Epimorphisms in Varieties of Residuated Structures

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JOINT WORK WITH

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In a concrete category $\mathcal{K}$, a morphism $h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is called a ($\mathcal{K}$-) **epimorphism** when, for any $\mathcal{K}$-morphisms $f, g\colon \boldsymbol{B} \longrightarrow \boldsymbol{C}$,

$$\text{if } f \circ h = g \circ h, \text{ then } f = g.$$

Note: (1) Surjective $\mathcal{K}$-morphisms are $\mathcal{K}$-epimorphisms.

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(2) We say that $\mathcal{K}$ has the **ES property** if all $\mathcal{K}$-epimorphisms are surjective.

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(3) A **variety** $\mathcal{K}$ has the ES property iff no $\boldsymbol{B} \in \mathcal{K}$ has a ($\mathcal{K}$-) **epic** (proper) **subalgebra** $\boldsymbol{D}$, i.e., one such that any $\mathcal{K}$-morphism $f\colon \boldsymbol{B} \longrightarrow \boldsymbol{C}$ is determined by $f|_{\boldsymbol{D}}$.

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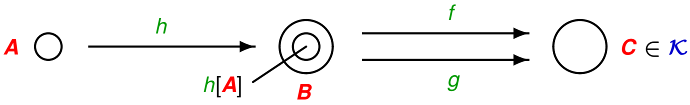

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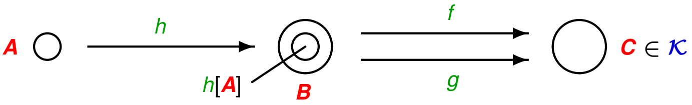

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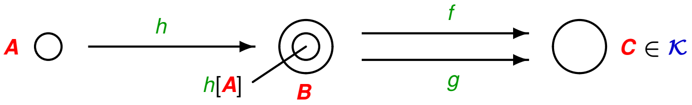

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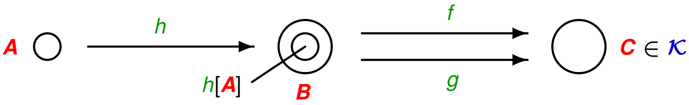

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(4) {Groups}, {*R*-modules}, {Lattices}, {Semilattices} and {Boolean algebras} are varieties with the ES property.

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(5) The variety {Rings} *lacks* ES, as $\mathbb{Z}$ is epic in $\mathbb{Q}$.

This is because, although multiplicative inverses needn't exist, they are **implicitly definable** in rings—i.e., *uniquely determined* **or** *non-existent*.

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(6) The ES property needn't persist in subvarieties:

it holds in {Lattices}, but not in {Distributive Lattices}, where

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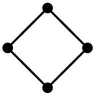

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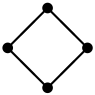

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Why study ES? Let $\mathcal{K}$ be a variety algebraizing a logic $\vdash$,

e.g., {Boolean algebras} $\longleftrightarrow$ classical propositional logic,
or {Heyting algebras} $\longleftrightarrow$ intuitionistic propositional logic.

Theorem. (Blok & Hoogland, 2006) $\mathcal{K}$ has the ES property iff $\vdash$ has the **infinite Beth** (definability) **property**, which means:

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whenever $\Gamma \subseteq \text{Form}(X \,\dot{\cup}\, Z)$ and

$$\Gamma \cup h[\Gamma] \vdash z \leftrightarrow h(z)$$

for all $z \in Z$ and all substitutions h (of formulas for variables) such that $h(x) = x$ for all $x \in X$,

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for each $z \in Z$, there's a formula $\varphi_z \in \text{Form}(X)$ such that

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The **finite Beth property** makes the same demand, but only when Z is finite.

Theorem. (Németi, 1984) $\vdash$ has the finite Beth property iff $\mathcal{K}$ has the **weak ES property**, which means:

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every ‘almost onto’ $\mathcal{K}$-epimorphism is onto,

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where ‘$h\colon \boldsymbol{A} \longrightarrow \boldsymbol{B}$ is *almost onto*’ means that $\boldsymbol{B}$ is generated by $h[A] \cup \{b\}$ for some $b \in B$.

Problem. Does the *finite Beth property* imply the *infinite* one?

Blok-Hoogland Conjecture: No.

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In algebraic terms:

Question. Does *weak ES* imply *ES* (at least for varieties)?

Yes, for *amalgamable* varieties (known), so we eschew these.

Where to look?

