algebras and Cancellative Residuated Lattices

Every integral GMV algebra may be viewed as the negative cone of an ℓ-group endowed with a suitable nucleus (namely one whose image generates the negative cone as semigroup).

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The preceding result implies the categorical equivalence between MV algebras and unital commutative ℓ-groups (D. Mundici; 1986), as well as the one between ΨMV algebras and unital ℓ-groups (A. Dvurečenskij; 2002).

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Any cancellative residuated lattice $\mathbf{L}$ whose monoid reduct is a right reversible monoid (Ore residuated lattice) may be viewed as an ℓ-group endowed with a suitable co-nucleus. In more detail, if $\mathbf{L}$ is an Ore residuated lattice and $\mathbf{G}$ is the ℓ-group of left fractions of $\mathbf{L}$, then the map $\eta\colon a^{-1}b \mapsto a\backslash b$ is a co-nucleus on $\mathbf{G}(\mathbf{L})$ and $\mathbf{L} = \eta[\mathbf{G}(\mathbf{L})]$.

- F. Montagna and C. Tsinakis, Ordered groups with a co-nucleus, Journal of Pure and Applied Algebra 214 (1) (2010), 71-88.

The Structure of the Lattice of Convex Subuniverses

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Key Properties for a Manageable Structure Theory

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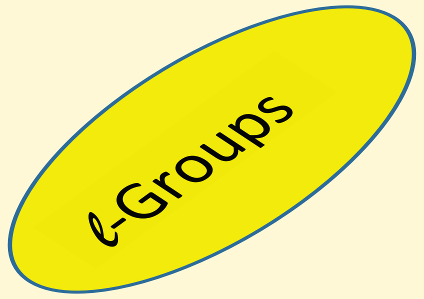
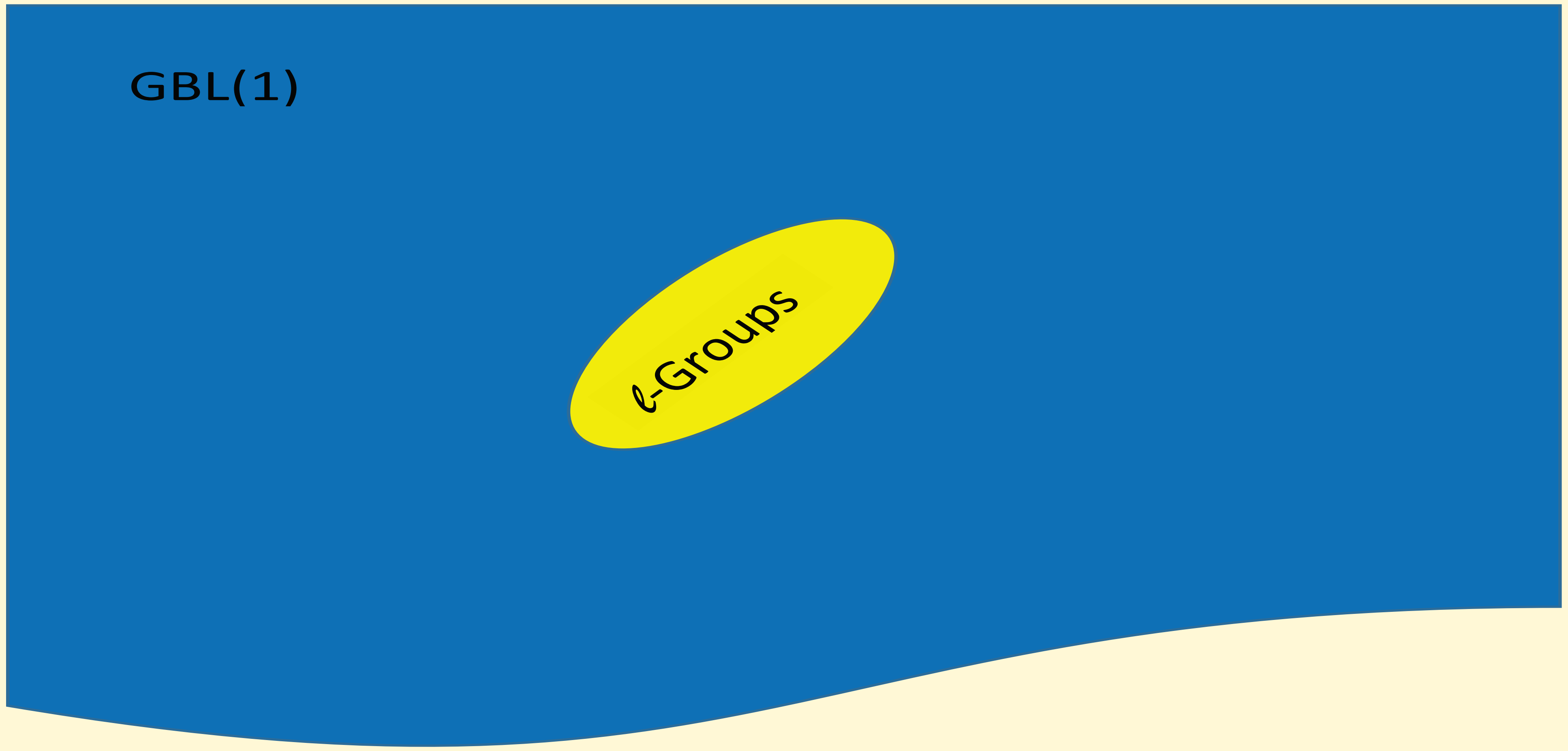


