How useful is proof theory for substructural logics

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What are substructural logics?

- Logics that may lack some of structural rules (exchange/weakening/contraction)
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- Study of the universe of logics
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- Common basis for various nonclassical logics
  linear, BI, relevant, fuzzy, superintuitionistic logics
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  lattice-ordered groups, relation algebras, ideal lattices of rings, MV algebras, Heyting algebras
- Abundance of weird logics/algebras
  pathology for proof theory
Main topic: cut elimination.

Difficulty 1: Not many consequences.

- Analyticity (subformula property)
How useful is proof theory for substructural logics?

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- . . .

Difficulty 2: Limitation on systematic approach.
Substructural hierarchy

Classification of axioms

\[ \mathcal{P}_0, \mathcal{N}_0 ::= \text{the set of variables} \]

\[ \mathcal{P}_n ::= \mathcal{N}_{n-1} \lor 1 \lor \mathcal{P}_n \lor \mathcal{P}_n \land \mathcal{P}_n \]

\[ \mathcal{N}_n ::= \mathcal{P}_{n-1} \lor 0 \lor \mathcal{N}_n \land \mathcal{N}_n \lor \mathcal{P}_n \rightarrow \mathcal{N}_n \]

Some \( \mathcal{N}_2 \) axioms:

- \( A \rightarrow 1, \ 0 \rightarrow A \) \quad \text{weakening}
- \( A \rightarrow A \cdot A \) \quad \text{contraction}
- \( A \cdot A \rightarrow A \) \quad \text{expansion}
- \( A^n \rightarrow A^m \) \quad \text{knotted axioms \((n, m \geq 0)\)}
- \( \neg (A \land \neg A) \) \quad \text{no-contradiction}
Substructural hierarchy

Classification of axioms

\[ \mathcal{P}_0, \mathcal{N}_0 ::= \text{the set of variables} \]
\[ \mathcal{P}_n ::= \mathcal{N}_{n-1} | 1 | \mathcal{P}_n \lor \mathcal{P}_n | \mathcal{P}_n \cdot \mathcal{P}_n \]
\[ \mathcal{N}_n ::= \mathcal{P}_{n-1} | 0 | \mathcal{N}_n \land \mathcal{N}_n | \mathcal{P}_n \rightarrow \mathcal{N}_n \]

Some \( \mathcal{P}_3 \) axioms:

- \((A \rightarrow B) \lor (B \rightarrow A)\) \hspace{1cm} \text{prelinearity}
- \(A \lor \neg A\) \hspace{1cm} \text{excluded middle}
- \(\neg A \lor \neg \neg A\) \hspace{1cm} \text{weak excluded middle}
- \(\neg (A \cdot B) \lor (A \land B \rightarrow A \cdot B)\) \hspace{1cm} \text{weak nilpotent minimum}
- \(\bigvee_{i=0}^k (A_i \rightarrow \bigvee_{j \neq i} A_j)\) \hspace{1cm} \text{bounded width} \leq k
- \(\bigvee_{i=0}^k (A_0 \land \cdots \land A_{i-1} \rightarrow A_i)\) \hspace{1cm} \text{bounded size} \leq k
Substructural hierarchy

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\[ \mathcal{P}_n ::= \mathcal{N}_{n-1} \mid 1 \mid \mathcal{P}_n \lor \mathcal{P}_n \mid \mathcal{P}_n \land \mathcal{P}_n \]
\[ \mathcal{N}_n ::= \mathcal{P}_{n-1} \mid 0 \mid \mathcal{N}_n \land \mathcal{N}_n \mid \mathcal{P}_n \rightarrow \mathcal{N}_n \]

Theorem (Ciabattoni, Galatos, T. 08)

Over \textbf{FLew},

- every \( \mathcal{N}_2 \) axiom can be transformed into \textit{sequent} structural rules,
- every \( \mathcal{P}_3 \) axiom can be transformed into \textit{hypersequent} structural rules,

so that the calculus admits cut elimination.
The next target would be $\mathcal{N}_3$, that contains

\[
((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A \quad \text{axiom } \mathcal{L}
\]

\[
A \land B \rightarrow A \cdot (A \rightarrow B) \quad \text{divisibility}
\]

\[
(A \rightarrow A \cdot B) \rightarrow B \quad \text{cancellativity}
\]
The next target would be $\mathcal{N}_3$, that contains

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Is there a good “hyper-hyper” sequent calculus for $\mathcal{N}_3$?
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Is there a good “hyper-hyper” sequent calculus for $\mathcal{N}_3$?

No! Here is an absolute limitation.
**Algebraic semantics:** To each logic $L$ corresponds a class $V(L)$ of algebras.

- $V(\text{Cl})$: Boolean algebras
- $V(\text{Int})$: Heyting algebras
- $V(\text{FLe})$: pointed commutative residuated lattices
- $V(\mathcal{L})$: MV algebras

A completion of an algebra $A$ is a complete algebra $B$ such that $A \hookrightarrow B$. 
Failure of completion

Algebraic semantics: To each logic $\mathbf{L}$ corresponds a class $V(\mathbf{L})$ of algebras.

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A completion of an algebra $A$ is a complete algebra $B$ such that $A \rlarrows B$.

Theorem (Chang’s chain)

There is an algebra $C$ in $V(\mathfrak{L})$ which has no completion in $V(\mathfrak{L})$.

$C$ can be syntactically described.
Theorem (Chang’s chain formalized)

There is a set $C$ of (finitary) formulas such that

- $\Box L + C$ is consistent,
- $L + C + \text{infinitary } \wedge$ is inconsistent.

\[
\begin{align*}
\{ \Rightarrow \Gamma, A_i \}_{i \in I} & \\
\Rightarrow \Gamma, \wedge_{i \in I} A_i
\end{align*}
\]
Failure of conservativity

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There is a set $C$ of (finitary) formulas such that

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If there were a good calculus with cut elimination for $\mathcal{L}$, it would allow us to remove $\text{infinitary } \land$ from the derivation of finitary assumptions and conclusions.
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- If there were a good calculus with cut elimination for $\mathcal{L}$, it would allow us to remove infinitary $\land$ from the derivation of finitary assumptions and conclusions.
- Proof theory was invented for Hilbert’s program, which aims at reducing ideal arguments to finitist ones.
Failure of conservativity

Theorem (Chang’s chain formalized)

There is a set $C$ of (finitary) formulas such that

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Proof theory was invented for Hilbert’s program, which aims at reducing ideal arguments to finitist ones.

NB: There is a calculus for $\mathbb{L}$ (GMO 2005), but it doesn’t allow to eliminate infinitary $\land$. 
Theorem (Jerabek 15)

Over FL^e, the hierarchy collapses to $\mathcal{N}_3$. 

$\mathcal{N}_3$ no systematic proof theory

$\mathcal{P}_3$ hypersequent calculus

$\mathcal{N}_2$ sequent calculus

$\mathcal{P}_2$ 

$\mathcal{N}_1$ 

$\mathcal{P}_1$ 

$\mathcal{N}_0$ 

$\mathcal{P}_0$
1. Are there other applications of proof theory?

