Embedding ExtIPC into ExtPLL via canonical formulas

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Introduction/Notation

- $\mathcal{L}_{\text{IPC}}$ denote the language of propositional logic.
- $\text{IPC}$ denotes the intuitionistic propositional calculus.
- $\text{ExtIPC}$ denotes the lattice of all superintuitionistic logics (si-logics).

- An intuitionistic modal logic is a collection of formulas in the language $\mathcal{L}_{\text{IPC}} \cup \{\lozenge\}$, closed under MP and substitution.
Propositional lax logic **PLL**

**Definition**

PLL is an intuitionistic modal logic with a peculiar modality $\bigcirc$ that is axiomatized by

- $p \rightarrow \bigcirc p$
- $\bigcirc\bigcirc p \rightarrow \bigcirc p$
- $\bigcirc(p \land q) \leftrightarrow (\bigcirc p \land \bigcirc q)$.

The modality $\bigcirc$ was studied in several contexts (see Fairtlough and Mendler [FM97]). Different semantics were studied by Goldblatt in [Gol81], by Dragalin in [Dra88] and in [FM97]. PLL has the finite model property and is decidable [Gol81], [FM97], [WZ98].
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Nuclear Heyting algebras

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Let $A$ be a Heyting algebra. A *nucleus* on $A$ is a function $j : A \rightarrow A$ such that for all $a, b \in A$

$$a \leq j(a), \quad j(j(a)) = j(a), \quad j(a \land b) = j(a) \land j(b).$$

A *nuclear Heyting algebra* is a pair $\mathfrak{A} = (A, j)$, where $A$ is a Heyting algebra and $j$ is a nucleus on $A$. 

**Theorem (Gol81)**

Every $M \in \text{ExtPLL}$ is sound and complete with respect to its corresponding variety of nuclear Heyting algebras.
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Goal of the talk

Let $\Gamma \subseteq \mathcal{L}_{\text{IPC}}$. Then

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Let $X \in \{\text{fmp, Kripke completeness, tabularity, decidability}\}$. 
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Suppose the si-logic $\text{IPC} + \Gamma$ has property $X$,

does $\text{PLL} + \Gamma$ have property $X$, too?

- Wolter and Zakharyaschev studied such preservation results by embedding intuitionistic modal logics into classical bi-modal logics.
Canonical formulas for si-logics

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- Let $A$ be a finite s.i. Heyting algebra, $D \subseteq A^2$. Then the canonical formula $\beta(A, D)$ encodes
  - the $(\land, \to, 0)$-structure of $A$ fully and
  - the behavior of $\lor$ partially on the set $D$. 
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- Every formula in $\mathcal{L}_{\text{IPC}}$ is equivalent to a finite conjunction of canonical formulas.

- Thus, all si-logics are axiomatizable by canonical formulas.
A nuclear Heyting algebra $\mathfrak{A} = (A,j)$ is subdirectly irreducible (s.i.) iff $A$ has a second largest element.
Canonical formulas for PLL

- A nuclear Heyting algebra $\mathcal{A} = (A, j)$ is subdirectly irreducible (s.i.) iff $A$ has a second largest element.

- Let $\mathcal{A} = (A, j)$ be finite and s.i., $D^\vee \subseteq A^2$ and $D^\ominus \subseteq A$. 
A nuclear Heyting algebra $\mathfrak{A} = (A, j)$ is subdirectly irreducible (s.i.) iff $A$ has a second largest element.

Let $\mathfrak{A} = (A, j)$ be finite and s.i., $D^\lor \subseteq A^2$ and $D^\circ \subseteq A$.

For $a \in A$ let $p_a$ be a propositional letter, let $s$ be the second largest element of $A$. 

$$\beta(\mathfrak{A}, D^\lor, D^\circ) := \bigwedge \{ p_a^* b \leftrightarrow (p_a^* p_b) | a, b \in A, \ast \in \{\land, \rightarrow\} \} \land \{ p_0 \leftrightarrow 0 \} \land \bigwedge \{ p_a \lor b \leftrightarrow (p_a \lor p_b) | a, b \in D^\lor \} \land \bigwedge \{ \# p_a \rightarrow p_{j(a)} | a \in D^\circ \} \rightarrow p_s.$$ 

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$\to p_s$.

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Refutation criterion

- Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a $(\land, \rightarrow, 0)$-morphism.
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  - If \( j(f(a)) \leq f(j(a)) \) for all \( a \in A \), then we call \( f \) stable.
  - If in addition,
    - \( f(a \lor b) = f(a) \lor f(b) \) for every \( (a, b) \in D^\lor \) and
    - \( f(j(a)) = j(f(a)) \) for every \( a \in D^\circ \),
  then \( f \) is called \((D^\lor, D^\circ)\)-stable.
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**Theorem**

*For every nuclear Heyting algebra \( \mathcal{B} = (B, j) \), the following are equivalent:*

1. \( \mathcal{B} \not\models \beta(\mathcal{A}, D^\lor, D^\bigcirc) \).