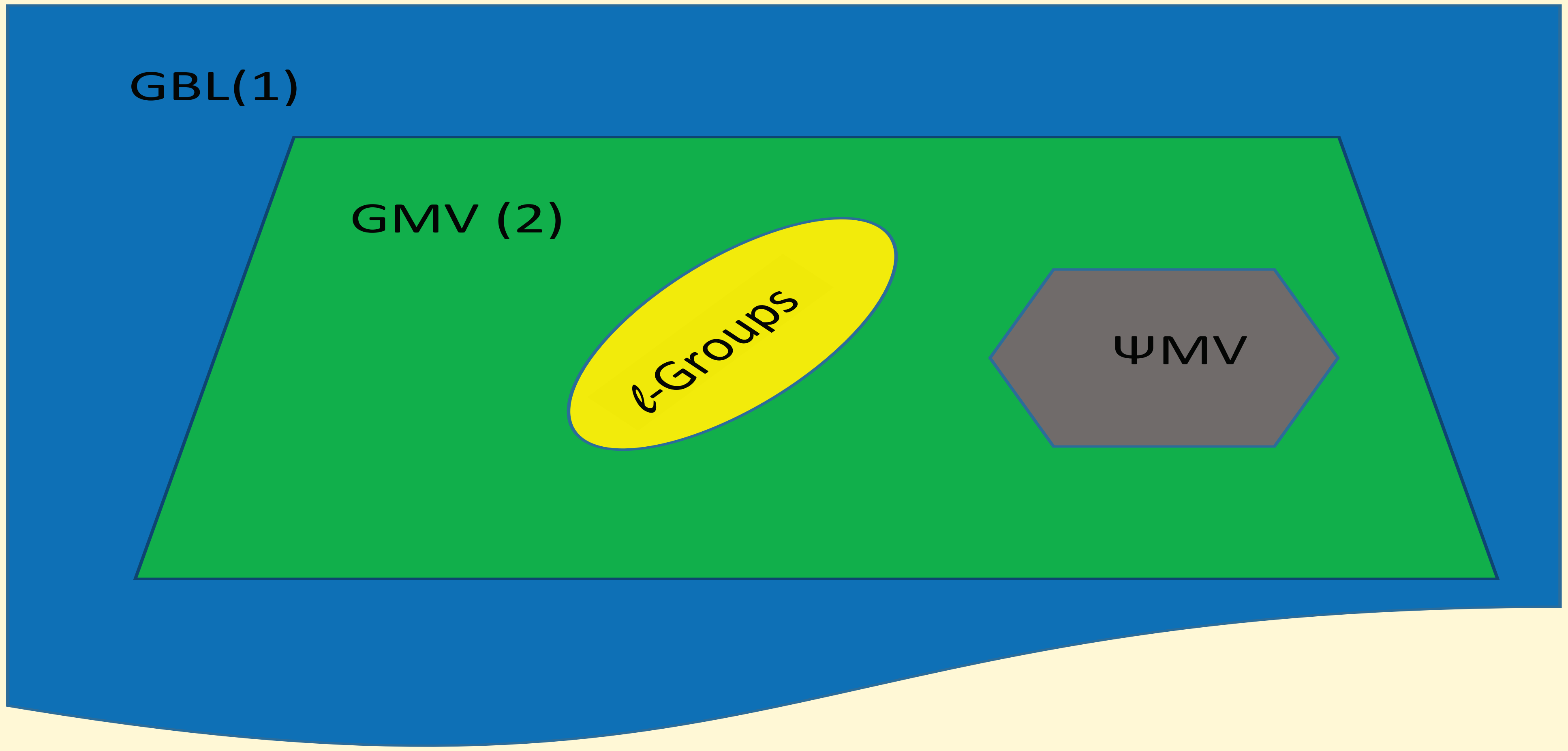
Key Properties for a Manageable Structure Theory

(1) Divisibility



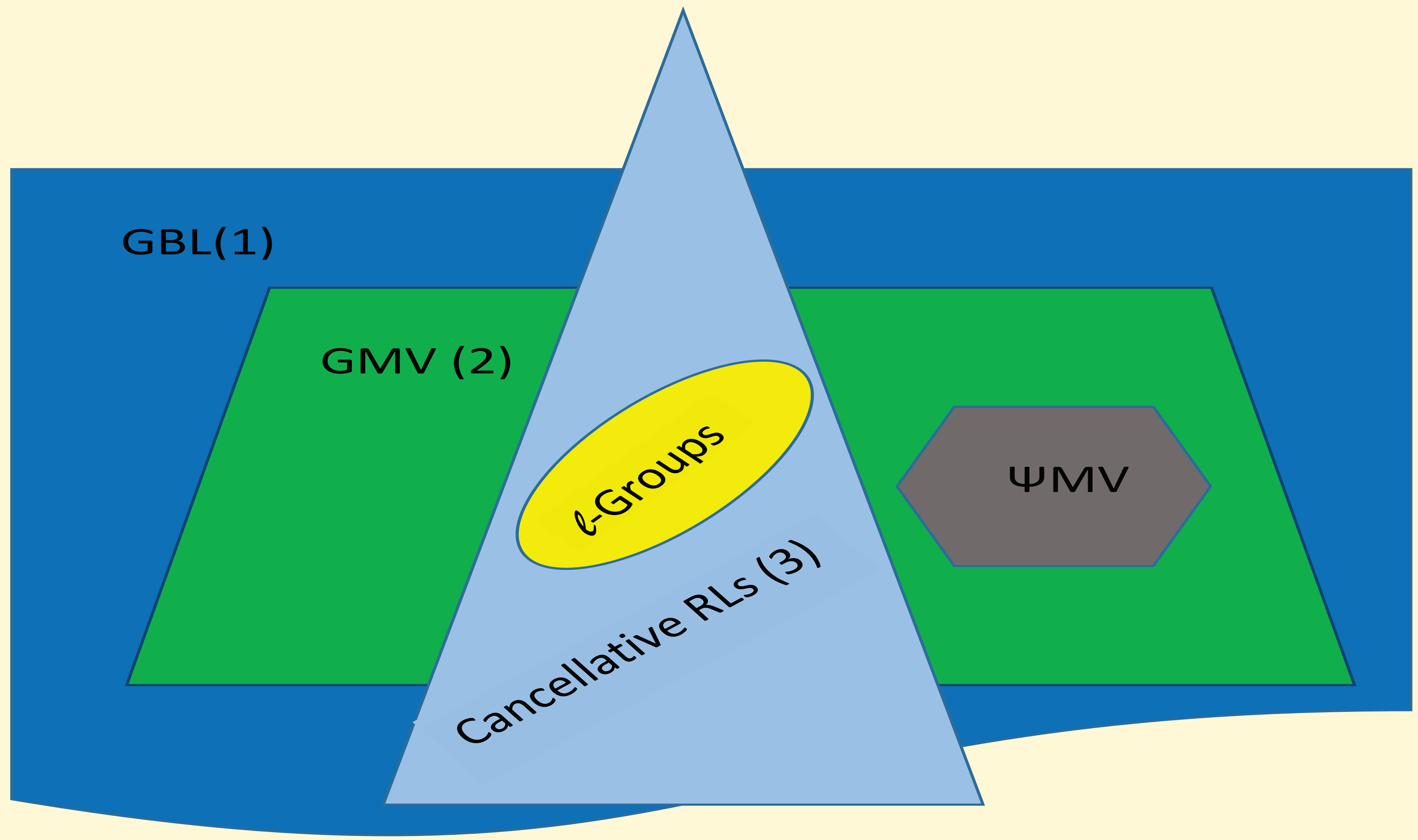
Key Properties for a Manageable Structure Theory