Although {Boolean algebras} have ES, the $2^{\aleph_0}$ varieties of **Heyting algebras** ALL have *weak ES* (Kreisel, 1960), but *only finitely many* of them are amalgamable (Maksimova, 1970s).

$\mathcal{HA}$:= {**all** Heyting algebras} has ES.

Question. Which *subvarieties* of $\mathcal{HA}$ have ES?

Answer. Not all. (Blok-Hoogland Conjecture confirmed.)

Some of the counter-examples are locally finite.

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Theorem. If a variety of Heyting algebras has *finite depth*, then it has surjective epimorphisms. ($2^{\aleph_0}$ examples.)

[Known: finitely generated $\Rightarrow$ finite depth $\Rightarrow$ locally finite.]

Corollary. Every *finitely generated* variety of Heyting algebras has surjective epimorphisms.

[In contrast, it's known that only finitely many subvarieties of $\mathcal{HA}$ have the so-called **strong ES property**: whenever $\boldsymbol{A} \leqslant \boldsymbol{B} \in \mathcal{K}$ and $b \in B \backslash A$, there are two $\mathcal{K}$-morphisms $f, g \colon \boldsymbol{B} \longrightarrow \boldsymbol{C}$ that agree on A but not at b (Maksimova, 2000).]

Corollary. Every variety of *Gödel algebras* (i.e., of subdirect products of totally ordered Heyting algebras) has ES.

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Everything said thus far applies equally to **Brouwerian algebras**, i.e., to possibly unbounded Heyting algebras.

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Theorem. If a super-intuitionistic [or positive] logic is **tabular**—or more generally if its theorems include a formula from the sequence

$$h_0 := y; \quad h_n := x_n \vee (x_n \to h_{n-1}) \ \ (0 < n \in \omega),$$

then it has the infinite Beth property. Likewise all Gödel logics.

Even the finite Beth property fails in all axiomatic extensions of Hajek's Basic Logic (BL), excepting the Gödel logics [Montagna, 2006]. Likewise many relevance logics [Urquhart, 1999], but new exceptions emerge here.

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More general than Heyting/BL algebras are ***residuated lattices***

$$\boldsymbol{A} = \langle A; \cdot, \rightarrow, \wedge, \vee, e \rangle.$$

[$\langle A; \wedge, \vee \rangle$ is a lattice and $\langle A; \cdot, e \rangle$ a commutative monoid with

$$x \cdot y \leqslant z \iff y \leqslant x \rightarrow z \qquad \text{(law of residuation).]}$$

Several varieties of these are categorically equivalent to varieties of (enriched) Gödel algebras [Galatos & R, 2012/15].

The ES property is categorical, so it transfers.

With more work, we obtain:

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Theorem. *Every* variety of *Sugihara monoids* has ES.

[A **Sugihara monoid** $\boldsymbol{A} = \langle A; \cdot, \rightarrow, \wedge, \vee, \neg, e \rangle$ is a residuated *distributive* lattice with an *involution* $\neg$, where $\cdot$ is idempotent.

It needn't be *integral*, i.e., e needn't be its top element.]

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From $\boldsymbol{A}$, we construct an *Esakia space* $\boldsymbol{A}_* := \langle \mathrm{Pr}\, \boldsymbol{A}; \subseteq, \tau \rangle$.

$\mathrm{Pr}\, \boldsymbol{A}$ is the set of all *prime* filters of $\boldsymbol{A}$ (i.e., all lattice filters F with $\top \in F$, such that $A \backslash F$ is closed under $\vee$), and τ is a certain topology on $\mathrm{Pr}\, \boldsymbol{A}$.

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A sub-basis for τ is then $\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}$.

For a $\mathcal{HA}$–morphism $h \colon \boldsymbol{A} \to \boldsymbol{B}$, define $h_* : \boldsymbol{B}_* \to \boldsymbol{A}_*$ by $F \mapsto h^{-1}[F]$.

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The proof of ES for varieties of Heyting algebras $\mathbf{A} = \langle A; \rightarrow, \wedge, \vee, \top, \bot \rangle$ of finite depth uses Esakia duality.

From $\mathbf{A}$, we construct an *Esakia space* $\mathbf{A}_* := \langle \operatorname{Pr} \mathbf{A}; \subseteq, \boldsymbol{\tau} \rangle$.

$\operatorname{Pr} \mathbf{A}$ is the set of all *prime* filters of $\mathbf{A}$ (i.e., all lattice filters F with $\top \in F$, such that $A \backslash F$ is closed under $\vee$), and $\boldsymbol{\tau}$ is a certain topology on $\operatorname{Pr} \mathbf{A}$.

