Order-Based and Continuous Modal Logics

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“Eat before shopping. If you go to the store hungry, you are likely to make unnecessary purchases.”

American Heart Association Cookbook
Many-valued modal logics with values in $\mathbb{R}$ fall loosely into two families:

- **Order-based modal logics** (e.g., Gödel modal logics)
- **Continuous modal logics** (e.g., Łukasiewicz modal logics)

Key problems include finding axiomatizations and algebraic semantics, and establishing decidability and complexity results.
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Let us say that an algebra $A = \langle A, \land, \lor, 0, 1, \ldots \rangle$ is order-based if

(a) $\langle A, \land, \lor, 0, 1 \rangle$ is a complete sublattice of $\langle [0, 1], \text{min}, \text{max}, 0, 1 \rangle$.

(b) Each operation of $A$ is definable by a quantifier-free first-order formula in a language with operations $\land, \lor$, and constants of $A$. 

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Order-Based Algebras

Let us say that an algebra \( A = \langle A, \wedge, \vee, 0, 1, \ldots \rangle \) is order-based if

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A Definable Operation

The Gödel implication

\[ a \rightarrow b = \begin{cases} 
1 & \text{if } a \leq b \\
b & \text{otherwise}
\end{cases} \]

can always be defined by the quantifier-free first-order formula

\[ F\rightarrow(x, y, z) = ((x \leq y) \Rightarrow (z \approx 1)) \& ((y < x) \Rightarrow (z \approx y)). \]

That is, for all \( a, b, c \in A \),

\[ A \models F\rightarrow(a, b, c) \iff a \rightarrow b = c. \]

Note also that we can also define \( \neg a := a \rightarrow 0 \).
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An **A-frame** $\mathcal{F} = \langle W, R \rangle$ consists of

- a non-empty set of **states** $W$
- an **A-valued accessibility relation** $R: W \times W \to A$.

$\mathcal{F}$ is called **crisp** if also $R_{xy} \in \{0, 1\}$ for all $x, y \in W$.

We extend the language of **A** with unary (modal) connectives $\Box, \Diamond$ and define the set of formulas $Fm$ inductively as usual.
Frames and Formulas

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An **A-model** $\mathcal{M} = \langle W, R, V \rangle$ adds a map $V: \text{Fm} \times W \rightarrow A$ satisfying

$$V(\star(\varphi_1, \ldots, \varphi_n), x) = \star^A(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each operation symbol $\star$ of $A$, and

$$V(\Box \varphi, x) = \bigwedge\{Rxy \rightarrow V(\varphi, y) : y \in W\}$$

$$V(\Diamond \varphi, x) = \bigvee\{Rxy \land V(\varphi, y) : y \in W\}.$$ 

$\mathcal{M}$ is called **crisp** if $\langle W, R \rangle$ is crisp, in which case,

$$V(\Box \varphi, x) = \bigwedge\{V(\varphi, y) : Rxy\}$$

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$$V(\ast(\varphi_1, \ldots, \varphi_n), x) = \ast^A(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each operation symbol $\ast$ of $A$, and

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Validity

A formula \( \varphi \) is called

- **valid** in an \( A \)-model \( \langle W, R, V \rangle \) if \( V(\varphi, x) = 1 \) for all \( x \in W \)
- \( K(A)\)-valid if it is valid in all \( A \)-models
- \( K(A)^C\)-valid if it is valid in all crisp \( A \)-models.
A formula $\varphi$ is called

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- $\mathcal{K}(\mathbf{A})$-**valid** if it is valid in all $\mathbf{A}$-models
- $\mathcal{K}(\mathbf{A})^C$-**valid** if it is valid in all crisp $\mathbf{A}$-models.
Consider the standard algebra for Gödel logic

\[ G = \langle [0, 1], \land, \lor, \rightarrow, 0, 1 \rangle. \]

An axiomatization for \( K(G) \) is obtained by adding the prelinearity axiom schema \((\varphi \rightarrow \psi) \lor (\psi \rightarrow \varphi)\) to the intuitionistic modal logic \( IK \).