(1) Divisibility (2) Involutive Implication



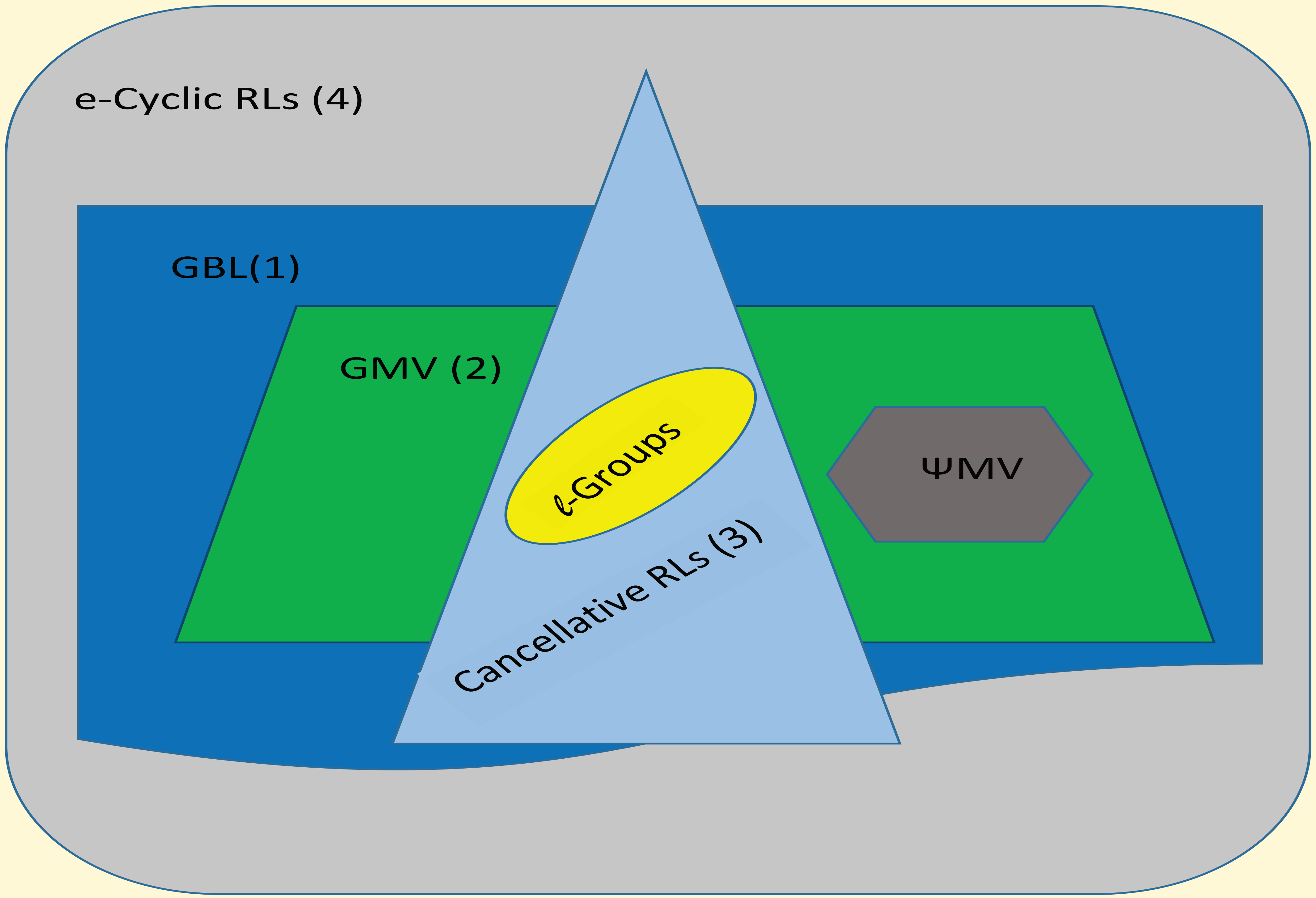
Key Properties for a Manageable Structure Theory