For $a \in A$, we define $\varphi(a) = \{F \in \operatorname{Pr} \mathbf{A} : a \in F\}$ and

$$\varphi(a)^c = \{F \in \operatorname{Pr} \mathbf{A} : a \notin F\}.$$

A sub-basis for $\boldsymbol{\tau}$ is then $\{\varphi(a) : a \in A\} \cup \{\varphi(a)^c : a \in A\}$.

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Theorem. [Esakia, 1974] A ***duality*** between $\mathcal{HA}$ and the category $\mathcal{ESP}$ of *Esakia spaces* (and *morphisms*) is established by the functor $\boldsymbol{A} \mapsto \boldsymbol{A}_*$; $h \mapsto h_*$. I.e.,

the categories $\mathcal{HA}$ and $\mathcal{ESP}^{\mathrm{op}}$ are equivalent.

In general, an **Esakia space** $\boldsymbol{X} = \langle X; \leqslant, \tau \rangle$ comprises a po-set $\langle X; \leqslant \rangle$ and a compact Hausdorff topology τ on X in which

every open set is a union of clopen sets;

$\uparrow x$ is closed, for all $x \in X$;

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An **Esakia morphism** $h \colon \boldsymbol{X} \to \boldsymbol{Y}$ between such spaces is a continuous function such that $h[\uparrow x] = \uparrow h(x)$, for all $x \in X$.

The reverse functor $\boldsymbol{X} \mapsto \boldsymbol{X}^* \in \mathcal{HA}$; $h \mapsto h^*$ sends $\boldsymbol{X}$ to its set of clopen up-sets (including X and $\emptyset$), equipped with operations $\cap, \cup$ and $U \to V \ := \ X \backslash \downarrow (U \backslash V)$, while $h^* \colon U \mapsto h^{-1}[U]$.

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Depth: Let $\boldsymbol{A}$ be a Heyting algebra, with dual $\boldsymbol{A}_* = \langle \mathrm{Pr}\, A; \subseteq, \tau \rangle$.

We say that $\boldsymbol{A}$ (and $\boldsymbol{A}_*$) have **depth** $n \in \omega$ if, in $\boldsymbol{A}_*$, there's a chain $p_1 < \ldots < p_n$, but no chain $q_1 < \ldots < q_{n+1}$.

Depths of *elements* of $\boldsymbol{A}_*$ are defined similarly.

We say that $\mathcal{K} \subseteq \mathcal{HA}$ has **depth** $\leqslant n$ if all $\boldsymbol{A} \in \mathcal{K}$ do.

Fact. $\mathcal{HA}_n := \{\boldsymbol{A} \in \mathcal{HA} : \mathrm{depth}(\boldsymbol{A}) \leqslant n\}$ is a variety, $\forall n \in \omega$.
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Theorem. Let $\mathcal{K} \subseteq \mathcal{HA}$ be a variety of finite depth, n say. Then $\mathcal{K}$ has surjective epimorphisms.

Proof sketch. First, $\mathcal{K}$ has ES iff all $\mathcal{K}_*$-**monomorphisms** h are injective. [Here, $h \circ f = h \circ g \implies f = g$.]

We induct on n, the case $n = 0$ being trivial. Let $n > 0$.

W.l.o.g., we can restrict to the following situation, in which $h\colon \boldsymbol{X} \to \boldsymbol{Y}$ is a $\mathcal{K}_*$-mono, with $x \neq y$ in $\boldsymbol{X}$, where $X = {\uparrow}\{x, y\}$ and — with a view to contradiction — $h(x) = h(y)$.

Here, $\boldsymbol{P} := \{u \in X : \mathrm{depth}(u) < n\}$. By the induction hypothesis, $h|_{\boldsymbol{P}}$ is one-to-one, so x or y has depth $= n$.

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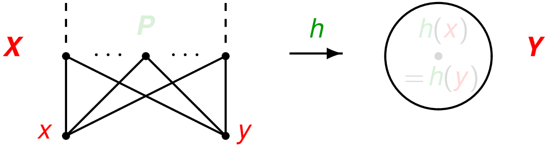

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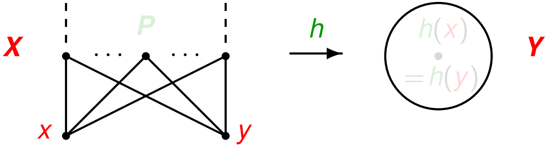