2. To what extent proof theory is useful for $\mathcal{N}_3$?
Problems to be addressed

1. Are there other applications of proof theory?
2. To what extent proof theory is useful for $\mathcal{N}_3$?

⇒ Brouwer’s fixed point theorem based on $\mathcal{L}$
Fixpoints in logic and computer science

Usually:

- Knaster-Tarski (or Banach)
  Given a complete lattice $L$, any monotone map $f : L \rightarrow L$ has a fixpoint.
- least/greatest fixpoints for monotone formulas
- foundation of induction/coinduction
Fixpoints in logic and computer science

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Today:

□ Brouwer

Any continuous map $f : [0, 1]^n \rightarrow [0, 1]^n$ has a fixpoint.

□ fixpoints for arbitrary formulas
□ related to naive comprehension
Knaster-Tarski fixpoints can be found by brutal force:

\[ \text{fix } f := f^\alpha(\bot), \quad \text{for some ordinal } \alpha. \]
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Brouwer fixpoints need some ingenuity:

- algebraic topology (no continuous map from a ball to its sphere)
- combinatorial argument (Sperner’s lemma)
- HEX (no draw)
- ...
Knaster-Tarski vs. Brouwer

**Knaster-Tarski fixpoints** can be found by **brutal force:**

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**Our attempt:** Prove it by **proof theory!**
### Systems to be discussed

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- Consistency of $\mathcal{L}_{fix}$ is equivalent to Brouwer’s fixpoint theorem.
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- Consistency of Łfix is equivalent to Brouwer’s fixpoint theorem.
- Consistency of Łset is a big open problem.
- MTLfix, MTLset are more tractable.
- Classification problem for extensions of FLewfix.
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- Consistency of $\mathcal{L}_{\text{fix}}$ is equivalent to **Brouwer’s fixpoint theorem**.
- Consistency of $\mathcal{L}_{\text{set}}$ is a **big open problem**.
- $\text{MTL}_{\text{fix}}$, $\text{MTL}_{\text{set}}$ are more tractable.
- **Classification problem** for extensions of $\text{FLew}_{\text{fix}}$.
- $\text{FLew}_{\text{set}}$ is a basis for **resource bounded set theory**.
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- Consistency of \( L_{fix} \) is equivalent to **Brouwer’s fixpoint theorem**.
- Consistency of \( L_{set} \) is a **big open problem**.
- \( MTL_{fix}, MTL_{set} \) are more tractable.
- **Classification problem** for extensions of \( FL_{ew\,fix} \).
- \( FL_{ew\,set} \) is a basis for **resource bounded set theory**.
\textbf{Int} is inconsistent with fixpoints

\textbf{Int} (intuitionistic logic) with \textbf{self-contradiction}

\((sc)\quad \alpha \leftrightarrow \neg\alpha\)

is inconsistent \((\alpha \text{ a propositional constant}).\)
Int is inconsistent with fixpoints

\textbf{Int} (intuitionistic logic) with \textbf{self-contradiction}

\[(sc) \quad \alpha \leftrightarrow \neg \alpha\]

is inconsistent ($\alpha$ a propositional constant).

\[
\begin{array}{c}
\alpha \Rightarrow \alpha \\
\neg \alpha, \alpha \Rightarrow \\
\alpha, \alpha \Rightarrow \\
\alpha \Rightarrow (c) \\
\alpha \Rightarrow \\
\alpha \Rightarrow \\
\alpha \Rightarrow \\
\Rightarrow
\end{array}
\]
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\[\Rightarrow \quad \Rightarrow \quad \Rightarrow \quad (c) \quad \Rightarrow \quad \Rightarrow \quad \Rightarrow \]

\[\Rightarrow \quad \Rightarrow \quad \Rightarrow \quad \alpha \]

\[\square \quad \text{Contraction is the criminal.}\]
Int is inconsistent with fixpoints

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\[
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\alpha, \alpha & \Rightarrow (c) \\
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\Rightarrow & \neg \alpha \\
\Rightarrow & \alpha
\end{align*}
\]

□ Contraction is the criminal.
□ Cut elimination procedure works stepwise, though does not terminate.
**Int** is inconsistent with fixpoints

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\]

- **Contraction** is the criminal.
- Cut elimination procedure works **stepwise**, though does not **terminate**.
- **Induction on the cut formula** is not available.
**FLew is consistent with fixpoints**

**FLew**: Int without contraction. 
= intuitionistic multiplicative-additive linear logic + weakening

**Fact**

**FLew** is consistent with \((sc)\).
**FLew** is consistent with fixpoints

**FLew**: Int without contraction.  
= intuitionistic multiplicative-additive linear logic + weakening

**Fact**  
**FLew** is consistent with \((sc)\).

- Proofs shrink by reducing (principal) cuts:

\[
\begin{align*}
\vdots \\
\Gamma \Rightarrow \neg \alpha & \quad \vdots \\
\Gamma \Rightarrow \alpha & \quad \neg \alpha \Rightarrow \Pi \\
\vdots \\
\Gamma \Rightarrow \Pi \\
\end{align*}
\]

\[
\vdots \\
\vdots \\
\Gamma \Rightarrow \neg \alpha & \quad \neg \alpha \Rightarrow \Pi \\
\vdots \\
\Gamma \Rightarrow \Pi \\
\]

even though the cut formula may become more complicated.
Weaker forms of contraction

Rule \((wc_n)\):

\[
\begin{array}{c}
\Gamma, \ldots, \Gamma, \Sigma \Rightarrow \Gamma, \ldots, \Gamma, \Sigma \\
\Gamma, \ldots, \Gamma, \Sigma \Rightarrow (wc_n)
\end{array}
\]
Weaker forms of contraction

Rule \((wc_n)\):

\[
\begin{array}{c}
\{ n+1 \} \\
\{ \Gamma, \ldots, \Gamma, \Sigma \Rightarrow \} \\
\{ \Gamma, \ldots, \Gamma, \Sigma \Rightarrow \} \\
\{ n \}
\end{array}
\]

\((wc_n)\)

Rule \((c')\):

\[
\begin{array}{c}
\Gamma, \Gamma, \Sigma \Rightarrow \Pi \\
\Delta, \Delta, \Sigma \Rightarrow \Pi \\
\Gamma, \Delta, \Sigma \Rightarrow \Pi
\end{array}
\]

\((c')\)

Both admit stepwise cut elimination procedures. Do they terminate?
Weaker forms of contraction

Rule \((w_{cn})\):  
\[
\begin{array}{c}
\Gamma, \ldots, \Gamma, \Sigma \Rightarrow \\
\Gamma, \ldots, \Gamma, \Sigma \Rightarrow \\
\end{array}
\]
\[
\frac{\text{n+1}}{\text{n}}
\]

Rule \((c')\):  
\[
\frac{\Gamma, \Gamma, \Sigma \Rightarrow \Pi \quad \Delta, \Delta, \Sigma \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi}
\]

Both admit stepwise cut elimination procedures. Do they terminate?