(1) Divisibility (2) Involutive Implication (3) Cancellativity



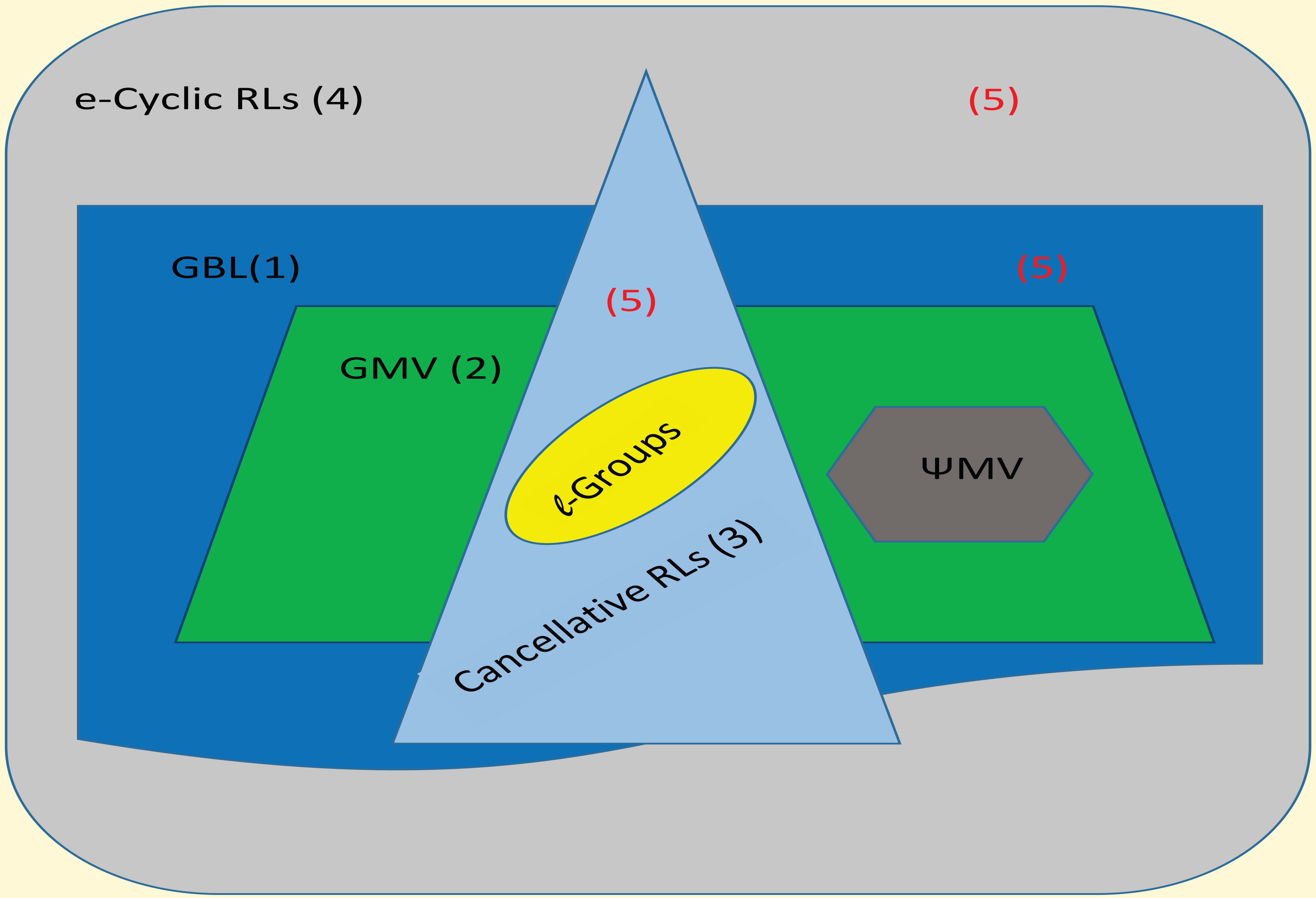
Key Properties for a Manageable Structure Theory

(1) Divisibility (2) Involutive Implication (3) Cancellativity
(4) e-Cyclicity (e/$x\approx x$\e)



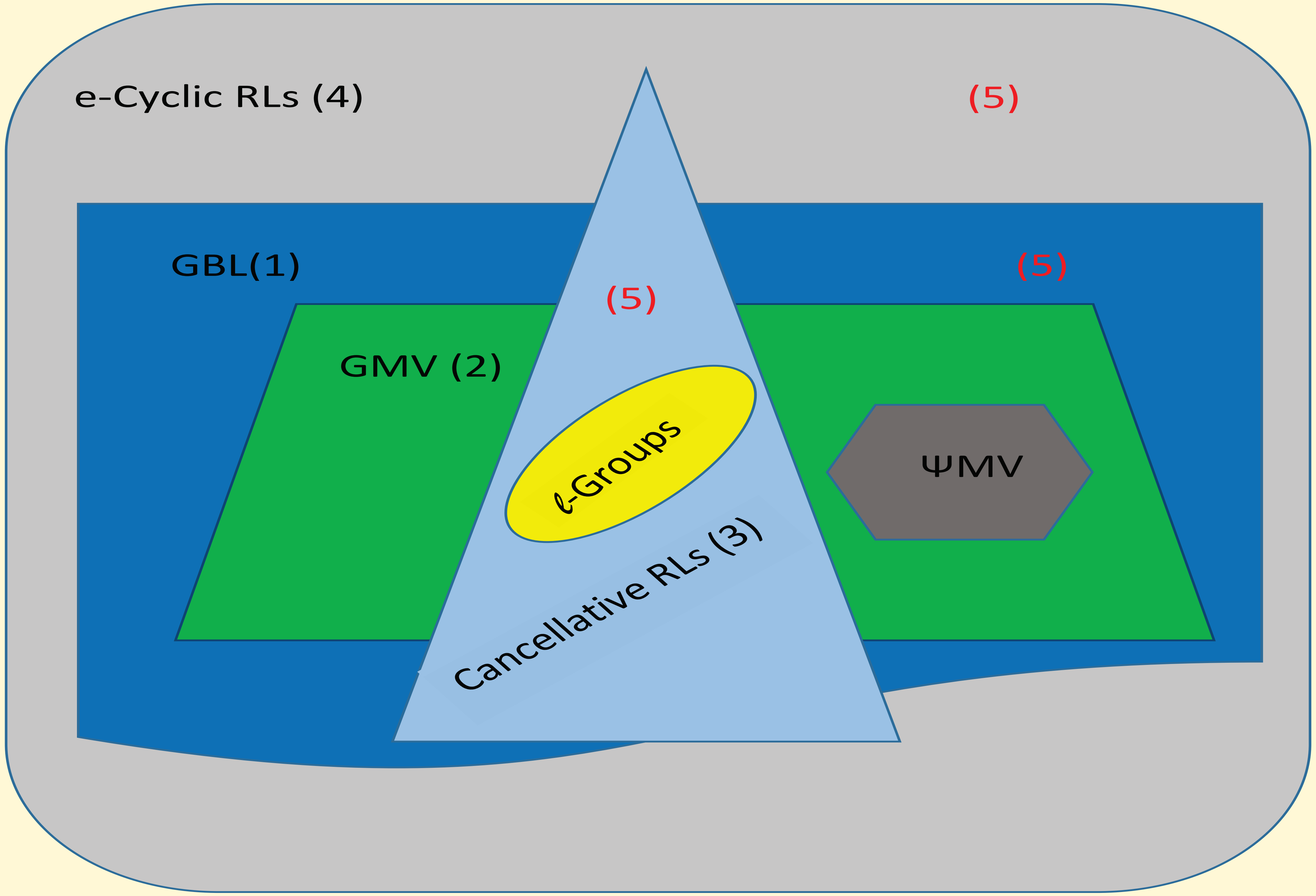
Key Properties for a Manageable Structure Theory