X. Caicedo and R. Rodríguez.
Bi-modal Gödel logic over \([0,1]\)-valued Kripke frames.

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More generally, we may consider (expansions of) Gödel modal logics $K(A)$ and $K(A)^C$ where $A$ is any complete subalgebra of $G$; e.g.,

$$A = \{0\} \cup \{\frac{1}{n+1} \mid n \in \mathbb{N}\} \quad \text{or} \quad A = \{1 - \frac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{1\}.$$  

Indeed, there are countably infinitely many different infinite-valued Gödel modal logics (considered as sets of valid formulas).
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More generally, we may consider (expansions of) \textbf{Gödel modal logics} \( K(\mathbf{A}) \) and \( K(\mathbf{A})^C \) where \( \mathbf{A} \) is any complete subalgebra of \( \mathbf{G} \); e.g.,

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\]

Indeed, there are countably infinitely many different infinite-valued Gödel modal logics (considered as sets of valid formulas).
The following formula is valid in all finite \( K(G) \)-models

\[ \square
\neg
\neg p \rightarrow \neg \neg \square p \]

but not in the infinite \( K(G) \)-model \( \langle N, N^2, V \rangle \) where \( V(p, x) = \frac{1}{x+1} \).

\[
\left( V(\square \neg \neg p \rightarrow \neg \neg \square p, 0) = \left( \bigwedge_{x \in N} V(\neg \neg p, x) \right) \rightarrow \left( \neg \neg \bigwedge_{x \in N} V(p, x) \right) \right.
\]

\[
= \left( \bigwedge_{x \in N} 1 \right) \rightarrow \left( \neg \neg \bigwedge_{x \in N} \frac{1}{x+1} \right)
\]

\[
= 1 \rightarrow 0 = 0.
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The following formula is valid in all finite $K(G)$-models

$$\Box \neg \neg p \to \neg \neg \Box p$$

but not in the infinite $K(G)$-model $\langle \mathbb{N}, \mathbb{N}^2, V \rangle$ where $V(p, x) = \frac{1}{x+1}$.

$$
\begin{align*}
V(\Box \neg \neg p \to \neg \neg \Box p, 0) &= (\bigwedge_{x \in \mathbb{N}} V(\neg \neg p, x)) \to (\neg \neg \bigwedge_{x \in \mathbb{N}} V(p, x)) \\
&= (\bigwedge_{x \in \mathbb{N}} 1) \to (\neg \neg \bigwedge_{x \in \mathbb{N}} \frac{1}{x+1}) \\
&= 1 \to 0 = 0.
\end{align*}
$$
Failure of the Finite Model Property

The following formula is valid in all **finite** $K(G)$-models

$$\square \neg \neg p \rightarrow \neg \neg \square p$$

but not in the **infinite** $K(G)$-model $\langle \mathbb{N}, \mathbb{N}^2, V \rangle$ where $V(p, x) = \frac{1}{x+1}$.

$$\left( V(\square \neg \neg p \rightarrow \neg \neg \square p, 0) = \left( \bigwedge_{x \in \mathbb{N}} V(\neg \neg p, x) \right) \rightarrow \left( \neg \neg \bigwedge_{x \in \mathbb{N}} V(p, x) \right) = \left( \bigwedge_{x \in \mathbb{N}} 1 \right) \rightarrow \left( \neg \neg \bigwedge_{x \in \mathbb{N}} \frac{1}{x+1} \right) = 1 \rightarrow 0 = 0. \right)$$
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The following formula is valid in all finite $K(G)$-models

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but not in the infinite $K(G)$-model $\langle \mathbb{N}, \mathbb{N}^2, V \rangle$ where $V(p, x) = \frac{1}{x+1}$.

$$V(\Box \neg\neg p \to \neg\neg \Box p, 0) = (\bigwedge_{x \in \mathbb{N}} V(\neg\neg p, x)) \to (\neg\neg \bigwedge_{x \in \mathbb{N}} V(p, x))$$

$$= (\bigwedge_{x \in \mathbb{N}} 1) \to (\neg\neg \bigwedge_{x \in \mathbb{N}} \frac{1}{x+1})$$

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$$
We prove decidability (indeed PSPACE-completeness) for order-based modal logics satisfying a certain topological property by providing new semantics that admit the finite model property.