Fact

1. \( \text{FLeW} + (w_{cn}) \) is inconsistent with \( \beta \leftrightarrow \neg \beta^n \).
2. \( \text{FLeW} + (c') \) is consistent with any fixpoints.
System FLew $f_{ix}$

Failure of completion

Failure of conservativity

Connectives:\quad \land, \lor, \cdot, \rightarrow, 1, 0
Connectives: $\land$, $\lor$, $\cdot$, $\rightarrow$, 1, 0

We identify a fixpoint constant with its unfoldings:

$$\alpha = \neg \alpha = \neg \neg \alpha = \neg \neg \neg \alpha = \cdots$$
Connectives: $\land$, $\lor$, $\cdot$, $\rightarrow$, $1$, $0$

We identify a fixpoint constant with its unfoldings:

$$\alpha = \neg\alpha = \neg\neg\alpha = \neg\neg\neg\alpha = \cdots$$

We also consider mutual fixpoints: $\alpha = A(\beta)$, $\beta = B(\alpha)$

$$\alpha = A(B(\alpha)) = A(B(A(\beta))) = A(B(A(B(\alpha)))) = \cdots$$
Connectives: $\wedge$, $\vee$, $\cdot$, $\rightarrow$, $1$, $0$

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$$\alpha = \neg \alpha = \neg \neg \alpha = \neg \neg \neg \alpha = \cdots$$

We also consider mutual fixpoints: $\alpha = A(\beta)$, $\beta = B(\alpha)$

$$\alpha = A(B(\alpha)) = A(B(A(\beta))) = A(B(A(B(\alpha)))) = \cdots$$

More generally we assume: given $n$ formulas in $n$ variables $A_1(\vec{p}), \ldots, A_n(\vec{p})$, there are constants $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1 = A_1(\alpha_1, \ldots, \alpha_n)$$
$$\vdots$$
$$\alpha_n = A_n(\alpha_1, \ldots, \alpha_n)$$

This defines **System FLew$^\text{fix}$**.
Consistency of \( \text{FLew}_{fix} + (c') \)

Fact

\( \text{FLew}_{fix} + (c') \) is consistent.

\[
\frac{\Gamma, \Gamma, \Sigma \Rightarrow \Pi \quad \Delta, \Delta, \Sigma \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi} \quad (c')
\]
Consistency of $\text{FLew}_{fix} + (c')$

**Proof Idea:** any proof of contradiction shrinks by stepwise cut elimination.

\[
\begin{array}{c}
\vdash d_A \quad \vdash d_B \\
\Rightarrow A \\ \\
\vdash d_{AA} \\
\Rightarrow B \\
\vdash d_{BB} \\
\Rightarrow A, A \Rightarrow B, B \Rightarrow (c') \\
\Rightarrow A, B \Rightarrow (cut)
\end{array}
\]
Consistency of $\text{FLew}_{fix} + (c')$

Proof Idea: any proof of contradiction shrinks by stepwise cut elimination.

$$
\begin{array}{ccc}
\vdots d_A & \vdots d_B & \vdots d_{AA} & \vdots d_{BB} \\
\Rightarrow A & \Rightarrow B & A, A \Rightarrow B, B \Rightarrow (c') \\
\hline \\
\end{array}
$$

reduces to

$$
\begin{array}{ccc}
\vdots d_A & \vdots d_A & \vdots d_{AA} \\
\Rightarrow A & \Rightarrow A & A, A \Rightarrow (cut) \\
\hline \\
\end{array} \quad \text{AND} \quad 
\begin{array}{ccc}
\vdots d_B & \vdots d_B & \vdots d_{BB} \\
\Rightarrow B & \Rightarrow B & B, B \Rightarrow (cut) \\
\hline \\
\end{array}
$$
Consistency of $\text{FLew}_v^{\text{fix}} + (c')$

**Proof Idea:** any proof of contradiction shrinks by stepwise cut elimination.

\[
\begin{align*}
\vdots & \vdots & \vdots \\
\Rightarrow A & \Rightarrow B & A, A \Rightarrow B, B \Rightarrow (c') \\
\end{align*}
\]

reduces to

\[
\begin{align*}
\vdots & \vdots & \vdots \\
\Rightarrow A & \Rightarrow A & A, A \Rightarrow (\text{cut}) \\
\end{align*}
\]

AND

\[
\begin{align*}
\vdots & \vdots & \vdots \\
\Rightarrow B & \Rightarrow B & B, B \Rightarrow (\text{cut}) \\
\end{align*}
\]

Compare $|d_A|$ and $|d_B|$. If $|d_A| \leq |d_B|$, the left proof is smaller than the original one.
Actually we have a more general result. Note that $\text{FLew} + (c')$ is a sublogic of $\mathcal{L}$ (blackboard).

**Theorem**

Let $L$ be an axiomatic extension of $\text{FLew}$.

1. If $L$ is above $\text{FLew} + (wc_n)$ for some $n$, $L_{fix}$ is inconsistent.
2. If $L$ is below $\mathcal{L}$, $L_{fix}$ is consistent.
Actually we have a more general result. Note that $\text{FLe}w + (c')$ is a sublogic of $\mathcal{L}$ (blackboard).

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Let $\mathcal{L}$ be an axiomatic extension of $\text{FLe}w$.

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- 1 subsumes all superintuitionistic logics and all finite valued logics extending $\text{FLe}w$. 
Actually we have a more general result. Note that $\text{FLew} + (c')$ is a sublogic of $\mathcal{L}$ (blackboard).

**Theorem**

Let $\mathcal{L}$ be an axiomatic extension of $\text{FLew}$.

1. If $\mathcal{L}$ is above $\text{FLew} + (wc_n)$ for some $n$, $\mathcal{L}_{fix}$ is inconsistent.
2. If $\mathcal{L}$ is below $\mathcal{L}$, $\mathcal{L}_{fix}$ is consistent.

- 1 subsumes all superintuitionistic logics and all finite valued logics extending $\text{FLew}$.
- 2 is to be discussed later.
Actually we have a more general result. Note that $\mathbf{FLe}w + (c')$ is a sublogic of $\mathcal{L}$ (blackboard).

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- □ 2 is to be discussed later.
- □ It is not a full classification.
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**Theorem**

Let $\mathcal{L}$ be an axiomatic extension of $\text{FLew}$.