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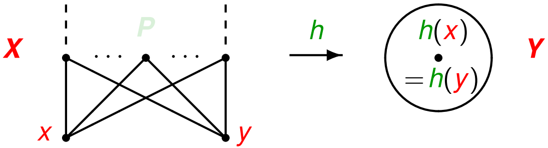

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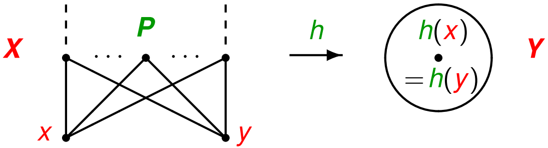

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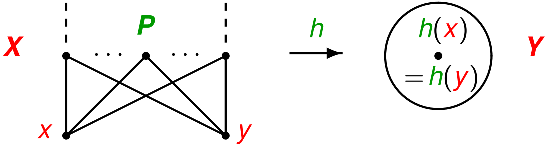

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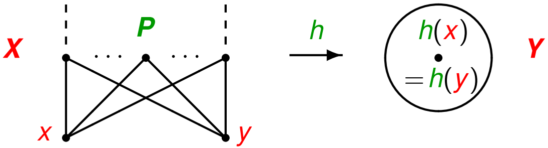

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It follows that $\uparrow x$ and $\uparrow y$ are isomorphic Esakia spaces.

Let $\boldsymbol{W}$ be the disjoint union of $\uparrow x$, $\uparrow y$ and a copy $\uparrow z$ of $\uparrow x$.

Each strict upper bound a of x in $\boldsymbol{X}$ yields copies $a_x > x$, $a_y > y$ and $a_z > z$ of itself in $\boldsymbol{W}$. Sending these back to a, we get Esakia morphisms $g_1, g_2 \colon \boldsymbol{W} \to \boldsymbol{X}$ differing only in that

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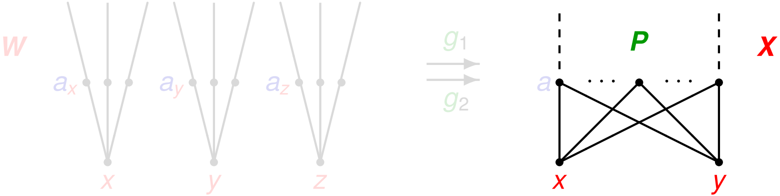

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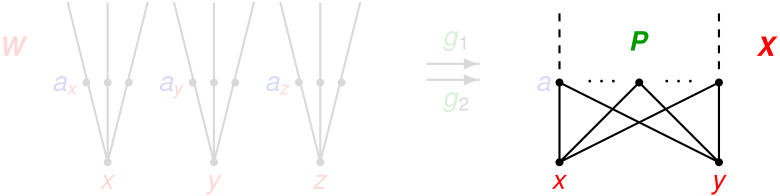

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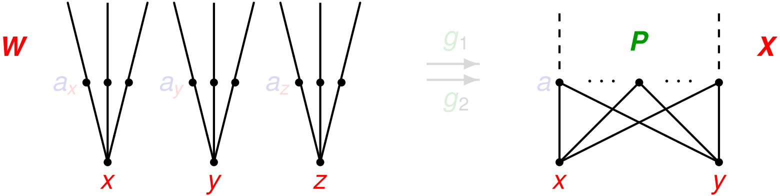

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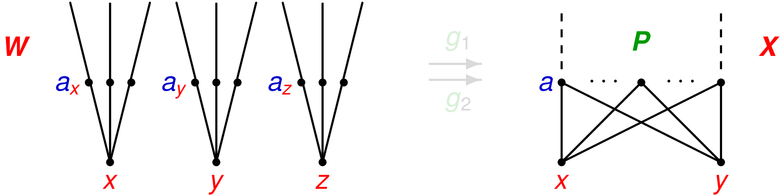

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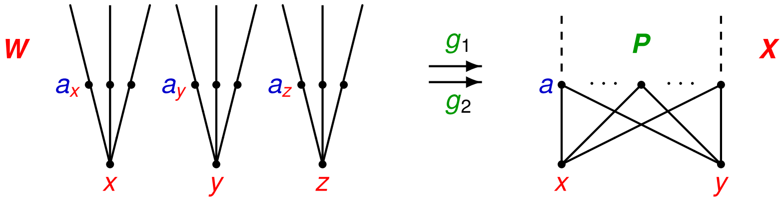

