Some observations regarding cut-free hypersequent calculi for intermediate logics

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Proof theory for intermediate logics

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Remark

Of course decidability is a necessary requirement, but other than that not much seems to be known.

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Definition

A sequent rule (r) is *structural* if its of the form

$$\frac{\Gamma_{11}, \ldots, \Gamma_{1n_1} \Rightarrow \Pi_1 \qquad \ldots \qquad \Gamma_{1m}, \ldots, \Gamma_{1n_m} \Rightarrow \Pi_m}{\Gamma_{01}, \ldots, \Gamma_{0n_0} \Rightarrow \Pi_0}\,(r)$$

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Examples

$$\frac{\Gamma \Rightarrow \Pi}{\Gamma, \varphi \Rightarrow \Pi} \, (lw) \qquad\qquad \frac{\Gamma_1 \Rightarrow \Pi_1 \qquad \Gamma_2 \Rightarrow \Pi_2}{\Gamma_1 \Rightarrow \Pi_2} \, (\lightning)$$

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Structural rules

Proposition (Ciabattoni, Galatos & Terui 2008)

Any structural sequent rule is either derivable in LJ *or derives every formula in* LJ.

Consequently, this approach is *not* helpful when trying to give an (partial) answer to Question I.

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Structural hypersequent rules may be defined in the evident way.

Examples

$$\frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi_1 \qquad H \mid \Sigma_1, \Sigma_2 \Rightarrow \Pi_2}{H \mid \Gamma_1, \Sigma_1 \Rightarrow \Pi_1 \mid \Gamma_2, \Sigma_2 \Rightarrow \Pi_2} \qquad \frac{H \mid \Gamma_1, \Gamma_2 \Rightarrow}{H \mid \Gamma_1 \Rightarrow \mid \Gamma_2 \Rightarrow}$$

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Every structural hypersequent rule (r) *is equivalent to a (so-called completed) structural hypersequent rule* (r') *such that* $\vdash_{\mathrm{HLJ}+(r')} H$ *implies* $\vdash^{cf}_{\mathrm{HLJ}+(r')} H$*, for any hypersequent* H.

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Question III:

For which intermediate logics can we find structural hypersequent calculi? That is, which intermediate logics are determined by hypersequent calculi of the form $\mathrm{HLJ} + \mathscr{R}$, for some set $\mathscr{R}$ of structural hypersequent rules?

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Examples

The logics

$$\mathbf{LC}, \mathbf{KC}, \mathbf{BTW}_n, \mathbf{BW}_n \quad \& \quad \mathbf{BC}_n, \qquad (n \geq 2)$$

can all be axiomatised by formulas belonging to $\mathcal{P}_3$.

The level $\mathcal{P}_3$ of the substructural hierarchy

Theorem (Ciabattoni, Galatos & Terui 2008)

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In fact it is not very difficult to show that any intermediate logic admitting a structural hypersequent calculus can be axiomatised by $\mathcal{P}_3$-formulas.

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$$\begin{aligned}\mathbf{BTW}_n &= \mathbf{IPC} + \bigwedge_{0\leq i<j\leq n}\left(\neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j\neq i} \neg p_j)\right)\\ &= \mathbf{IPC} + \bigvee_{i=0}^{n}\left(\bigwedge_{j<i} p_j \to \neg\neg p_i\right).\end{aligned}$$

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What we need are semantic criteria determining whether or not an intermediate logic can be axiomatised by $\mathcal{P}_3$-formulas. We provide such criteria, both in terms of the algebraic and the Kripke semantics.

Stable intermediate logics

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Definition

We say that an intermediate logic L is $(0, \wedge, 1)$-stable if whenever $h\colon \mathfrak{A} \hookrightarrow \mathfrak{B}$ is an $(0, \wedge, 1)$-embedding of Heyting algebras with $\mathfrak{B} \in \mathbb{V}(L)_{si}$ then $\mathfrak{A} \in \mathbb{V}(L)$.

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Key lemma

An intermediate logic L is $(0, \wedge, 1)$-stable iff $\mathbb{V}(L)$ is generated by a universal class of Heyting algebras closed under $(0, \wedge, 1)$-subalgebras.

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Corollary

For $n \geq 2$ the logic $\mathbf{BD}_n$ does not admit any structural hypersequent calculus, i.e., a calculus of the form $\mathrm{HLJ} + \mathscr{R}$, with $\mathscr{R}$ a set of structural rules.

Semantic characterisation: Kripke frames

Definition

A first-order formula is a *geometric implication* if it a conjunction of formulas the form

$$\forall \vec{w}(\varphi(\vec{w}) \Longrightarrow \exists \vec{v}(\psi_1(\vec{w}, \vec{v}) \text{ or } \ldots \text{ or } \psi_n(\vec{w}, \vec{v}))),$$

where φ and ψ_k are conjunctions of atomic formulas and the variables $\vec{v}$ do not occur (free) in φ.

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Geometric implications can be used to construct labeled sequent calculi for intermediate and modal logics.

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(in the language of partial orders) is said to be *simple* if

1. there is $w_0 \in \vec{w}$ such that every atomic subformula of φ is of the form $w_0 \leq w_l$;
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Examples

$\forall w_1, \ldots, w_{n+1}(\text{AND}_{i=1}^{n}(w_i \leq w_{i+1}) \implies \text{OR}_{i \neq j}(w_i = w_j))$ &

$\forall w_0, w_1, w_2((w_0 \leq w_1 \text{ and } w_0 \leq w_1) \implies \exists v\ (w_1 \leq v \text{ and } w_2 \leq v)).$

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1. Cardinality of $(0, \wedge, 1)$-stable logics;
2. Substructural logics: Semantic characterisation of $\mathcal{P}_3$-formulas ($\mathcal{P}'_3$-formulas) over $\mathbf{FL}_{ew}$ and $\mathbf{FL}_e$;
3. Hypersequent calculi for modal logics and stable modal logics.

Thank you for your attention.