(1) Divisibility (2) Involutive Implication (3) Cancellativity
(4) e-Cyclicity (e/x≈x\e) (5) ADD PRE-LINEARITY



Left Prelinearity Law LP and Right Prelinearity Law RP

$$(\mathrm{LP}) \quad ((x\backslash y) \wedge e) \vee ((y\backslash x) \wedge e) \approx e$$
$$(\mathrm{RP}) \quad ((y/x) \wedge e) \vee ((x/y) \wedge e) \approx e$$



We call a (pointed) residuated lattice e-cyclic if it satisfies the identity $e/x \approx x\backslash e$.
Unless stated otherwise, all residuated lattices under consideration will be e-cyclic.
This variety encompasses most, but not all, varieties of notable significance.

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NOTATION

- $\langle S \rangle$, the submonoid generated by $S \subseteq L$
- $\mathrm{C}[S]$, the convex subuniverse generated by $S \subseteq L$
- $\mathrm{C}[a] = \mathrm{C}[\{a\}]$, the principal convex subuniverse generated by $a \in L$

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The absolute value of $a \in L$ is the element $|a| = a \wedge (\mathrm{e}/\mathrm{a}) \wedge \mathrm{e}$. If $S \subseteq L$, we set $|S| = \{|a| \colon a \in S\}$.

If $S \subseteq L$, then

$$\mathrm{C}[S] = \mathrm{C}[|S|] = \{x \in L\colon h \leq |x|, \text{ for some } h \in \langle |S| \rangle\}.$$

In particular, if $a \in L$, then

$$\mathrm{C}[a] = \mathrm{C}[|a|] = \{x \in L\colon |a|^n \leq |x|, \text{ for some } n \in \mathbb{N}\}.$$

(Note that if H is a convex subuniverse of $\mathbf{L}$, then $H = \mathrm{C}[H^-]$.)

Distributivity

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THEOREM

If $\mathbf{L}$ is an e-cyclic residuated lattice, then $\mathcal{C}(\mathbf{L})$ is a distributive algebraic lattice. The poset $\mathcal{K}(\mathcal{C}(\mathbf{L}))$ of compact elements of $\mathcal{C}(\mathbf{L})$ consists of the principal convex subuniverses of $\mathbf{L}$ and is a sublattice of $\mathcal{C}(\mathbf{L})$. More specifically, for all $a, b \in L$,

$$\mathrm{C}[a] \cap \mathrm{C}[b] = \mathrm{C}[|a| \vee |b|] \text{ and } \mathrm{C}[a] \vee \mathrm{C}[b] = \mathrm{C}[|a| \wedge |b|] = \mathrm{C}[|a||b|]$$

.

Relative Pseudo-complements and Polars

The lattice $\mathcal{C}(\mathbf{L})$ (with $\mathbf{L}$ e-cyclic) satisfies the join-infinite distributive law

$$H \cap \bigvee_{i\in I} K_i = \bigvee_{i\in I} (H \cap K_i).$$

Hence, for all $H, K \in \mathcal{C}(\mathbf{L})$, the relative pseudo-complement $H \to K$ of H relative to K exists:

$$H \to K = \max\{J \in \mathcal{C}(\mathbf{L})\colon H \cap J \subseteq K\}.$$

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An element-wise description of $H \to K$ is

$$H \to K = \{a \in L\colon |a| \vee |x| \in K, \text{ for all } x \in H\}.$$

In particular,

$$H^{\perp} = H \to \{\mathrm{e}\} = \{\mathrm{a} \in \mathrm{L}\colon |\mathrm{a}| \vee |\mathrm{x}| = \mathrm{e}, \text{ for all } \mathrm{x} \in \mathrm{H}\}.$$

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$X^{\perp}$ can be defined for any non-empty subset $X \subseteq L$, using the preceding equality. Then $X^{\perp} = \mathrm{C}[X]^{\perp}$. We refer to $X^{\perp}$ as the polar of X, and $x^{\perp} = \{x\}^{\perp}$ $(= \mathrm{C}[x]^{\perp})$ as the principal polar of x.

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By Glivenko's classical result, ${}^{\perp\perp} \colon \mathcal{C}(\mathbf{L}) \to \mathcal{C}(\mathbf{L})$ is an intersection-preserving map (i.e., a nucleus with respect to $\cap$), and $\mathcal{B}(\mathbf{L}) ={}^{\perp\perp} [\mathbf{L}]$ is a Boolean algebra.