1. If $\mathcal{L}$ is above $\text{FLew} + (WC_n)$ for some $n$, $\mathcal{L}_{fix}$ is inconsistent.
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- 2 is to be discussed later.
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**Problem 1**

Sharpen the above theorem.
## Systems to be discussed

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- $\text{FLew}_{set}$ is a basis for resource bounded set theory.
Naive set theory over FLew

Terms and formulas:

\[ t ::= x \mid \{ x : \varphi \} \]
\[ \varphi ::= t \in t \mid 0 \mid \varphi \rightarrow \varphi \mid \forall x. \varphi \]
Naive set theory over $\text{FLe}w$

Terms and formulas:
\[
  t ::= x \mid \{x : \varphi\} \\
  \varphi ::= t \in t \mid 0 \mid \varphi \to \varphi \mid \forall x.\varphi
\]

$\text{FLe}w_{set}$: extension of $\text{FLe}w\forall$ with naive comprehension:
\[
t \in \{x : \varphi(x)\} \iff \varphi(t).
\]
Naive set theory over FLew

Terms and formulas:

\[
\begin{align*}
t & ::= x \mid \{x : \varphi\} \\
\varphi & ::= t \in t \mid 0 \mid \varphi \to \varphi \mid \forall x.\varphi
\end{align*}
\]

**FLew\textsubscript{set}:** extension of FLew\textsubscript{∀} with naive comprehension:

\[
t \in \{x : \varphi(x)\} \iff \varphi(t).
\]

Eg. let \( R := \{x : x \not\in x\} \). Then

\[
R \in R \iff R \not\in R.
\]
Naive set theory over $\text{FLew}$

**Terms and formulas:**
\[
\begin{align*}
t & ::= \ x \mid \{x : \varphi\} \\
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\end{align*}
\]

$\text{FLew}_{\text{set}}$: extension of $\text{FLew}_{\forall}$ with naive comprehension:
\[
t \in \{x : \varphi(x)\} \iff \varphi(t).
\]

Eg. let $R := \{x : x \notin x\}$. Then
\[
R \in R \iff R \notin R.
\]

More generally, any set $\{A_1(\vec{p}), \ldots, A_n(\vec{p})\}$ admits a mutual fixpoint.

Hence $\text{FLew}_{\text{fix}}$ is embeddable into $\text{FLew}_{\text{set}}$. 
There is a shrinking cut elimination procedure:

\[ \Gamma \Rightarrow \varphi(t) \quad \varphi(t) \Rightarrow \Pi \]
\[ \Gamma \Rightarrow t \in \{x : \varphi(x)\} \quad t \in \{x : \varphi(x)\} \Rightarrow \Pi \]
\[ \Gamma \Rightarrow \Pi \]

Theorem (Grisin 1982)

**FLew**_{set} is consistent.
We may define

- Leipniz equality
- logical connectives
- union, intersection, complement
- natural numbers
We may define

- Leipniz equality
- logical connectives
- union, intersection, complement
- natural numbers

**Theorem**

For any formula $A(x, y)$ there is a term $t_A$ such that

$$x \in t_A \leftrightarrow A(x, t_A).$$
Expressivity of $\text{FLew}_{set}$

We may define

- Leibniz equality
- logical connectives
- union, intersection, complement
- natural numbers

**Theorem**

For any formula $A(x, y)$ there is a term $t_A$ such that

$$x \in t_A \iff A(x, t_A).$$

This allows us to define a term $\mathbb{N}$ such that

$$x \in \mathbb{N} \iff x = 0 \lor \exists y \in \mathbb{N}. x = y + 1.$$
Expressivity of $\text{FLe}w_{set}$

**Fact**

$\text{FLe}w_{set} \vdash t \in \mathbb{N} \iff t$ is a natural number.
Expressivity of $\text{FLew}_{set}$

**Fact**

$\text{FLew}_{set} \vdash t \in \mathbb{N} \iff t$ is a natural number.

We may also define all r.e. sets.

**Theorem**

Provability in $\text{FLew}_{set}$ is $\Sigma^0_1$-complete.

However, $\text{FLew}_{set}$ is a very weak theory, which is analogous to Robinson’s $Q$ in arithmetic.
Expressivity of $\text{FLew}_{set}$

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In arithmetic, one extends $\text{Q}$ with inductions
Expressivity of $\text{FLew}_\text{set}$

**Fact**

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However, $\text{FLew}_\text{set}$ is a very weak theory, which is analogous to Robinson’s $\text{Q}$ in arithmetic.

In arithmetic, one extends $\text{Q}$ with inductions.

In naive set theory, we extend $\text{FLew}_\text{set}$ with controlled contractions.
We may extend $\mathbf{FLew}_{set}$ with $K$-modality $!$:

\[
\Gamma \Rightarrow B \\
!\Gamma \Rightarrow !B \\
!A, !A, \Gamma \Rightarrow \Pi \\
!A, \Gamma \Rightarrow \Pi
\]

This is called the elementary affine set theory.
We may extend $\text{FLew}_{\text{set}}$ with $K$-modality $!$:

\[
\begin{align*}
\Gamma \Rightarrow B & \quad \quad !A, !A, \Gamma \Rightarrow \Pi \\
!\Gamma \Rightarrow !B & \quad \quad !A, \Gamma \Rightarrow \Pi
\end{align*}
\]

This is called the elementary affine set theory.

$$\mathbb{N} := \{x : \forall X. \forall y (y \in X \rightarrow y+1 \in X) \rightarrow !(0 \in X \rightarrow x \in X)\}$$

It supports elementary induction principle:

\[
\begin{align*}
A(0) & \quad \forall y. A(y) \rightarrow A(y + 1) \\
\forall x \in \mathbb{N}. !A(x)
\end{align*}
\]
We may extend $\mathbf{FLew}_{\text{set}}$ with $K$-modality $!$:

$$
\Gamma \Rightarrow B \\
!\Gamma \Rightarrow !B \\
!A, !A, \Gamma \Rightarrow \Pi \\
!A, \Gamma \Rightarrow \Pi
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$$
\mathbb{N} := \{x : \forall X. !\forall y (y \in X \rightarrow y+1 \in X) \rightarrow !(0 \in X \rightarrow x \in X)\}
$$

It supports elementary induction principle:

$$
\begin{align*}
A(0) & \forall y.A(y) \rightarrow A(y+1) \\
\forall x \in \mathbb{N}.!A(x)
\end{align*}
$$

Theorem (Girard 98, T. 04)

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is elementary recursive iff it is provably total in elementary affine set theory.
We may also extend $\text{FLew}_\text{set}$ with two modalities $!$, $\$\$ with

\[
\begin{align*}
A \Rightarrow B & \quad !A, !A, \Gamma \Rightarrow \Pi \\
!A \Rightarrow !B & \quad !A, \Gamma \Rightarrow \Pi \\
\Gamma, \Delta \Rightarrow B & \quad !\Gamma, \$\$\Delta \Rightarrow \$\$B
\end{align*}
\]

This is called the light affine set theory.
We may also extend $\text{FLeu}^{\text{set}}$ with two modalities $!, \&$ with

\[
\begin{align*}
  A \Rightarrow B & \quad \vdash !A \Rightarrow !B \\
  !A, !A, \Gamma \Rightarrow \Pi & \quad \vdash !A, \Gamma \Rightarrow \Pi \\
  \Gamma, \Delta \Rightarrow B & \quad \vdash !\Gamma, \&\Delta \Rightarrow \&B
\end{align*}
\]

This is called the light affine set theory.

**Theorem (Girard 98, T. 04)**

A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is polynomial time computable iff it is provably total in light affine set theory.
\[ \square \quad \text{FL}_{\text{ew}_{\text{set}}} \text{ is a very weak naive set theory.} \]
Summary

- **FLew**$_{set}$ is a very weak naive set theory.
- It can be extended by modality $!$ controlling contraction.
- **FLew_{set}** is a **very weak** naive set theory.
- It can be extended by modality \( ! \) controlling contraction.
- If \( ! \) is K, it captures **elementary recursive** functions.
- FLew_set is a very weak naive set theory.
- It can be extended by modality ! controlling contraction.
- If ! is K, it captures elementary recursive functions.
- If ! is functorial and bounded by K-modality §, it captures polynomial time functions.
□ \textbf{FLe} is a very weak naive set theory.