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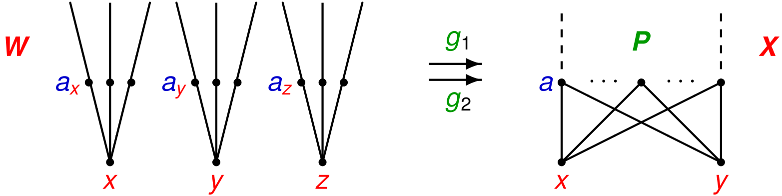

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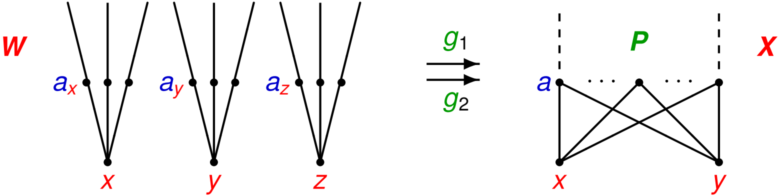

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Now $h \circ g_1 = h \circ g_2 \colon \boldsymbol{W} \to \boldsymbol{Y} \in \mathcal{K}_*$ (as $h(x) = h(y)$).

Since $g_1 \neq g_2$, this will contradict the fact that h is a $\mathcal{K}_*$-monomorphism, *provided* that $\boldsymbol{W} \in \mathcal{K}_*$.

As $\uparrow x$ is a closed up-set of $\boldsymbol{X}$, the inclusion $i \colon (\uparrow x) \to \boldsymbol{X}$ is an $\mathcal{ESP}$-morphism, so $i_* \colon \boldsymbol{X}^* \to (\uparrow x)^*$ is onto, i.e.,

$$(\uparrow x)^* \in \mathbb{H}(\boldsymbol{X}^*) \subseteq \mathbb{H}(\mathcal{K}) \subseteq \mathcal{K} \text{ (since } \mathcal{K}_* \text{ is a variety).}$$

So, $(\uparrow x)^*, (\uparrow y)^*, (\uparrow z)^* \in \mathcal{K}$.

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A proper epic subalgebra in a Heyting algebra variety

The variety $\mathbb{V}(\boldsymbol{A})$ generated by the Heyting algebra $\boldsymbol{A}$ on the left lacks the ES property, confirming the Blok-Hoogland conjecture. The red elements form a $\mathbb{V}(\boldsymbol{A})$-epic subalgebra.

$\mathbb{V}(\boldsymbol{A})$ is locally finite and has a fairly simple finite axiomatization.

An explicit failure of the infinite Beth property can be extracted from this example.

In the finitely subdirectly irreducible (but not all) members of $\mathbb{V}(\boldsymbol{A})$, the 'incomparable companion' of an element is implicitly definable, but *not* explicitly.

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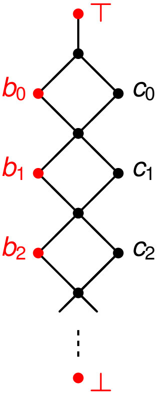

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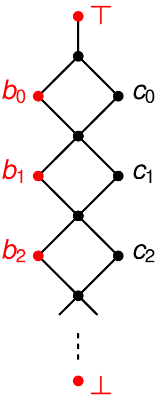

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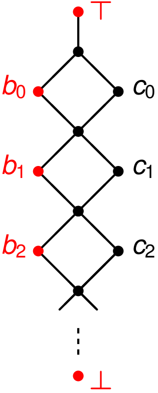

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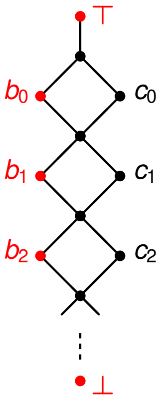

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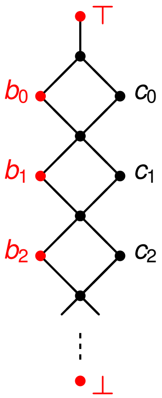

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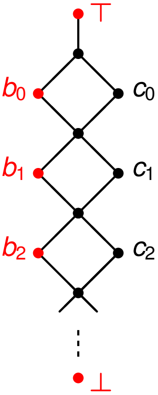

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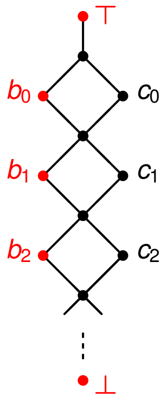

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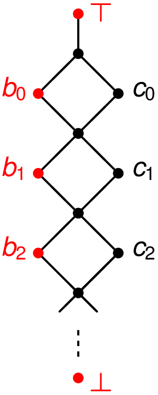

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