Primes

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Let $\mathbf{L}$ be an e-cyclic residuated lattice that satisfies LP or RP. Then for every $H \in \mathcal{C}(\mathbf{L})$, the following are equivalent:

(1) H is a prime convex subuniverse of $\mathbf{L}$.
(2) For all $a, b \in L$, if $|a| \vee |b| \in H$, then $a \in H$ or $b \in H$.
(3) For all $a, b \in L$, if $|a| \vee |b| = \mathrm{e}$, then $a \in H$ or $b \in H$.
(4) The set of all convex subuniverses exceeding H is a chain under set-inclusion.

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Proposition

Let $\mathbf{L}$ be an e-cyclic residuated lattice that satisfies either prelinerity law. If $\mathcal{C}(\mathbf{L})$ – equivalently, $\mathcal{K}(\mathcal{C}(\mathbf{L}))$ – is totally ordered, then so is $\mathbf{L}$.

Values

A convex subuniverse $H \in \mathcal{C}(\mathbf{L})$ is said to be completely meet-irreducible in $\mathcal{C}(\mathbf{L})$ if $H \neq L$ and whenever $(K_i \colon i \in I)$ is a family of convex subuniverses of $\mathbf{L}$ and $H = \bigcap_{i \in I} K_i$, then $H = K_i$ for some $i \in I$.

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A completely meet-irreducible subuniverse H has a unique cover H^* in $\mathcal{C}(\mathbf{L})$, namely the intersection of all convex subuniverses that properly contain it.

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This is all lattice theory. To take advantage of the full structure of $\mathbf{L}$ we need the concept of a normal convex subuniverse.

Normality

Let $\mathbf{L}$ be a residuated lattice. Given an element $u \in L$, we define

$$\lambda_u(x) = (u\backslash xu) \wedge \mathrm{e} \quad \text{and} \quad \rho_u(x) = (ux/u) \wedge \mathrm{e},$$

for all $x \in L$. We refer to λ_u and ρ_u as the left conjugation map and the right conjugation map by u.

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The normal convex subuniverses of $\mathbf{L}$ form an algebraic distributive lattice $\mathcal{NC}(\mathbf{L})$ with respect to set-inclusion, and this lattice is isomorphic to the congruence lattice of $\mathbf{L}$. Specifically, the maps $H \mapsto \theta_H$ and $\theta \mapsto [\mathrm{e}]_\theta$, where $\theta_H := \{\langle x, y\rangle \in L^2 \colon x\backslash y \wedge y\backslash x \wedge \mathrm{e} \in \mathrm{H}\}$ and $[a]_\theta := \{x \in L \colon \langle x, a\rangle \in \theta\}$ for $a \in L$, are mutually inverse isomorphisms between the lattice $\mathcal{NC}(\mathbf{L})$ and the congruence lattice of $\mathbf{L}$.

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Corollary

Let $\mathbf{L}$ be an e-cyclic residuated lattice that satisfies one of the prelinearity laws. If H is a normal prime convex subuniverse of $\mathbf{L}$, then $\mathbf{L}/H$ is totally ordered.

A Few Consequences

Semilinearity

Theorem

For a variety $\mathcal{V}$ of residuated lattices, the following statements are equivalent.

(1) $\mathcal{V}$ is semilinear.
(2) $\mathcal{V}$ satisfies either of the equations below.

$$\lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y) \approx \mathrm{e}$$
$$\lambda_u(x/(x \vee y)) \vee \rho_v(y/(x \vee y)) \approx \mathrm{e}$$

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If in addition $\mathcal{V}$ is a variety of e-cyclic residuated lattices and satisfies either of the prelinearity laws, the preceding conditions are equivalent to each of the following conditions.

(3) **For all $\mathbf{L} \in \mathcal{V}$, all polars in $\mathcal{C}(\mathbf{L})$ are normal.**
(4) For all $\mathbf{L} \in \mathcal{V}$, all minimal prime convex subuniverses of $\mathbf{L}$ are normal.

Extensions

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A residuated lattice $\mathbf{L}$ is said to be

- projectable if $x^{\perp} \vee^{\mathcal{C}(\mathbf{L})} x^{\perp\perp} = L$, for all $x \in L$;
- strongly projectable if $H^{\perp} \vee^{\mathcal{C}(\mathbf{L})} H^{\perp\perp} = L$, for all $H \in \mathcal{C}(\mathbf{L})$; and
- laterally complete if all its orthogonal subsets have a greatest lower bound.

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Theorem Any member of a semilinear variety $\mathcal{V}$ of e-cyclic residuated lattices can be densely embedded into a laterally complete and projectable member of $\mathcal{V}$.

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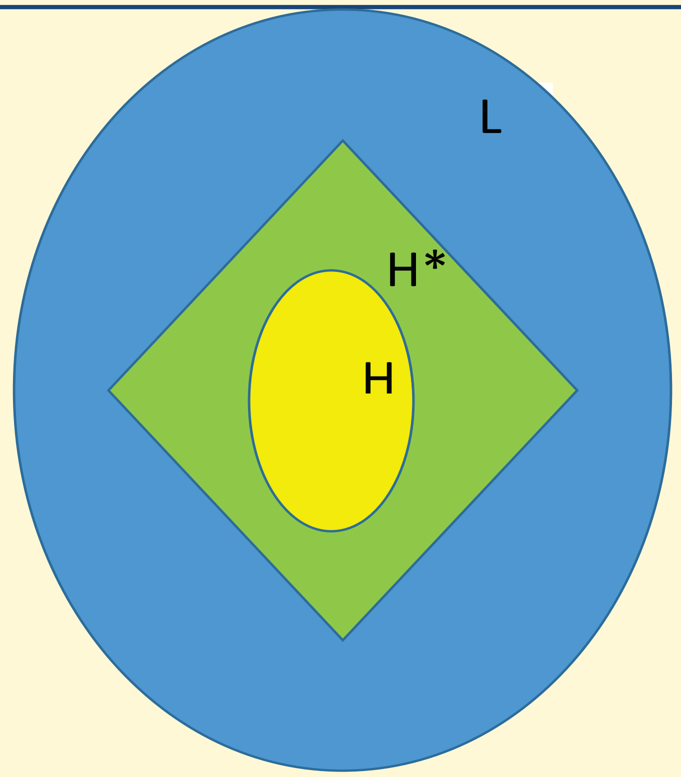
Theorem Any member of a semilinear variety $\mathcal{V}$ of e-cyclic residuated lattices can be densely embedded into a laterally complete and projectable member of $\mathcal{V}$.