□ It can be extended by modality \(!\) controlling contraction.

□ If \(!\) is K, it captures \textbf{elementary recursive} functions.

□ If \(!\) is functorial and bounded by K-modality \(\S\), it captures \textbf{polynomial time} functions.

□ If \(!\) is T, it is \textbf{inconsistent}.

\[
\begin{align*}
!\alpha \Rightarrow !\alpha \\
\hline
!\alpha, \neg !\alpha \Rightarrow \\
\hline
!\alpha, \alpha \Rightarrow \\
\hline
!\alpha, !\alpha \Rightarrow (T) \\
\hline
!\alpha \Rightarrow (c) \\
\hline
\Rightarrow \neg !\alpha \\
\Rightarrow \alpha
\end{align*}
\]

\((\alpha = \neg !\alpha)\)
- FLew_{set} is a very weak naive set theory.
- It can be extended by modality $!$ controlling contraction.
- If $!$ is K, it captures elementary recursive functions.
- If $!$ is functorial and bounded by K-modality $\Box$, it captures polynomial time functions.
- If $!$ is T, it is inconsistent.

\[
\begin{align*}
   !\alpha & \Rightarrow !\alpha \\
   !\alpha, \neg !\alpha & \Rightarrow 
\end{align*}
\]

**Problem 2**

Is K4 consistent? What about other modalities?

\[
\begin{align*}
   !\alpha & \Rightarrow \\
   \Rightarrow \neg !\alpha \\
   \Rightarrow \alpha \\
   (\alpha = \neg !\alpha)
\end{align*}
\]
Systems to be discussed

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- Consistency of $\mathcal{L}_{fix}$ is equivalent to Brouwer’s fixpoint theorem.
Łukasiewicz’s infinite-valued logic

\[ \mathcal{L} := \text{FLeW} + \]

(axiom \( \mathcal{L} \)) \quad ((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A
Łukasiewicz’s infinite-valued logic

\[ \mathcal{L} := \mathsf{FLew} + \]

\[(\text{axiom } \mathcal{L}) \quad ((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A\]

It allows us to define

\[ A \lor B := (A \rightarrow B) \rightarrow B \]

| \( \frac{A \Rightarrow A}{A, A \rightarrow B \Rightarrow B} \) | \( \frac{B \Rightarrow B \quad A \Rightarrow A}{B, B \rightarrow A \Rightarrow A} \) |
| \( \frac{B \Rightarrow (B \rightarrow A) \rightarrow A}{B \Rightarrow (A \rightarrow B) \rightarrow B} \) |
| \( \frac{A \Rightarrow C}{C \rightarrow B \Rightarrow A \rightarrow B} \) | \( \frac{B \Rightarrow C}{(B \rightarrow C) \rightarrow C \Rightarrow C} \) |

\[ (A \rightarrow B) \rightarrow B \Rightarrow (C \rightarrow B) \rightarrow B \]

\[ (A \rightarrow B) \rightarrow B \Rightarrow C \]
Łukasiewicz’s infinite-valued logic

Failure of completion
Failure of conservativity

(axiom Ł) \((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A\)
Łukasiewicz’s infinite-valued logic

Failure of completion
Failure of conservativity

(axiom Ł) \[ ((A \rightarrow B) \rightarrow B) \rightarrow (B \rightarrow A) \rightarrow A \]

For linear logicians: Ł is an extension of MLL in which additives are multiplicatively definable.
Łukasiewicz’s infinite-valued logic

Failure of completion

Failure of conservativity

(axiom Ł) \[ ((A \to B) \to B) \to (B \to A) \to A \]

For linear logicians: Ł is an extension of MLL in which addtives are multiplicatively definable.

Theorem (Kowalski 2012)

Let \( A, B \) be \( \to \)-only formulas.

- If \( (A \to B) \to B \) is provable in FLew, either \( A \) or \( B \) is provable.
- The following inference is admissible in FLew

\[
\Rightarrow (A \to B) \to B \\
\Rightarrow (B \to A) \to A
\]
Łukasiewicz interpretation

Łukasiewicz and Tarski (1930) assigned to each formula

\[ B \equiv B(\beta_1, \ldots, \beta_n) \]

a function

\[ [B] : [0, 1]^n \rightarrow [0, 1] \]

defined by

\[
\begin{align*}
[\beta_i](\vec{x}) & := x_i \\
[0](\vec{x}) & := 0 \\
[B \rightarrow C](\vec{x}) & := \min(1, 1 - [B](\vec{x}) + [C](\vec{x})) \\
[B \cdot C](\vec{x}) & := \max(0, [B](\vec{x}) + [C](\vec{x}) - 1)
\end{align*}
\]
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\[ B \equiv B(\beta_1, \ldots, \beta_n) \]

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[B \cdot C](\vec{x}) := \max(0, [B](\vec{x}) + [C](\vec{x}) - 1)
\]

**Theorem**

This is the **only** assignment on \([0, 1] \) which is both **FLew**-sound and **continuous**.
Brower’s fixpoint theorem $\Rightarrow$ Consistency of $\text{L}_{fix}$

**Theorem (Brouwer 1910)**

Every continuous map $f : [0, 1]^n \rightarrow [0, 1]^n$ has a fixed point.
Brower’s fixpoint theorem \( \Rightarrow \) Consistency of \( \mathcal{L}_{fix} \)

**Theorem (Brouwer 1910)**

Every continuous map \( f : [0, 1]^n \rightarrow [0, 1]^n \) has a fixed point.

**Corollary**

\( \mathcal{L}_{fix} \) is consistent.

Given \( A_1(\vec{\alpha}), \ldots, A_n(\vec{\alpha}) \), consider

\[
([A_1], \ldots, [A_n]) : [0, 1]^n \rightarrow [0, 1]^n
\]

and let \((r_1, \ldots, r_n)\) be a fixed point.

Then valuation \( v(\alpha_i) := r_i \) satisfies all \( \alpha_i \Leftrightarrow A_i(\vec{\alpha}_i) \). Hence \( \mathcal{L}_{fix} \) is consistent.
Towards a proof theory of $\mathcal{L}_{fix}$

Two reasons to study proof theory of $\mathcal{L}_{fix}$:

1. $\text{Con}(\mathcal{L}_{fix})$ implies BFT.
2. First step to the consistency of $\mathcal{L}_{set}$, which is a big open problem in fuzzy logic.
Towards a proof theory of $\mathcal{L}_{fix}$

Two reasons to study proof theory of $\mathcal{L}_{fix}$:

1. $\text{Con}(\mathcal{L}_{fix})$ implies BFT.
2. First step to the consistency of $\mathcal{L}_{set}$, which is a big open problem in fuzzy logic.