The idea is to restrict the values at each state that can be taken by box and diamond formulas; $\Box \varphi$ and $\Diamond \varphi$ can then be “witnessed” at states where the value of $\varphi$ is “sufficiently close” to the value of $\Box \varphi$ or $\Diamond \varphi$. 
Towards Decidability

We prove decidability (indeed PSPACE-completeness) for order-based modal logics satisfying a certain topological property by providing new semantics that admit the finite model property.

X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger.
Decidability of Order-Based Modal Logics.

The idea is to restrict the values at each state that can be taken by box and diamond formulas; □φ and ◊φ can then be “witnessed” at states where the value of φ is “sufficiently close” to the value of □φ or ◊φ.
We augment \( G \)-frames with a map \( T \) from states to finite subsets of \([0, 1]\) containing 0 and 1, and \( G \)-models are defined as before except that

\[
V(\Box \varphi, x) = \max\{r \in T(x) : r \leq \bigwedge_{y \in W} (Rxy \rightarrow V(\varphi, y))\}
\]

\[
V(\Diamond \varphi, x) = \min\{r \in T(x) : r \geq \bigvee_{y \in W} \min(Rxy, V(\varphi, y))\}.
\]
A New Semantics

We augment $G$-frames with a map $T$ from states to finite subsets of $[0, 1]$ containing 0 and 1, and $G$-models are defined as before except that

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A Finite Counter Model

We find a finite counter-model for $\square \neg\neg p \rightarrow \neg\neg \square p$:

$$\langle \{a\}, \{(a, a)\}, T, V \rangle \quad \text{where} \quad V(p, a) = \frac{1}{2} \quad \text{and} \quad T(a) = \{0, 1\}.$$  

\[
\begin{align*}
V(\square \neg\neg p, a) & = \max\{r \in T(a) : r \leq V(\neg\neg p, a)\} = 1 \\
V(\neg\neg \square p, a) & = \neg\neg \max\{r \in T(a) : r \leq V(p, a)\} = 0 \\
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More Generally...

We consider an order-based algebra $\mathbf{A}$ that is "locally homogeneous"; roughly, for any right (or left) accumulation point $a$ of $\mathbf{A}$, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

We augment an $\mathbf{A}$-frame $\langle W, R \rangle$ with maps

$$T\Box : W \rightarrow \mathcal{P}(A) \quad \text{and} \quad T\Diamond : W \rightarrow \mathcal{P}(A)$$

such that for each $x \in W$,

- the constants of $\mathbf{A}$ are contained in both $T\Box(x)$ and $T\Diamond(x)$
- $T\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite $I$, where each $c_i \in A$ witnesses homogeneity at a right accumulation point $a_i$ of $\mathbf{A}$
- $T\Diamond(x) = A \setminus \bigcup_{j \in J}(d_j, b_j)$ for some finite $J$, where each $d_j \in A$ witnesses homogeneity at a left accumulation point $b_j$ of $\mathbf{A}$. 
More Generally...

We consider an order-based algebra $A$ that is "locally homogeneous"; roughly, for any right (or left) accumulation point $a$ of $A$, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

We augment an $A$-frame $\langle W, R \rangle$ with maps

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- $T \lozenge (x) = A \setminus \bigcup_{j \in J} (d_j, b_j)$ for some finite $J$, where each $d_j \in A$ witnesses homogeneity at a left accumulation point $b_j$ of $A$. 