Theorem Any member of a semilinear variety $\mathcal{V}$ of GMV-algebras has a unique lateral, lateral and projectable, projectable, or strongly projectable hull in $\mathcal{V}$.

Normal Values

$\mathbf{L}$ is said to be normal-valued if every completely meet-irreducible convex subuniverse of $\mathbf{L}$ is normal in its unique cover H^* (in $\mathcal{C}(\mathbf{L})$).



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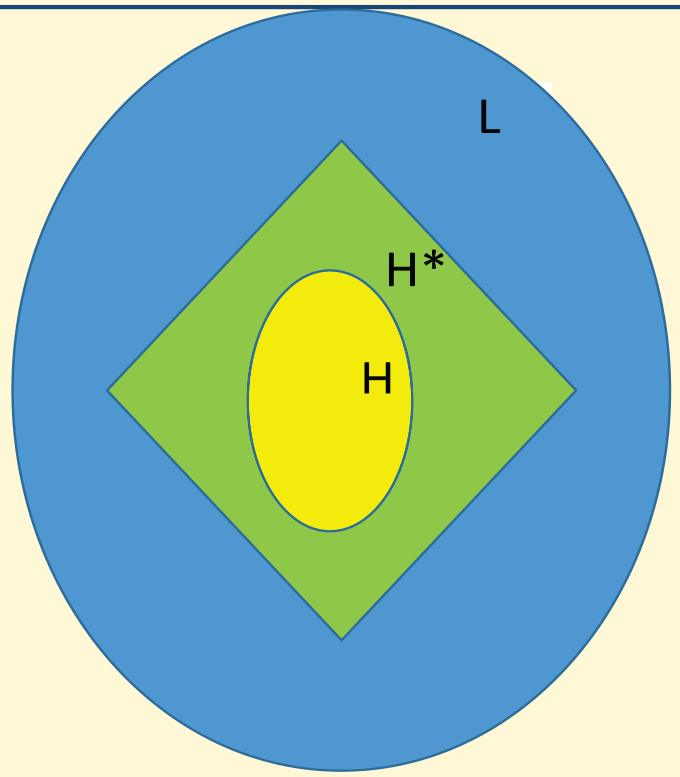

Theorem

Let $\mathbf{L}$ be an e-cyclic residuated lattice satisfying either prelinearity law. Then $\mathbf{L}$ is normal-valued if and only if $\mathbf{L}$ satisfies the following equations, for all $n \in \mathbb{N}$.

$$(x \wedge \mathrm{e})^2(\mathrm{y} \wedge \mathrm{e})^2 \leq (y \wedge \mathrm{e})(\mathrm{x} \wedge \mathrm{e})$$

(It suffices for GMV algebras)

$$\left((y/x \wedge \mathrm{e})^{\mathrm{n}} \backslash |\mathrm{x}||\mathrm{y}| \wedge \mathrm{e}\right)^2 \leq |x||y|/(x\backslash y \wedge \mathrm{e})^{4\mathrm{n}}$$

$$\left(|x||y|/(x\backslash y \wedge \mathrm{e})^{\mathrm{n}} \wedge \mathrm{e}\right)^2 \leq (y/x \wedge \mathrm{e})^{4\mathrm{n}} \backslash |\mathrm{x}||\mathrm{y}|$$

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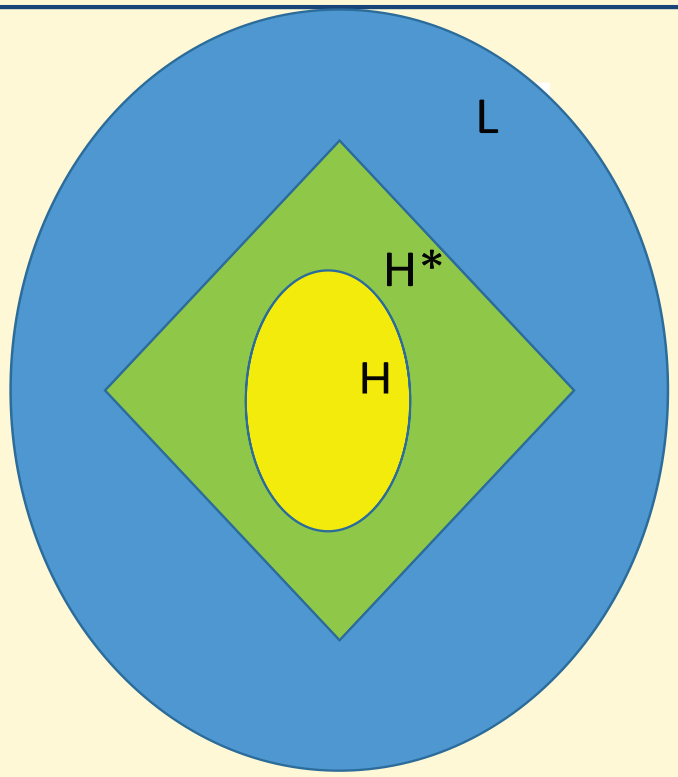

Theorem

Let $\mathbf{L}$ be an e-cyclic residuated lattice satisfying either prelinearity law. Then $\mathbf{L}$ is normal-valued if and only if $\mathbf{L}$ satisfies the following equations, for all $n \in \mathbb{N}$.

$$(x \wedge \mathrm{e})^2(y \wedge \mathrm{e})^2 \leq (y \wedge \mathrm{e})(x \wedge \mathrm{e})$$

(It suffices for GMV algebras)

$$\left((y/x \wedge \mathrm{e})^{n}\backslash|x||y| \wedge \mathrm{e}\right)^2 \leq |x||y|/(x\backslash y \wedge \mathrm{e})^{4n}$$
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Theorem [A. Dvurečenskij, 2007]

Any integral totally ordered GBL algebra is normal-valued.