Note: White (1979) introduced a natural deduction system for $\mathcal{L}_{set}$ and “proved” its consistency. It has been believed correct until recently. But it turned out incorrect (look at a note on my webpage).
A McNaughton function is a continuous piecewise-linear function $f : [0, 1]^n \rightarrow [0, 1]$ with integer coefficients. I.e., there is a partition

$$[0, 1]^n = X_0 \cup \cdots \cup X_m$$

and on each $X_i$

$$f(\vec{x}) = a_1 x_1 + \cdots + a_n x_n + a_0$$

for some $a_0, \ldots, a_n \in \mathbb{Z}$.

(blackboard)
A McNaughton function is a continuous piecewise-linear function \( f : [0, 1]^n \rightarrow [0, 1] \) with integer coefficients. I.e., there is a partition

\[
[0, 1]^n = X_0 \cup \cdots \cup X_m
\]

and on each \( X_i \)

\[
f(\vec{x}) = a_1 x_1 + \cdots + a_n x_n + a_0
\]

for some \( a_0, \ldots, a_n \in \mathbb{Z} \).

Theorem

\( f : [0, 1]^n \rightarrow [0, 1]^n \) is a product of McNaughton functions iff there are formulas \( A_1, \ldots, A_n \) with \( f = (\llbracket A_1 \rrbracket, \ldots, \llbracket A_n \rrbracket) \).
Quasi-McNaughton functions

Rational numbers are definable by fixpoints:

\[ \alpha \leftrightarrow \neg \alpha \quad \implies \quad \alpha = 1/2 \]
\[ \alpha \leftrightarrow \neg (\alpha \cdot \alpha) \quad \implies \quad \alpha = 2/3 \]
Rational numbers are definable by fixpoints:

\[
\begin{align*}
\alpha &\leftrightarrow \neg\alpha \quad \implies \quad \alpha = \frac{1}{2} \\
\alpha &\leftrightarrow \neg(\alpha \cdot \alpha) \quad \implies \quad \alpha = \frac{2}{3}
\end{align*}
\]

Given a (product of) McNaughton function
\[g : [0, 1]^{n+m} \longrightarrow [0, 1]^n\]
and \(q_1, \ldots, q_m \in [0, 1] \cap \mathbb{Q},\)

\[f(x) := g(x, \bar{q}) : [0, 1]^n \longrightarrow [0, 1]^n\]

is called a quasi-McNaughton function.
Quasi-McNaughton functions

Rational numbers are definable by fixpoints:

\[ \alpha \leftrightarrow -\alpha \implies \alpha = 1/2 \]
\[ \alpha \leftrightarrow - (\alpha \cdot \alpha) \implies \alpha = 2/3 \]

Given a (product of) McNaughton function
\[ g : [0, 1]^{n+m} \rightarrow [0, 1]^n \] and \( q_1, \ldots, q_m \in [0, 1] \cap \mathbb{Q} \),

\[ f(\vec{x}) := g(\vec{x}, \vec{q}) : [0, 1]^n \rightarrow [0, 1]^n \]

is called a quasi-McNaughton function.

Lemma

\( \text{Con}(\mathcal{L}_{\text{fix}}) \) implies BFT for quasi-McNaughton functions.
Lemma

Con($\mathcal{L}_{fix}$) implies BFT for quasi-McNaughton functions.

Proof. Given a quasi-McNaughton $f$, there are $A_1, \ldots, A_n$ and $q_1, \ldots, q_m \in [0, 1] \cap \mathbb{Q}$ such that

$$f(\bar{x}) = ([A_1](\bar{x}, \bar{q}), \ldots, [A_n](\bar{x}, \bar{q})).$$
Lemma

Con($\mathcal{L}_{\text{fix}}$) implies BFT for quasi-McNaughton functions.

Proof. Given a quasi-McNaughton $f$, there are $A_1, \ldots, A_n$ and $q_1, \ldots, q_m \in [0, 1] \cap \mathbb{Q}$ such that

$$f(\vec{x}) = ([A_1](\vec{x}, \vec{q}), \ldots, [A_n](\vec{x}, \vec{q})).$$

The rationals $q_1, \ldots, q_m$ are definable by $\beta_i \leftrightarrow B_i(\beta_i)$ for $i = 1, \ldots, m$. Consider fixpoint equations for $\vec{A}(\vec{\alpha}, \vec{\beta}), \vec{B}(\vec{\beta})$. 
Lemma

Con(Łfix) implies BFT for quasi-McNaughton functions.

Proof. Given a quasi-McNaughton \( f \), there are \( A_1, \ldots, A_n \) and \( q_1, \ldots, q_m \in [0, 1] \cap \mathbb{Q} \) such that

\[
f(\vec{x}) = ([A_1](\vec{x}, \vec{q}), \ldots, [A_n](\vec{x}, \vec{q})).
\]

The rationals \( q_1, \ldots, q_m \) are definable by \( \beta_i \leftrightarrow B_i(\beta_i) \) for \( i = 1, \ldots, m \). Consider fixpoint equations for \( \vec{A}(\vec{\alpha}, \vec{\beta}) \), \( \vec{B}(\vec{\beta}) \).

Since Łfix is consistent, there is an assignment

\[
(r_1, \ldots, r_n, q_1, \ldots, q_m) \in [0, 1]^{n+m}.
\]

satisfying \( \alpha_i \leftrightarrow A_i(\vec{\alpha}, \vec{\beta}) \), that is, \( \vec{r} = f(\vec{r}) \).
Theorem

Con(Ł_{fix}) implies Brouwer’s fixed point theorem.

Proof. Every continuous \( f : [0, 1]^n \rightarrow [0, 1]^n \) can be approximated by a sequence of quasi-McNaughton \( \{f_i\}_{i \in \mathbb{N}} \):

\[
f_i(x) \rightarrow f(x) \quad (i \rightarrow \infty).
\]
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Each $f_i$ has a fixed point $r_i$. 
**Theorem**

Con($\mathcal{L}_{fix}$) implies Brouwer’s fixed point theorem.

**Proof.** Every continuous $f : [0, 1]^n \rightarrow [0, 1]^n$ can be approximated by a sequence of quasi-McNaughton $\{f_i\}_{i \in \mathbb{N}}$:

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Each $f_i$ has a fixed point $r_i$. Since $[0, 1]^n$ is compact, we may assume $r_i \rightarrow r \quad (i \rightarrow \infty)$. 

Con($\mathcal{L}_{fix}$) $\Rightarrow$ BFT
Con(\(\mathcal{L}_{fix}\)) \Rightarrow \text{BFT}

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Con(\(\mathcal{L}_{fix}\)) implies Brouwer’s fixed point theorem.

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\[ f_i(x) \rightarrow f(x) \quad (i \rightarrow \infty). \]

Each \(f_i\) has a fixed point \(r_i\).
Since \([0, 1]^n\) is compact, we may assume \(r_i \rightarrow r \quad (i \rightarrow \infty)\).
We conclude \(f(r) = r\).
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Con($\mathcal{L}_{fix}$) implies Brouwer’s fixed point theorem.

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Since $[0, 1]^n$ is compact, we may assume $r_i \to r \quad (i \to \infty)$.