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We consider an order-based algebra $A$ that is “locally homogeneous”; roughly, for any right (or left) accumulation point $a$ of $A$, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

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$$T□: W \to \mathcal{P}(A) \quad \text{and} \quad T◊: W \to \mathcal{P}(A)$$

such that for each $x \in W$,

- the constants of $A$ are contained in both $T□(x)$ and $T◊(x)$
- $T□(x) = A \setminus \bigcup_{i \in I} (a_i, c_i)$ for some finite $I$, where each $c_i \in A$ witnesses homogeneity at a right accumulation point $a_i$ of $A$
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George Metcalfe (University of Bern)
More Generally...

We consider an order-based algebra $\mathbf{A}$ that is “locally homogeneous”; roughly, for any right (or left) accumulation point $a$ of $\mathbf{A}$, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

We augment an $\mathbf{A}$-frame $\langle W, R \rangle$ with maps

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- $T_\Box(x) = A \setminus \bigcup_{i \in I} (a_i, c_i)$ for some finite $I$, where each $c_i \in A$ witnesses homogeneity at a right accumulation point $a_i$ of $\mathbf{A}$
- $T_\Diamond(x) = A \setminus \bigcup_{j \in J} (d_j, b_j)$ for some finite $J$, where each $d_j \in A$ witnesses homogeneity at a left accumulation point $b_j$ of $\mathbf{A}$. 
For any locally homogeneous order-based algebra $A$:

- $K(A)$ and $K(A)^C$ are sound and complete with respect to the new semantics.
- The new semantics has the finite model property.
- If there is an oracle for checking consistency with finite models, then validity in $K(A)$ and $K(A)^C$ are both decidable.

In particular, validity in $K(G)$ and $K(G)^C$ are both PSPACE-complete.
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We have also obtained decidability (indeed, co-NP completeness) for order-based modal logics $S_5(A)^C$ based on crisp $K(A)$-models where $R$ is an equivalence relation.

This provides co-NP completeness also for one-variable fragments of first-order order-based logics (in particular, first-order Gödel logic).

Extending these results to a general theory seems to be difficult...
Beyond the Basic Modal Logics

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Łukasiewicz modal logics are defined with connectives on $[0, 1]$

$$x \rightarrow y = \min(1, 1-x+y) \quad \neg x = 1-x$$
$$x \oplus y = \min(1, x+y) \quad x \odot y = \max(0, x+y-1).$$

Łukasiewicz (multi-)modal logics can also be viewed as fragments of continuous logic and studied as fuzzy description logics.

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Hansoul and Teheux (2013) axiomatize Łukasiewicz modal logic over crisp Kripke frames by adding to an axiomatization of Łukasiewicz logic:

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\begin{align*}
\Box(\varphi \to \psi) & \to (\Box \varphi \to \Box \psi) \\
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\Box(\varphi \otimes \varphi) & \to (\Box \varphi \otimes \Box \varphi) \\
\hline
\varphi & \Box \varphi
\end{align*}
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and a rule with infinitely many premises:

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\begin{array}{cccc}
\varphi \oplus \varphi & \varphi \oplus \varphi^2 & \varphi \oplus \varphi^3 & \ldots \\
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\end{array}
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But is this infinitary rule really necessary?
An Axiomatization Problem

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Towards a Solution...

We have axiomatized a modal logic over $\mathbb{R}$ with abelian group operations (extending the multiplicative fragment of Abelian logic), whose validity problem is in EXPTIME.

Axiomatizing a Real-Valued Modal Logic.  

Extending this system with the additive (lattice) connectives would provide the basis for a finitary axiomatization for Łukasiewicz modal logic.
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Challenges

- Find an axiomatization of crisp Gödel modal logic.
- Develop a robust algebraic theory for order-based modal logics.
- Prove decidability for guarded fragments of order-based modal logics.
- Prove (un)decidability of two-variable fragments of first-order order-based modal logics.
- Find an axiomatization of crisp Łukasiewicz modal logic and investigate its algebraic semantics.
- Establish the complexity of validity in Łukasiewicz modal logics.
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