2. There is a homomorphically image \( \mathcal{C} \) of \( \mathcal{B} \) and a \((D^\lor, D^\bigcirc)\)-stable embedding from \( \mathcal{A} \) into \( \mathcal{C} \).*
Axiomatic completeness

Proposition

For every PLL-formula \( \varphi \), there is a finite collection \( \{ (A_i, D_i^\lor, D_i^\circ) \}_{1 \leq i \leq n} \) such that for each nuclear Heyting algebra \( \mathcal{B} \), TFAE:

1. \( \mathcal{B} \not\models \varphi \).
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Corollary

1. Every formula in the language of PLL is equivalent to a finite conjunction of canonical formulas.

2. Every \( M \in \text{ExtPLL} \) can be axiomatized by canonical formulas.
Comparison to Zakharyaschev’s canonical formulas

- By “deleting the parts with $\circ$”, we obtain an algebraic version of Zakharyaschev.’s canonical formulas:
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- By “deleting the parts with $\circ$”, we obtain an algebraic version of Zakharyaschev.'s canonical formulas:

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$$\rightarrow p_s.$$
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1. \( B \not\models \beta(A, D^\vee) \).

2. There is a homomorphically image \( C \) of \( B \) and a \((\land, \rightarrow, 0)\)-embedding from \( h : A \to C \) with \( h(a \lor b) = h(a) \lor h(b) \) for all \((a, b) \in D^\vee\).
Lemma

Let $\mathcal{A} = (A, j)$ be finite and s.i., $D^\vee \subseteq A^2$, $D^\bigcirc \subseteq A$. Let $L = \text{IPC} + \Gamma$. Then

$$L \vdash \beta(A, D^\vee) \text{ implies } \text{PLL} + \Gamma \vdash \beta(\mathcal{A}, D^\vee, D^\bigcirc).$$
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Proof.

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- There is $\mathcal{B} = (B, k)$ with $\mathcal{B} \models \Gamma$ and $\mathcal{B} \nvdash \beta(\mathcal{A}, D^\vee, D^\bigcirc)$.

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- So, $C$ is a homomorphic image of $B$ and $h : A \to C$ is a $(\wedge, \to, 0)$-embedding from $A$ into $C$ with $h(a \vee b) = h(a) \vee h(b)$ for all $(a, b) \in D^\vee$. 
Lemma

Let $\mathfrak{A} = (A, j)$ be finite and s.i., $D^\vee \subseteq A^2$, $D^\circ \subseteq A$. Let $L = \text{IPC} + \Gamma$. Then

$$L \models \beta(A, D^\vee) \text{ implies } \text{PLL} + \Gamma \models \beta(\mathfrak{A}, D^\vee, D^\circ).$$

Proof.

- Suppose $\text{PLL} + \Gamma \not\models \beta(\mathfrak{A}, D^\vee, D^\circ)$.
- There is $\mathfrak{B} = (B, k)$ with $\mathfrak{B} \models \Gamma$ and $\mathfrak{B} \not\models \beta(\mathfrak{A}, D^\vee, D^\circ)$.
- There is a homomorphically image $C$ of $\mathfrak{B}$ and a $(D^\vee, D^\circ)$-stable embedding $h : \mathfrak{A} \to C$.
- So, $C$ is a homomorphically image of $B$ and $h : A \to C$ is a $(\land, \to, 0)$-embedding from $A$ into $C$ with $h(a \lor b) = h(a) \lor h(b)$ for all $(a, b) \in D^\vee$.
- $B \not\models \beta(A, D^\vee)$. Since $B \models L$, $\beta(A, D^\vee) \not\in L$. 


Theorem

Let $L = \text{IPC} + \Gamma$ be a si-logic. If $L$ has one of the properties
- tabularity,
- the fmp,
- Kripke completeness,
- decidability and Kripke completeness,

then $\text{PLL} + \Gamma$ also enjoys the same property.
Sketch: fmp is preserved.

Suppose $PLL + \Gamma \not\vdash \beta(A, D^\vee, D^\Box)$. Then $L \not\vdash \beta(A, D^\vee)$. 

Since $C$ is an $L$-algebra, $C$ validates $\Gamma$ and is finite.
Sketch: fmp is preserved.

- Suppose $\text{PLL} + \Gamma \not\vdash \beta(A, D^\vee, D^\bigcirc)$. Then $L \not\vdash \beta(A, D^\vee)$.
- Some finite $L$- Heyting algebra $B$ refutes $\beta(A, D^\vee)$. 

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- Suppose $\textbf{PLL} + \Gamma \not\models \beta(A, D^\lor, D^\land)$. Then $L \not\models \beta(A, D^\lor)$.
- Some finite $L$-Heyting algebra $B$ refutes $\beta(A, D^\lor)$.
- $\exists$ homomorphic image $C$ of $B$ and a $(\land, \rightarrow, 0)$-embedding $f : A \rightarrow C$ preserving $\lor$ for $(a, b) \in D^\lor$. 
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- Define a nucleus $k$ on $C$ so that $f : \mathfrak{A} \rightarrow \mathfrak{C} = (C, k)$ preserves its structure.
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- Define a nucleus $k$ on $C$ so that $f : \mathcal{A} \to \mathcal{C} = (C, k)$ preserves its structure.

- Then $f : \mathcal{A} \to \mathcal{C}$ is $(D^\vee, D^\circ)$-stable, so $\mathcal{C}$ refutes $\beta(\mathcal{A}, D^\vee, D^\circ)$. 

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- Define a nucleus $k$ on $C$ so that $f : \mathcal{A} \to C = (C, k)$ preserves its structure.
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Future work and open problems

- Is the $(\land, \rightarrow, j)$-fragment of nuclear Heyting algebras locally finite?
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- Thank you!