We conclude $f(r) = r$. 
(axiom $\mathcal{L}$) is equivalent to

$$\begin{align*}
\Gamma, A \Rightarrow B & \Rightarrow B & \Delta, B \Rightarrow A & \Sigma, A \Rightarrow \Pi \\
\hline
\Gamma, \Delta, \Sigma & \Rightarrow \Pi
\end{align*}$$

(\mathcal{L})
A proof system for $\mathcal{L}_{fix}$

(axiom $\mathcal{L}$) is equivalent to

$$
\frac{\Gamma, A \rightarrow B \Rightarrow B, \Delta, B \Rightarrow A, \Sigma, A \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi} \quad \text{(}$\mathcal{L}$\text{)}
$$

(cut) and ({$\mathcal{L}$}) can be eliminated stepwise from derivations of $\Rightarrow$:

\[\begin{array}{c}
\vdots d_1 \\
\vdots d_2 \\
\vdots d_3 \\
A \rightarrow B \Rightarrow B \\
B \Rightarrow A \\
A \Rightarrow
\end{array}\]

reduces to

\[\begin{array}{c}
\vdots d_3 \\
A \Rightarrow \\
A \Rightarrow B \\
\Rightarrow A \rightarrow B \\
A \rightarrow B \Rightarrow B \\
B \Rightarrow A \\
A \Rightarrow
\end{array}\]

\[\text{(cut)}\]
A proof system for $\mathcal{L}_{fix}$

(axiom $\mathcal{L}$) is equivalent to

$$
\frac{\Gamma, A \rightarrow B \Rightarrow B \quad \Delta, B \Rightarrow A \quad \Sigma, A \Rightarrow \Pi}{\Gamma, \Delta, \Sigma \Rightarrow \Pi} \quad (\mathcal{L})
$$

(cut) and ($\mathcal{L}$) can be eliminated stepwise from derivations of $\Rightarrow$:

$$
\vdots \; d_1 \quad \vdots \; d_2 \quad \vdots \; d_3
\quad A \rightarrow B \Rightarrow B \quad B \Rightarrow A \quad A \Rightarrow
\Rightarrow
$$

reduces to

$$
\vdots \; d_3
\quad A \Rightarrow
\Rightarrow A \rightarrow B \quad A \rightarrow B \Rightarrow B \quad B \Rightarrow A \quad A \Rightarrow \quad (\text{cut})
$$

Problem 3

Does the procedure terminate? If so, we would obtain a proof-theoretic proof of Brouwer's fixpoint theorem.
Systems to be discussed

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- Consistency of $\mathcal{L}_{set}$ is a big open problem.
Terms and formulas:

\[ t ::= x \mid \{x : \varphi(x)\} \]
\[ \varphi ::= t \in t \mid 0 \mid \varphi \to \varphi \mid \forall x.\varphi \]

\[ \mathcal{L}_{set}: \] extension of \( L\forall \) with naive comprehension axiom:

\[ t \in \{x : \varphi(x)\} \iff \varphi(t). \]
Łukasiewicz interpretation can be extended:

\[
[\forall x.\varphi(x)] := \bigwedge_{a \in D} [\varphi(a)].
\]
Łukasiewicz interpretation can be extended:

$$\left[ \forall x. \varphi(x) \right] := \bigwedge_{a \in D} \left[ \varphi(a) \right].$$

**Problem 4**

Is $\mathcal{L}_{set}$ consistent?
Łukasiewicz interpretation can be extended:

\[
\lbrack \forall x. \varphi(x) \rbrack := \bigwedge_{a \in D} \lbrack \varphi(a) \rbrack.
\]

**Problem 4**

Is $\mathcal{L}_{set}$ consistent?

**Two obstacles:**

- Infinitary $\bigwedge$ breaks **continuity**.
Łukasiewicz interpretation can be extended:

$$\left[ \forall x. \varphi(x) \right] := \bigwedge_{a \in D} [\varphi(a)].$$

Problem 4
Is $Ł_{set}$ consistent?

Two obstacles:

- Infinitary $\bigwedge$ breaks continuity.
- Has to consider an infinite dimensional vector space.
Łukasiewicz interpretation can be extended:

\[ \forall x. \varphi(x) ] := \bigwedge_{a \in D} \varphi(a). \]

**Problem 4**

Is \( \mathcal{L}_{set} \) consistent?

**Two obstacles:**

- Infinitary \( \bigwedge \) breaks **continuity**.
- Has to consider an **infinite** dimensional vector space.

BFT no more available. Forced to work **proof-theoretically**.
Systems to be discussed

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- $\mathbf{MTL}_{fix}$, $\mathbf{MTL}_{set}$ are more tractable.
Monoidal t-norm logic with fixpoints

$\textbf{MTL} := \textbf{FLew}$ with prelinearity:

$$(pl) \quad A \to B \lor B \to A.$$ 

$\textbf{MTL}_{fix}$: given $n$ formulas $A_1(\vec{p}), \ldots, A_n(\vec{p})$, there are constants $\alpha_1, \ldots, \alpha_n$ such that

\[
\begin{align*}
\alpha_1 &= A_1(\alpha_1, \ldots, \alpha_n) \\
\vdots &= \vdots \\
\alpha_n &= A_n(\alpha_1, \ldots, \alpha_n)
\end{align*}
\]

Formulas are identified modulo the equivalence.
Monoidal t-norm logic with fixpoints

\[ \text{MTL} := \text{FLew with prelinearity:} \]

\[ (pl) \quad A \rightarrow B \lor B \rightarrow A. \]

\[ \text{MTL}_{fix}: \text{ given } n \text{ formulas } A_1(\vec{p}), \ldots, A_n(\vec{p}), \text{ there are constants } \alpha_1, \ldots, \alpha_n \text{ such that} \]

\[ \alpha_1 = A_1(\alpha_1, \ldots, \alpha_n) \]

\[ \vdots \]

\[ \alpha_n = A_n(\alpha_1, \ldots, \alpha_n) \]

Formulas are identified modulo the equivalence.

\text{MTL} \text{ is a sublogic of } \mathcal{L}, \text{ so:}

\text{Fact}

\text{MTL}_{fix} \text{ is consistent.}
Hypersequent calculus for $\text{MTL}_{fix}$

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_n$ with $\Theta_i$ a sequent.

Hypersequent calculus for $\text{FL}$ consists of

Rules of $\text{FL}$

\[
\begin{align*}
\Xi \mid A, \Gamma \Rightarrow B & \quad \Xi \mid \Gamma \Rightarrow A \Rightarrow B \\
\Xi \mid \Gamma \Rightarrow A \Rightarrow B & \quad \Xi \mid \Gamma \Rightarrow \Pi \Rightarrow \Pi
\end{align*}
\]

Ext-Contraction

\[
\Xi \mid \Gamma \Rightarrow \Pi \Rightarrow \Pi
\]

Communication

\[
\begin{align*}
\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi & \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda \\
\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \Rightarrow \Pi & \quad \Delta_1, \Delta_2 \Rightarrow \Lambda (\text{com})
\end{align*}
\]
Hypersequent calculus for MTL$_{fix}$

Hypersequents: $\Theta_1 \mid \cdots \mid \Theta_n$ with $\Theta_i$ a sequent.