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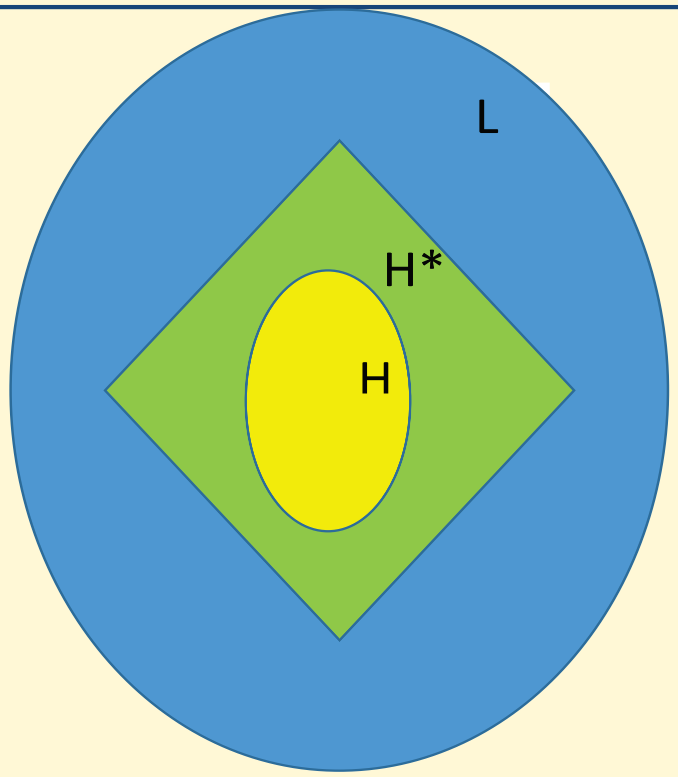

L is said to be normal-valued if every completely meet-irreducible convex subuniverse of **L** is normal in its unique cover H^* (in $\mathcal{C}(\mathbf{L})$).

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Corollary

Any semilinear GBL algebra is normal-valued.

Strongly Simple Residuated Lattices

An e-cyclic residuated lattice $\mathbf{L}$ is said to be strongly simple provided its only convex subuniverses are $\{\mathrm{e}\}$ and L. (Simple and subdirectly irreducible residuated lattices are too complicated in general to be amenable to useful description of their structure.)

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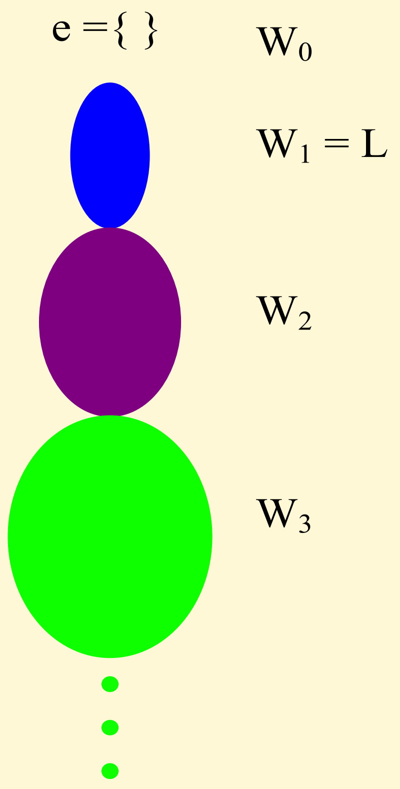

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States on GMV algebras

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Given a GMV algebra $\mathbf{L}$, an order preserving map $s\colon L \to \mathbb{R}$ is said to be a (Riečan) state on $\mathbf{L}$ if for all $x, y \in L$,

(1) $s(xy) = s(x) + s(y),$ whenever $x\backslash xy = y$ (equivalently, $xy/y = x$).

(Note that $s(\mathrm{e}) = 0.$)

If $\mathbf{L}$ has a least element f, we also require that

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The situation with GBL algebras is less satisfactory.

States on Cancellative Residuated Lattices

Let $\mathbf{L}$ be an integral Ore residuated lattice and let $\mathbf{G}(\mathbf{L})$ be its ℓ-group of left fractions. Then the map $\eta \colon a^{-1}b \mapsto a\backslash b$ is a co-nucleus on $\mathbf{G}(\mathbf{L})$ and $\mathbf{L} = \eta[\mathbf{G}(\mathbf{L})]$.

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Theorem

(1) If $s\colon \mathbf{L} \to \mathbb{R}$ is an order-preserving monoid homomorphism (i.e., a "Riečan state" on $\mathbf{L}$), then the map $\hat{s}\colon a^{-1}b \mapsto s(b) - s(a)$ is an order-preserving group homomorphism from $\mathbf{G}(\mathbf{L})$ to $\mathbb{R}$, and $\hat{s}{\restriction}_L = s$.

(2) If g is an order-preserving group homomorphism from $\mathbf{G}(\mathbf{L})$ to $\mathbb{R}$, then $s = g{\restriction}_L$ is an order-preserving monoid homomorphism from $\mathbf{L}$ to $\mathbb{R}^-$ and $\hat{s} = g$.