Hypersequent calculus for FL consists of

Rules of FL
\[
\Xi \mid A, \Gamma \Rightarrow B \\
\Xi \mid \Gamma \Rightarrow A \rightarrow B
\]

Ext-Contraction
\[
\Xi \mid \Gamma \Rightarrow \Pi | \Gamma \Rightarrow \Pi \\
\Xi \mid \Gamma \Rightarrow \Pi
\]

Communication
\[
\Xi \mid \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi \mid \Gamma_2, \Delta_2 \Rightarrow \Lambda
\]
\[
\Xi \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Delta_1, \Delta_2 \Rightarrow \Lambda \quad (com)
\]

\[
\begin{align*}
\alpha &\Rightarrow \alpha & \beta &\Rightarrow \beta & (com) \\
\alpha &\Rightarrow \beta & \beta &\Rightarrow \alpha & (\Rightarrow r)
\end{align*}
\]
\[
\Rightarrow (\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha) \quad (\lor r) \quad (EC)
\]
\[
\Rightarrow (\alpha \rightarrow \beta) \lor (\beta \rightarrow \alpha)
\]
Cut elimination for $\text{MTL}_{fix}$

**Goal:** define a notion of *size* and design a *shrinking* cut elimination procedure.
Cut elimination for $\text{MTL}_{fix}$

**Goal:** define a notion of **size** and design a **shrinking** cut elimination procedure.

A **slice** of derivation $d$ is a selection of 0 or 1 sequent from each hypersequent in $d$ such that:

\[
\begin{align*}
\Xi | A, \Gamma \Rightarrow B & \quad \Xi | \Gamma \Rightarrow \Pi | \Gamma \Rightarrow \Pi \\
\Xi | \Gamma \Rightarrow A \rightarrow B & \quad \Xi | \Gamma \Rightarrow \Pi \\
\Xi | \Gamma \Rightarrow \Pi | \Gamma \Rightarrow \Pi & \quad \text{or} \quad \Xi | \Gamma \Rightarrow \Pi \\
\Xi | \Gamma_1, \Delta_1 \Rightarrow \Pi & \quad \Xi | \Gamma_2, \Delta_2 \Rightarrow \Lambda \\
\Xi | \Gamma_1, \Gamma_2 \Rightarrow \Pi | \Delta_1, \Delta_2 \Rightarrow \Lambda & \quad \text{(com)}
\end{align*}
\]
**Cut elimination for** $\text{MTL}_{fix}$

**Goal:** define a notion of size and design a shrinking cut elimination procedure.

A slice of derivation $d$ is a selection of 0 or 1 sequent from each hypersequent in $d$ such that:

\[
\begin{align*}
\Xi | A, \Gamma \Rightarrow B & \quad \Xi | \Gamma \Rightarrow \Pi | \Gamma \Rightarrow \Pi \\
\Xi | \Gamma \Rightarrow A \Rightarrow B & \quad \Xi | \Gamma \Rightarrow \Pi
\end{align*}
\]

or

\[
\Xi | \Gamma \Rightarrow \Pi
\]

\[
\Xi | \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi | \Gamma_2, \Delta_2 \Rightarrow \Lambda \\
\Xi | \Gamma_1, \Gamma_2 \Rightarrow \Pi | \Delta_1, \Delta_2 \Rightarrow \Lambda
\]

(\text{com})

\[
\Xi | \Gamma_1, \Delta_1 \Rightarrow \Pi \quad \Xi | \Gamma_2, \Delta_2 \Rightarrow \Lambda \\
\Xi | \Gamma_1, \Gamma_2 \Rightarrow \Pi | \Delta_1, \Delta_2 \Rightarrow \Lambda
\]

The size $|d|$ is a multiset of natural numbers defined by:

\[
|d| := \{|d'| : d' \text{ is a slice of } d\}.
\]

where $|d'|$ is the number of inference rules visible in $d'$. We consider multiset ordering (which is well founded).
There is a shrinking cut elimination procedure for derivations of contradiction $\Rightarrow$.

\[
\begin{array}{c}
\vdash d_A \Rightarrow A \\
\vdash d_B \Rightarrow B \\
\Rightarrow A \Rightarrow B \\
\end{array}
\quad
\begin{array}{c}
\vdash d_{AA} \Rightarrow A, A \\
\vdash d_{BB} \Rightarrow B, B \\
\Rightarrow A, B \Rightarrow (cut) \\
\end{array}
\quad
\begin{array}{c}
\vdash A, A \Rightarrow (com) \\
\vdash A, B \Rightarrow | A, B \Rightarrow (EC) \\
\Rightarrow A, B \Rightarrow (cut) \\
\end{array}
\]

reduces to

\[
\begin{array}{c}
\vdash d_A \Rightarrow A \\
\vdash d_A \Rightarrow A \\
\Rightarrow A, A \Rightarrow (cut) \\
\end{array}
\quad
\begin{array}{c}
\vdash d_B \Rightarrow B \\
\vdash d_B \Rightarrow B \\
\Rightarrow B, B \Rightarrow (cut) \\
\end{array}
\quad
\begin{array}{c}
\vdash d_{AA} \Rightarrow A, A \\
\vdash d_{BB} \Rightarrow B, B \\
\Rightarrow A, B \Rightarrow (cut) \\
\end{array}
\]

AND
There is a shrinking cut elimination procedure for derivations of contradiction $\Rightarrow$.

\[
\begin{align*}
&\vdash d_A \Rightarrow A \\
&\vdash d_B \Rightarrow B \\
\hline
&\vdash A \land B \\
\end{align*}
\]

reduces to

\[
\begin{align*}
&\vdash d_A \Rightarrow A \\
&\vdash d_A \Rightarrow A, A \Rightarrow (cut) \\
\hline
&\vdash \vdash (cut) \\
\end{align*}
\]

AND

\[
\begin{align*}
&\vdash d_B \Rightarrow B \\
&\vdash d_B \Rightarrow B, B \Rightarrow (cut) \\
\hline
&\vdash \vdash (cut) \\
\end{align*}
\]

**Theorem**

Any proof of $\Rightarrow$ can be reduced to a cut-free proof (which does not exist).
The previous argument works for $\mathbf{MTL}_{\text{set}}$ as well.

**Theorem**

$\mathbf{MTL}_{\text{set}}$ is consistent.
### Summary

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- Consistency of Ł_{fix} is equivalent to Brouwer’s fixpoint theorem.
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- Consistency of $\mathcal{L}_{fix}$ is equivalent to Brouwer’s fixpoint theorem.
- Hence by proving the former proof-theoretically, we obtain a new proof of the latter.
Consistency of $\mathcal{L}_{fix}$ is equivalent to Brouwer’s fixpoint theorem.

Hence by proving the former proof-theoretically, we obtain a new proof of the latter.

Moreover, such a proof most likely extends to naive set theory, which would lead to the consistency of $\mathcal{L}_{set}$, a big open problem in fuzzy mathematics.